Expected Value and Mixed Strategies

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Expected Value

Definition

- Given a sample space with n outcomes
- With probability distribution $P = (p_1, p_2, \ldots, p_n)$.
- Let X be a random variable that assigns the payoff x_k to outcome k.
- We define the expected value of X to be

$$E = E(X) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

- Idea: average payoff per play if you play a bunch of times.
- If you run your process a hundred times, you will get about $100 \cdot E(X)$.

Expected Value

Definition

The expected value principle says a rational player will choose the option with the largest expected value.

- Can be seen as a definition of “rational”
- Often taken as prescriptive: a player *should* choose
- Has many caveats! Just one model of decision-making.

Mixed Strategies

Definition

- Consider an $m \times n$ zero-sum two-player game
- The strategies corresponding to rows are columns are now called pure strategies
- A mixed strategy for Row is a probability distribution $P = (p_1, \ldots, p_m)$ on their set of pure strategies
  - Row chooses a pure strategy at random, choosing row k with probability p_k
- A mixed strategy for Column is a probability distribution $Q = (q_1, \ldots, q_n)$ on their set of pure strategies.

Mixed Strategies

- Take $Q = (1/4, 1/2, 1/4)$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0
	$1/4$	$1/2$	$1/4$

- If Row plays Rock: $E = \frac{1}{4} \cdot (0) + \frac{1}{2} \cdot (-1) + \frac{1}{4} \cdot (1) = -\frac{1}{4}$
  - On average, Row will lose one point per four games.
- If Row plays Paper: $E = \frac{1}{4} \cdot (1) + \frac{1}{2} \cdot (0) + \frac{1}{4} \cdot (-1) = 0$
- If Row plays Scissors: $E = \frac{1}{4} \cdot (-1) + \frac{1}{2} \cdot (1) + \frac{1}{4} \cdot (0) = \frac{1}{4}$
- What should Row do? Play Scissors.

Mixed Strategies

Discussion Question

Is "Play Rock every time" a mixed strategy?

Definition

- A basic mixed strategy is a probability distribution with every probability except one equal to 0.
- Write P_i for Row's basic mixed strategy that sets $p_i = 1$ and always plays row i
- Write Q_j for Column's basic mixed strategy that sets $q_j = 1$ and always plays column j.

- We'll start by studying what happens when one player uses a pure strategy against the other player's mixed strategy
- Later we'll generalize.

Mixed Strategies and Expected Value

Lemma

Consider a $m \times n$ matrix game.

- *If Row plays P_i against Column's mixed $Q = (q_1, \ldots, q_n)$, then the expected value of the payoff is*

$$E(P_i, Q) = q_1 u_{i,1} + q_2 u_{i,2} + \cdots + q_n u_{i,n}.$$

- *Similarly, if Column plays P_j against Row's mixed $P = (p_1, \ldots, p_m)$, then the expected value of the payoff is*

$$E(P, Q_j) = p_1 u_{1,j} + p_2 u_{2,j} + \cdots + p_m u_{m,j}.$$

Corollary

If P_i and Q_j are basic mixed strategies, then $E(P_i, Q_j) = u_{i,j}$.

Independent Probabilities

- Often want to think about two separate random processes
- E.g. Row chooses a strategy and Column chooses a strategy
- Sometimes they choices are entangled but sometimes not.

Definition

- Suppose we have two random processes:
  - $P = (p_1, \ldots, p_m)$
  - $Q = (q_1, \ldots, q_n)$
- The processes are independent if the probability of the compound outcome (i, j) is $p_i q_j$.

Independent Probabilities

Example (Two coins)

- Flip two (fair) coins
  - Each has sample space $\{H, T\}$
  - Each has $P = (1/2, 1/2)$
- Flip both coins:
  - Sample space is $\{HH, HT, TH, TT\}$
  - Each has probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
  - $P = (1/4, 1/4, 1/4, 1/4)$.

Independent Probabilities

Example

- Tape two coins together head-to-tail
  - Each has sample space $\{H, T\}$
  - Each has $P = (1/2, 1/2)$
- But overall sample space is $\{HH, TT\}$
- Probability *both* coins land heads-up is $1/2$
- Overall probability is $(1/2, 1/2)$.
- Not independent.

Independent Probabilities

Example

- Class year and age?
  - Not independent
- Height and last digit of birthday?
  - Independent
- Political party and willingness to answer political polls?
  - Not independent unfortunately

Independent Probabilities

- Lots of things we want to measure are *correlated*, and thus not independent
- Independence is a major assumption underlying a lot of statistical tools
- A lot of statistical work goes into dealing with that problem
- When you read a study, think about independence
- But when I give two distributions in a class exercise, assume they're independent.
- In particular, Row's mixed strategy and Column's mixed strategy are independent
  - Unless Column can read Row's mind.

Independent Mixed Stratgies

Example (Mixed Strategies in Rock Paper Scissors)

- Row plays $P = (1/4, 1/2, 1/4)$
- Column plays $Q = (1/6, 1/3, 1/2)$

	R	P	S	
R	0	-1	1	1/4
P	1	0	-1	1/2
S	-1	1	0	1/4
	1/6	1/3	1/2	

	R	P	S	
R	1/24	1/12	1/8	1/4
P	1/12	1/6	1/4	1/2
S	1/24	1/12	1/8	1/4
	1/6	1/3	1/2	

Independent Mixed Stratgies

Example (Mixed Strategies in Rock Paper Scissors)

	R	P	S	
R	0	-1	1	1/4
P	1	0	-1	1/2
S	-1	1	0	1/4
	1/6	1/3	1/2	

	R	P	S	
R	1/24	1/12	1/8	1/4
P	1/12	1/6	1/4	1/2
S	1/24	1/12	1/8	1/4
	1/6	1/3	1/2	

$$\begin{aligned} E(P,Q) = {} & \frac{1}{24} \cdot (0) + \frac{1}{12} \cdot (-1) + \frac{1}{8} \cdot (1) \\ & + \frac{1}{12} \cdot (1) + \frac{1}{6} \cdot (0) + \frac{1}{4} \cdot (-1) \\ & + \frac{1}{24} \cdot (-1) + \frac{1}{12} \cdot (1) + \frac{1}{8} \cdot (0) = -1/12. \end{aligned}$$

Independent Mixed Stratgies

Lemma

- *$m \times n$ game with payoffs $u_{i,j}$*
- *Row plays $P = (p_1, \ldots, p_m)$*
- *Column plays $Q = (q_1, \ldots, q_n)$*
- *Then the probability of the outcome (i, j) is $p_i q_j$,*
- *Expected value of the payoff is the sum of the numbers $p_i \cdot q_j \cdot u_{i,j}$ for all values of i and j.*

$$
\begin{aligned}
E(P, Q) = {} & p_1 \cdot q_1 \cdot u_{1,1} + p_2 \cdot q_1 \cdot u_{2,1} + \cdots + p_m \cdot q_1 \cdot u_{m,1} \\
& + p_1 \cdot q_2 \cdot u_{1,2} + p_2 \cdot q_2 \cdot u_{2,2} + \cdots + p_m \cdot q_2 \cdot u_{m,2} \\
& \vdots \\
& + p_1 \cdot q_n \cdot u_{1,n} + p_2 \cdot q_n \cdot u_{2,n} + \cdots + p_m \cdot q_n \cdot u_{m,n}
\end{aligned}
$$

Independent Mixed Stratgies

Lemma

- *$m \times n$ game with payoffs $u_{i,j}$*
- *Row plays $P = (p_1, \ldots, p_m)$*
- *Column plays $Q = (q_1, \ldots, q_n)$*

Then the expected value of the payoff is

$$E(P,Q) = p_1 E(P_1, Q) + \cdots + p_m E(P_m, Q)$$
$$E(P,Q) = q_1 E(P, Q_1) + \cdots + q_n E(P, Q_n).$$

Proof.

Expand out either of these sums, to see that either one adds up all the $p_i q_j u_{i,j}$. □

Optimal Responses to Mixed Strategies

- Some games have saddle points
- In that case, both players will want to play the saddle point
- Game will converge to a stable equilibrium
- If there's no saddle point, no pure strategy gives a stable equilibrium
- John von Neumann proved every two-person zero sum game has an optimal, equilibrium mixed strategy.
- First step: find the optimal response to a given mixed strategy.

Optimal Responses to Mixed Strategies

Example

- In Rock Paper Scissors, Column plays $Q = (2/7, 4/7, 1/7)$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0
	$2/7$	$4/7$	$1/7$

- What should Row do?
- What do you guess before computing anything?
- Let's compute expected value of each response.

Optimal Responses to Mixed Strategies

Example

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0
	2/7	4/7	1/7

$$E(P_1, Q) = \frac{2}{7} \cdot (0) + \frac{4}{7} \cdot (-1) + \frac{1}{7} \cdot (1) = -3/7$$
$$E(P_2, Q) = \frac{2}{7} \cdot (1) + \frac{4}{7} \cdot (0) + \frac{1}{7} \cdot (-1) = 1/7$$
$$E(P_3, Q) = \frac{2}{7} \cdot (-1) + \frac{4}{7} \cdot (1) + \frac{1}{7} \cdot (0) = 2/7$$

- Best pure strategy: play Scissors

Optimal Responses to Mixed Strategies

- Can find the best pure strategy response

Discussion Question

Can we do better with a mixed strategy?