Criteria for Voting Systems with Multiple Candidates

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May's Theorem

Theorem (May's Theorem)

In two-candidate election, the only anonymous, neutral, monotone, and nearly decisive method is the simple majority method.

Proof.

- Suppose we have an anonymous, neutral, monotone, nearly decisive social choice function for two candidates.
- Anonymous so we only need to consider tabulated profiles
- Suppose a voters support candidate A, and b voters support candidate B. Set $t = a + b$ the total number of voters.
- Want to show that the method we are imagining must be the simple majority method. (Can't assume it's majority method!)

May's Theorem

Theorem (May's Theorem)

In two-candidate election, the only anonymous, neutral, monotone, and nearly decisive method is the simple majority method.

Proof.

- Suppose t is even.
- If $a = b = t/2$ neutrality implies we have a tie, as in Simple Majority.
- Suppose A has a majority, $a > t/2$. Want to show A wins.
- Can't assume majority wins; that's what we want to prove.
- Want to show A must win for any neutral, monotone, nearly decisive method.

Proving May's Theorem

Claim

Assume a voting method is neutral, monotone, nearly decisive. If t is even and $a > t/2$ then A has to win.

Proof.

- Since $a > t/2$, we know $a \neq b$. By nearly decisive, not a tie.
- Want to show B can't win. So think about what would happen B wins with $b = t - a$ votes.
- Since $b < t/2$, then B would win with $t/2$ votes by monotone.
- But we showed that if $a = b = t/2$ the election is a tie. So that can't be true.
- Election isn't a tie, and B doesn't win, so A wins.

Proving May's Theorem

What have we shown so far?

- Anonymous: only consider tabulated profiles.
- If t is even:
  - If $a = t/2 = b$ then the election is a tie. ✓
  - If $a > t/2$ then A wins. ✓
  - If $a < t/2$ then $b > t/2$, so B wins by neutrality. ✓
- If t is even, the method must be simple majority.

What next?

What if t is odd?

Proving May's Theorem

Claim

Assume a voting method is neutral, monotone, nearly decisive. If t is odd then the result is the same as the Simple Majority Method.

Proof.

- Neither candidate can get $t/2$ votes
- By near decisiveness, someone must win.
- Suppose A gets $a > t/2$ votes. Want to show A has to win.

Proving May's Theorem

Claim

Assume a voting method is neutral, monotone, nearly decisive. If t is odd and $a > t/2$ then A wins.

Proof.

- By near decisiveness, can't be a tie.
- We claim B can't win.
- B gets $b = t - a < t/2$ votes, so $b < a$.
- If B wins with $b < a$ votes, then B would also win with a votes by monotonicity.
- But then by neutrality A would win with a votes.

Proving May's Theorem

Claim

Assume a voting method is neutral, monotone, nearly decisive. If t is odd and $a > t/2$ then A wins.

Proof.

- By near decisiveness, can't be a tie.
- Just showed B can't win.
- So A wins.

- Showed that if $a > t/2$ then A wins. ✓
- By neutrality, if $b > t/2$ then B wins. ✓
- So if t is odd, the results match Simple Majority.

Theorem (May's Theorem)

In two-candidate election, the only anonymous, neutral, monotone, and nearly decisive method is the simple majority method.

Proof.

- By Anonymity, can just look at vote counts.
- If $a = b$, tie by Neutrality.
- If $a > t/2$:
  - Can't be a tie, by Near Decisiveness
  - B can't win, by Neutrality & Monotonicity
  - So A wins.
- If $b > t/2$, B wins by Neutrality.
- So this is precisely the Simple Majority method.

$\square$

What does May's Theorem mean?

Theorem (May's Theorem)

In two-candidate election, the only anonymous, neutral, monotone, and nearly decisive method is the simple majority method.

- Simple Majority is anonymous, neutral, monotone, nearly decisive
- No other method we've talked about is all four
- Better than that: we *cannot* find another method that is all four.
- In a real sense, Simple Majority is the "best" method for a two-candidate race.

An Impossibility Result

Corollary

It is impossible for a voting system with two candidates to be anonymous, neutral, monotone, and decisive.

Proof.

- If it's decisive, then it's nearly decisive.
- Anonymous, neutral, monotone, and nearly decisive, must be Simple Majority
- But Simple Majority isn't decisive.
- So this is impossible.

□

An exercise for you

Theorem

In an election with two candidates, a voting method that is anonymous, neutral, and monotone must be the simple majority method, a supermajority method, or the all-ties method.

Proof.

- Think about how you'd prove this.
- Similar outline as proof of May's Theorem.

□

Back to Multi-Candidate Elections

Discussion Question

- What do we want out of a multi-candidate election?
- Which of these criteria make sense?
- What other criteria might we want?

Multi-Candidate Election methods

Multi-Candidate Voting Systems

- Plurality
- Hare's method (Instant Runoff Voting)
- Coombs's Method
- Borda Count
- Copeland's Method

Discussion Question

- How do we decide which of these are good?
- What do we want out of a multi-candidate election?

Some Bad Options

Definition

In the dictatorship method, one voter is the dictator. Their first-choice candidate is the unique winner.

Definition

In the monarchy method, one candidate is the monarch. That candidate is the unique winner regardless of how anyone votes.

Definition

In the all-ties method, every candidate is selected as a winner.

Discussion Question

- When might each of these be a good idea?
- Why is each of these usually a bad idea?

Some Easy Criteria

Definition

A method satisfies the Unanimity criterion, or is unanimous, if, whenever all voters place the same candidate at the top of their preference orders, that candidate is the unique winner.

- Obviously desirable, but too easy.

Definition

A method is decisive if it always selects a unique winner.

- Obviously desirable, but too hard!

Definition

A method satisfies the majority criterion if, whenever a candidate receives a majority of the first-place votes, that candidate must be the unique winner.

Anonymity

Definition

A method is anonymous if the outcome is unchanged whenever two voters exchange their ballots.

Lemma

A social choice function is anonymous if and only if it depends only on the tabulated profile.

Proof.

The same as the proof in the two-candidate case. □

Neutrality

Definition

A method is neutral if it treats all candidates the same:

- Suppose we have some profile that names A to be a winner.
- Now suppose there is another candidate B, and all voters exactly swap their preferences for A and B.
- In the new profile, B should be a winner.

- Almost every method we consider is anonymous and neutral.
- Important to name and articulate, especially for proofs.
- Which methods aren't anonymous?
- Which methods aren't neutral?

Monotonicity

- Idea: gaining more support shouldn't hurt you.

Definition

A method is monotone if:

- Suppose there is a profile in which
  - Candidate A wins
  - But some voter puts another candidate B immediately ahead of A.
- If that voter moves A up one place to be ahead of B,
- Then A must be a winner in the new profile.

Corollary

In a monotone method, if A moves up any number of places on any number of ballots, they should still win.

Pareto

Discussion Question

A	C	B
B	A	C
C	D	D
D	B	A

Who *shouldn't* win?

Definition

A method is Pareto or satisfies the Pareto criterion if whenever every voter prefers a candidate A to another candidate B, then the method does not select B as a winner.

Pareto

Definition

A method is Pareto or satisfies the Pareto criterion if whenever every voter prefers a candidate A to another candidate B, then the method does not select B as a winner.

- Named after Italian economist Vilfredo Pareto (1848-1923)
- In general: a choice is Pareto optimal if no other choice is better for everyone.
- Pareto criterion guarantees winner is Pareto optimal.
- Does *not* guarantee A wins!

Pareto

Definition

A method is Pareto or satisfies the Pareto criterion if whenever every voter prefers a candidate A to another candidate B, then the method does not select B as a winner.

Example

7	6
D	C
C	D
A	A
B	B

- Does not guarantee A wins!
- All prefer A to B, so B can't win
- But all prefer C to A, so A can't win either
- Do we know who does win?

Condorcet

Definition

- A candidate is a Condorcet candidate if they beat every other candidate in a head-to-head (by simple majority).
- They are an anti-Condorcet candidate if they lose to every other candidate in a head-to-head.

- Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet (1743-1794)
- Nicolas de Condorcet or Marquis de Concorcet
- One of the first theorists of voting
- *Essay on the Application of Analysis to the Probability of Majority Decisions* (1785)

Condorcet

Definition

- A candidate is a Condorcet candidate if they beat every other candidate in a head-to-head (by simple majority).
- They are an anti-Condorcet candidate if they lose to every other candidate in a head-to-head.

Example

A	B	C
B	C	A
C	A	B

- A beats B
- B beats C
- C beats A
- No Condorcet candidate
- No anti-Condorcet candidate

Condorcet

Definition

- A candidate is a Condorcet candidate if they beat every other candidate in a head-to-head (by simple majority).
- They are an anti-Condorcet candidate if they lose to every other candidate in a head-to-head.

Definition

- A method satisfies the Condorcet criterion if whenever there's a Condorcet candidate, they're the unique winner.
- A method satisfies the anti-Condorcet criterion if whenever there's an anti-Condorcet candidate, they don't win.

Independence

Definition

- A method is independent if it satisfies the following (somewhat complicated) property:
- Suppose there are two profiles where no voter changes their mind about whether candidate A is preferred to candidate B: if a voter ranks A above B in the first profile, they also rank A above B in the second profile
- If A a wins in the first profile but B doesn't, then B cannot win in the second profile.
- (This also works backwards: if B wins in the second profile, they can't lose in the first.)

Independence

- Kenneth Arrow (1950)
- "Independence of irrelevant alternatives" or "IIA"

> After finishing dinner, [Columbia philosopher] Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Independence

Example

Profile 1						
A	C	B	C	C	A	B
B	A	C	B	B	B	A
C	B	A	A	A	C	C

Profile 2						
A	C	B	C	C	A	B
B	A	C	B	B	B	C
C	B	A	A	A	C	A

- Relative ranks of A and B haven't changed.
- If A wins and B loses in Profile 1, then B shouldn't win in Profile 2.

Independence

- Why is the phrasing so complicated?
- *Idea*: if A beats B in profile 1, A still beats B in profile 2.
- But a move could make A and B both lose.

Example

Profile 1		
A	A	C
C	C	A
B	B	B

Profile 2		
A	C	C
C	A	A
B	B	B

- Change shouldn't make B win but it can make A lose
- We don't name second place, so A doesn't "beat" B.

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

- Since our method is anonymous, can consider tabulated profiles
- Suppose we have $2n$ voters and 2 candidates
- Consider two profiles:

n	n
A	B
B	A

n	n
B	A
A	B

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

n	n
A	B
B	A

n	n
B	A
A	B

- At least one candidate wins.
- Assume A wins in the top profile.
- By neutrality B wins in the bottom.
- Profiles are the same, so B wins in top.
- Top profile does not have a unique winner.
- Therefore the method can't be decisive.

□

Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

- Suppose we have an independent Condorcet method.
- Consider this profile:

A	C	B
B	A	C
C	B	A

- Claim no candidate can be a winner.

□

Taylor's Theorem

A	C	B
B	A	C
C	B	A

A	C	C
B	A	B
C	B	A

Claim

A can't win under independence and Condorcet

Proof.

- Consider this profile 2
- By Condorcet, C must be unique winner in profile 2
- C wins and A loses in profile 2
- Only swapped C and B, so by independence A loses in profile 1.

□

Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

A	C	B
B	A	C
C	B	A

- Assume we have an independent Concorcet method for this profile
- Showed A can't win this profile
- Same argument shows B can't win
- Same argument shows C can't win
- Our method can't name a winner for this profile.

□