

Solving 2×2 Games

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November 19, 2025

Solving 2×2 games

Theorem

Any 2×2 zero-sum game has a Nash equilibrium, and there's a formula to find it.

Proof.

Consider the most general possible 2×2 game:

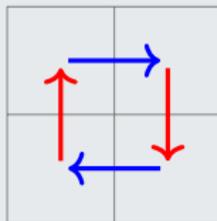
	1	2
1	a	b
2	c	d

- If there's a saddle point, that gives a Nash Equilibrium.
- Assume there's no saddle point. What happens?

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Proof.

	1	2
1	a	b
2	c	d

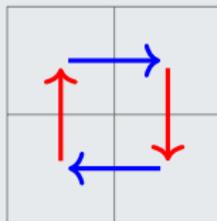


- Assume without loss of generality that $a \geq c$
 - (If $c \geq a$ swap the columns around)
- That gives a flow diagram with the left arrow pointing up
- Since there's no saddle point, we must get this flow diagram.
 - Can't have $a = c$, so $a > c$
 - Also $d > b$
 - $a > b, d > c$.

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Proof.

	1	2
1	a	b
2	c	d



- Suppose Column adopts strategy $Q = (1 - q, q)$. Then
$$E(P_1, Q) = a(1 - q) + bq = (b - a)q + a$$
$$E(P_2, Q) = c(1 - q) + dq = (d - c)q + c.$$
- These are equations of lines
- $b < a$, so first line slopes down
- $d > c$, so second line slopes up
- They intersect somewhere; find the intersection algebraically.

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Proof.

$$(b - a)q + a = (d - c)q + c$$

$$a - c = aq - bq + dq - cq$$

$$= ((a - c) + (d - b))q$$

$$q = \frac{a - c}{(a - c) + (d - b)}.$$

$$1 - q = 1 - \frac{a - c}{(a - c) + (d - b)}$$

$$= \frac{(a - c) + (d - b)}{(a - c) + (d - b)} - \frac{a - c}{(a - c) + (d - b)}$$

$$= \frac{d - b}{(a - c) + (d - b)}.$$

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Proof.

- Column's strategy $Q = (1 - q, q)$ is neutralizing when

$$q = \frac{a - c}{(a - c) + (d - b)} \quad 1 - q = \frac{d - b}{(a - c) + (d - b)}$$

$$\begin{aligned} E(P_1, Q) &= a(1 - q) + bq \\ &= a \frac{d - b}{(a - c) + (d - b)} + b \frac{a - c}{(a - c) + (d - b)} \\ &= \frac{ad - ab + ab - bc}{(a - c) + (d - b)} = \frac{ad - bc}{(a - c) + (d - b)} \end{aligned}$$

$$\begin{aligned} E(P_2, Q) &= c(1 - q) + dq \\ &= \dots = \frac{ad - bc}{(a - c) + (d - b)}. \end{aligned}$$

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Proof.

- Set $q = \frac{a - c}{(a - c) + (d - b)}$
- The strategy $Q = (1 - q, q)$ is neutralizing
- Both of Row's pure strategies have expected value $v = \frac{ad - bc}{(a - c) + (d - b)}$.
- We can make the same series of arguments for Row.
- Set $p = \frac{a - b}{(a - b) + (d - c)}$
- Then the strategy $R = (1 - p, p)$ is neutralizing
- $E(P, Q_1) = E(P, Q_2) = \frac{ad - bc}{(a - b) + (d - c)}$.



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Corollary

- Any 2×2 zero-sum game has a Nash equilibrium.
- If there's no saddle point, we have a formula for the mixed strategy Nash equilibrium.
- Set $p = \frac{a - b}{(a - b) + (d - c)}$ and $q = \frac{a - c}{(a - c) + (d - b)}$
- Then Row's equilibrium strategy is $P = (1 - p, p)$
- Column's equilibrium strategy is $Q = (1 - q, q)$
- The value of the game is $v = \frac{ad - bc}{(a - b) + (d - c)}$.

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Remark

The number $ad - bc$ is sometimes called the **determinant** of the

matrix

a	b
c	d

. It is frequently important in math contexts. It is

sometimes notated Δ .

- Want to show how this sort of solution works in practice
- Also want to show why we should care.
- Rarely wind up having to play a 2×2 matrix game in real life
- But many situations are *like* 2×2 matrix games.

Example (The Hunt for Osama bin Laden)

- Osama bin Laden organized the September 11, 2001 terror attack on the World Trade Center
- The US occupied Afghanistan pursuing him
- December 2002: hiding on the Afghanistan-Pakistan border
- The US could search in Afghanistan or in Pakistan
- Bin Laden could hide in either Afghanistan or Pakistan
- Need to estimate likelihood of catching him

An Application to Counterterrorism

Example (The Hunt for Osama bin Laden)

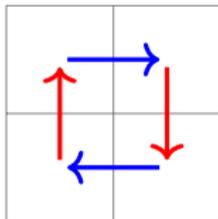
- If US searches Afghanistan and he's there: 60% chance
- If US searches Pakistan and he's there: 40% chance
- If US searches Pakistan and he's in Afghanistan: 0% chance
- If US searches Afghanistan and he's in Pakistan: 10% chance
Pakistani security catches him

	A	P
A	60	10
P	0	40

- What should the US do? What should bin Laden do?

An Application to Counterterrorism

	A	P
A	60	10
P	0	40



- Is there a saddle point? No
- Use our formula to find an equilibrium mixed strategy

$$p = \frac{a - b}{(a - b) + (d - c)} = \frac{60 - 10}{60 - 10 + 40 - 0} = \frac{50}{90} = \frac{5}{9}$$

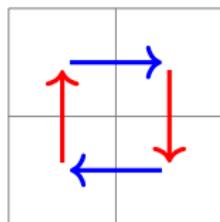
$$1 - p = \frac{9}{9} - \frac{5}{9} = \frac{4}{9}$$

$$q = \frac{a - c}{(a - c) + (d - b)} = \frac{60 - 0}{60 - 0 + 40 - 10} = \frac{60}{90} = \frac{2}{3}$$

$$1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

An Application to Counterterrorism

	A	P
A	60	10
P	0	40



- US should play *A* $4/9$ of the time and *P* $5/9$ of the time.
- ObL should play *A* $1/3$ of the time and *P* $2/3$ of the time.
- Value of the game is

$$\begin{aligned}v &= \frac{ad - bc}{(a - b) + (d - c)} = \frac{(60)(40) - (10)(0)}{60 - 10 + 40 - 0} \\ &= \frac{2400 - 0}{90} = \frac{240}{9} = \frac{80}{3} = 26\frac{2}{3}.\end{aligned}$$

An Application to Counterterrorism

- $P = (4/9, 5/9)$ $Q = (1/3, 2/3)$ $v = 26^{2/3}$
- How do we interpret this?
- One option: US randomly chooses whether to search in Afghanistan or in Pakistan, with non-uniform probabilities
- Another option: send $4/9$ of resources to Afghanistan and $5/9$ of resources to Pakistan.
- In any event: all these numbers are estimates. Don't take "26^{2/3}%" too seriously.
- But this can guide how we think about these sorts of decisions.

Example: Battle of the Bismarck Sea

	North	South
North	2	2
South	1	3

$$\bullet p = \frac{a - b}{(a - b) + (d - c)}$$

$$\bullet q = \frac{a - c}{(a - c) + (d - b)}$$

$$\bullet v = \frac{ad - bc}{(a - b) + (d - c)}$$

$$\bullet p = \frac{2 - 2}{(2 - 2) + (3 - 1)} = 0$$

$$\bullet q = \frac{2 - 1}{(2 - 1) + (3 - 2)} = 1/2$$

$$\bullet v = \frac{2 \cdot 3 - 1 \cdot 2}{(2 - 2) + (3 - 1)} = 4/2 = 2.$$

- Row plays North all the time; Column plays 50-50. The expected payout is 2.
- How does this compare to our original answers?

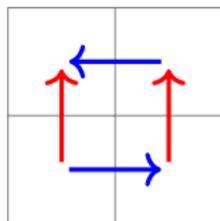
Another example

-1	1
-3	-4

- $p = \frac{a - b}{(a - b) + (d - c)}$
 - $q = \frac{a - c}{(a - c) + (d - b)}$
 - $v = \frac{ad - bc}{(a - b) + (d - c)}$
-
- $p = \frac{-1 - 1}{(-1 - 1) + (-4 + 3)} = \frac{-2}{-3} = \frac{2}{3}$
 - $q = \frac{-1 + 3}{(-1 + 3) + (-4 - 1)} = \frac{2}{-3} = -\frac{2}{3}$
 - $v = \frac{(-1)(-4) - (-3)(1)}{(-1 - 1) + (-4 + 3)} = \frac{7}{-3} = -\frac{7}{3}$
- How can a probability be negative?
 - Something went wrong.

Another example

-1	1
-3	-4



- Saddle point at (1,1)
- Our mixed strategy formula assumed no saddle point
- Always check first!

Escaping the conflict

- This is a framework for conflict with another player
- Zero sum: no room for cooperation
- Can win or lose a war, but can't avoid it
- Can we model the possibility of cooperation?