Voting Systems

Jay Daigle

jaydaigle@gwu.edu

https://jaydaigle.net/politics

The George Washington University

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Discussion Question

Alex, Bailey, and Casey are running for office.

33 voters like Alex best	18 prefer Bailey to Casey
	15 prefer Casey to Bailey
32 voters like Bailey best	24 prefer Alex to Casey
	8 prefer Casey to Alex
34 voters like Casey best	16 prefer Alex to Bailey
	18 prefer Bailey to Alex

Question

Who should win?

Another view

18	15	24	8	16	18
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Questions to think about

- Who has the most first-place votes? C has 34
- Who has the least first-place votes? B has 32
- Who has the most *last*-place votes? C has 42
- Who has the least last-place votes? A has 26

Another view

18	15	24	8	16	18
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

More Questions

- Who wins between A and B? B wins 50 to 49
- Who wins between A and C? A wins 57 to 42
- Who wins between B and C? B wins 50 to 49

Terminology

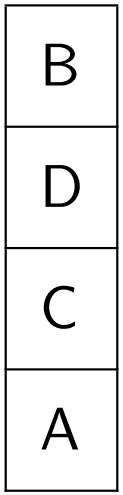
Consider an election with multiple candidates

Definition

The set of candidates is the slate. A, B, C, ...

The set of voters is the electorate.

Each voter submits a preference ballot, listing candidates in descending order of preference.



Basic Assumptions

Logistics: not our problem

We assume the votes are already collected and counted

- Who gets to vote?
- How do they submit their votes?
- What if voters make mistakes?
- What if we count the votes wrong?

Important questions! But we'll ignore them.

Basic Assumptions

Rational voters

We assume voter preferences rational or transitive

If $B > D$ and $D > C$ then $B > C$.

True for numbers. Is it true for people?

Voter profiles

A	B	A	C	A	B	C
B	C	B	A	B	C	A
C	A	C	B	C	A	B

Tabulated Profiles

3	2	2
A	B	C
B	C	A
C	A	B

Social Choice Functions

Definition

A social choice function for a slate of candidates takes in a voter profile, and outputs a non-empty subset of the slate.

We think of this subset as the list of "winners" of the election. We must have at least one winner, but we can have multiple winners (which you might interpret as a tie).

- Lots of possible functions!
- With three candidates and four voters, number of possible functions is a thousand digit number.
- But most aren't very interesting.

Plurality Voting

Definition

A candidate who gets more votes than any other candidate is said to have a plurality of the votes.

In the plurality method we select as the winner the candidate who is ranked first by the largest number of voters. In the case that there is a tie for the most first-choice votes, we select all the candidates who tie for the most first-choice votes.

- The good: simple and familiar
- The bad: doesn't use much information!

Plurality Voting Example

3	2	2
A	B	C
B	C	A
C	A	B

Who wins?

A has three first-place votes, while B and C each have two. So A wins.

Plurality Voting Problems

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

Who wins?

A has five first-place votes, while B, C, and D each have four and E has three. So A wins.

But fifteen people hate A. Is this a good idea?

Hare's Method

Definition

Hare's method operates as follows. Unless all candidates have the same number of first-place votes, identify the candidate (or candidates) who have the fewest first-place votes, and eliminate them from consideration. Create a new profile where that candidate is removed, and each voter moves up their lower-ranked candidates by one place.

Repeat the process, either until only one candidate remains, or until all remaining candidates have the same number of first-place votes. The remaining candidates are the winners.

Hare's Method

- Used in Australia, Papua New Guinea, Alaska, Maine, NYC
- Rewards strong base of support
- Simulates multi-round election with runoffs
- Referred to as *Instant Runoff Voting*, *Single Transferable Vote*
- Sometimes just *ranked choice voting*

  But lots of voting methods use ranked choice!

Hare's Method Example

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

$\rightarrow^{E}$

5	4	4	4	3
A	B	C	D	D
B	C	B	B	B
C	D	D	C	C
D	A	A	A	A

$\rightarrow^{B,C}$

5	4	4	4	3
A	D	D	D	D
D	A	A	A	A

$=$

5	15
A	D
D	A

So D wins.

Who would win in a race between just B and D?

Hare's Method Example 2

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

$\rightarrow^{E}$

5	4	4	4	3
B	C	A	D	A
C	A	B	A	B
D	B	D	B	D
A	D	C	C	C

$\rightarrow^{C,D}$

5	4	4	4	3
B	A	A	A	A
A	B	B	B	B

$=$

5	15
B	A
A	B

So A wins.

Coombs's Method

Definition

Coombs's method operates as follows. Unless all candidates have the same number of last-place votes, identify the candidate (or candidates) with the most last-place votes, and eliminate them. As in Hare's method, when a candidate is eliminated, remove them from the profile and let each voter move the candidates they ranked below the eliminated candidate up a spot.

Repeat the process, eliminating candidates, until either one candidate remains, or all remaining candidates have the same number of last-place votes. The candidate or candidates who remain at the end are the winners.

Rewards lack of strong opposition

Coombs's Method Example

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

$\rightarrow^{A}$

5	4	4	4	3
B	B	C	D	E
C	C	B	B	D
E	E	D	E	B
D	D	E	C	C

$=$

9	4	4	3
B	C	D	E
C	B	B	D
E	D	E	B
D	E	C	C

$\rightarrow^{D}$

9	4	4	3
B	C	B	E
C	B	E	B
E	E	C	C

$\rightarrow^{E}$

9	4	4	3
B	C	B	B
C	B	C	C

$=$

16	4
B	C
C	B

So B wins.

Coombs's Method Example 2

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

$\rightarrow^{C}$

5	4	4	4	3
B	A	A	D	E
E	B	B	A	A
D	E	E	B	B
A	D	D	E	D

$=$

5	8	4	3
B	A	D	E
E	B	A	A
D	E	B	B
A	D	E	D

$\rightarrow^{D}$

5	8	4	3
B	A	A	E
E	B	B	A
A	E	E	B

$=$

5	12	3
B	A	E
E	B	A
A	E	B

$\rightarrow^{E}$

5	12	3
B	A	A
A	B	B

$=$

5	15
B	A
A	B

so A wins.

Borda Count

Definition

The Borda count method works as follows. If there are n candidates, give each candidate $n-1$ points for each voter who ranks them first; $n-2$ points for each voter who ranks them second; $n-3$ points for each candidate who ranks them third; and so on, until they get 1 point for each voter who ranks them second-to-last (and 0 points for each voter who ranks them last.) Add up all the points; the candidate who gets the most points wins. If more than one candidate ties for the most points, all of them win.

Rewards broad moderate support

Borda Count Example

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

	5	4	4	4	3
4	A	B	C	D	E
3	B	C	B	B	D
2	C	E	D	E	B

- A gets 20 points
- B gets 61 points
- C gets 45 points
- D gets 37 points
- E gets 37 points.

B wins by the Borda count method.

Borda Count Example 2

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

- A gets 49 points
- B gets 54 points
- C gets 31 points
- D gets 28 points
- E gets 38 points.

B wins by the Borda count method.

Questions

- Who wins by plurality?
- Who wins in a race between A and B?

Copeland's Method

Definition

Copeland's method is the social choice function in which each candidate earns one point for every candidate they beat in a head-to-head matchup (using a simple majority method). A candidate earns half a point for every candidate they tie.

The candidate with the most points at the end becomes the winner. If there's a tie for the most points, all those candidates are winners.

Rewards strong head-to-head matchups

Rewards candidates who are preferred to many other individual candidates

Copeland's Method Example

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

	A	B	C	D	E	Total
A	-	0	0	0	0	0
B	1	-	1	1	1	4
C	1	0	-	1	1	3
D	1	0	0	-	0	1
E	1	0	0	1	-	2

	AvB	AvC	AvD	AvE	BvC	BvD	BvE	CvD	CvE	DvE
1st	5	5	5	5	16	13	17	13	13	8
2nd	15	15	15	15	4	7	3	7	7	12
Winner	B	C	D	E	B	B	B	C	C	E

B wins with four total points.

Copeland's Method Example 2

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

	A	B	C	D	E	Total
A	-	1	1	1	1	4
B	0	-	1	1	1	3
C	0	0	-	0	0	0
D	0	0	1	-	0	1
E	0	0	1	1	-	2

A wins with four points.

Slate 1 scoreboard

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

Plurality	A
Hares	D
Coombs's	B
Borda Count	B
Copeland's Method	B

Slate 2 scoreboard

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

Plurality	B
Hares	A
Coombs's	A
Borda Count	B
Copeland's Method	A

Which Method is Best?

- Plurality is simple
- Hare's method rewards strong support
- Coombs's method rewards lack of opposition
- Borda Count rewards broad moderate support
- Copeland's method rewards good head-to-head results

Question for you

Which is best?

Some Bad Options

Definition

In the dictatorship method, one voter is the dictator. Their first-choice candidate is the unique winner.

Definition

In the monarchy method, one candidate is the monarch. That candidate is the unique winner regardless of how anyone votes.

Definition

In the all-ties method, every candidate is selected as a winner.

Discussion Question

When might each of these be a good idea?