The Problem of Apportionment

Jay Daigle

jaydaigle@gwu.edu

https://jaydaigle.net/politics

The George Washington University

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The Problem of Apportionment

Article I, Section 2 of the Constitution

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons.

Amendment XIV, Section 2

Representatives shall be apportioned among the several States according to their respective numbers, counting the whole number of persons in each State, excluding Indians not taxed.

The Problem of Apportionment

US constitution

Representatives shall be apportioned among the several States according to their respective numbers...

- How exactly should we do this?
- "Fair share" seems easy
- But we have to give a *whole number*
- Rounding is more complicated than it seems.

The Problem of Apportionment: Maryland

- US population in 2020: 331,108,434
- Maryland population: 6,185,278
- $\frac{6{,}185{,}278}{331{,}108{,}434} \approx 0.01868 = 1.868\%$
- Maryland "deserves" 1.868% of Congressional seats
- $0.01868 \times 435 \approx 8.126$.

Discussion Question

How many representatives should Maryland get?

The Problem of Apportionment: Maryland

- Maryland "deserves" 1.868% of Congressional seats
- $0.01868 \times 435 \approx 8.126$.
- Currently has 8 seats. $\frac{8}{435} \approx 0.01839 \approx 1.839\%$ Too low!
- 9 seats: $\frac{9}{435} \approx 0.02069 \approx 2.069\%$ Too high!

Discussion Question

What should we do?

The Problem of Apportionment: Maryland

Idea: Be Generous

- Maryland "deserves" 8.126 Congressional seats.
- Round up: give them 9
- To be fair, round everyone up
- Need to give out 460 seats
- Can we do that?

- Kentucky has 4,509,342 people.
- With 435 seats, should get $435 \cdot \frac{4{,}509{,}342}{331{,}108{,}434} \approx 5.924$.
- Round up to 6.
- But with 460 seats, should get $460 \cdot \frac{4{,}509{,}342}{331{,}108{,}434} \approx 6.265$.
- Do we round up *again*? Where do we stop?

What is Apportionment?

- Need to set out goals. What are we doing?

Notation

- Assume we have n states.
- Allocate h Congressional seats
- List states in fixed order. State number k has population p_k
- $p = p_1 + p_2 + \cdots + p_n$ population of the country.

The USA

- We'll mostly be talking about the US
- $n = 50$
- $h = 435$
- $p = 331{,}108{,}434$

Census Data 2020

k	State	p_k	k	State	p_k
1	Alabama	5,030,053	11	Hawaii	1,460,137
2	Alaska	736,081	12	Idaho	1,841,377
3	Arizona	7,158,923	13	Illinois	12,822,739
4	Arkansas	3,013,756	14	Indiana	6,790,280
5	California	39,576,757	15	Iowa	3,192,406
6	Colorado	5,782,171	16	Kansas	2,940,865
7	Connecticut	3,608,298	17	Kentucky	4,509,342
8	Delaware	990,837	18	Louisiana	4,661,468
9	Florida	21,570,527	19	Maine	1,363,582
10	Georgia	10,725,274	20	Maryland	6,185,278
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

What is Apportionment?

Definition

A census is the collection of information:

- h the house size
- n the number of states
- $p_1, \ldots, p_n$ the population of each state.

Definition

- An apportionment method is a function whose input is a census $h, n, p_1, \ldots, p_n$, and whose output is a collection of positive integers $a_1, a_2, \ldots, a_n$ that add up to h.
- (That is, $a_1 + a_2 + \cdots + a_n = h$.)
- Think of a_k as the number of representatives given to state k.

Apportionment 2020

k	State	p_k	a_k	k	State	p_k	a_k
1	Alabama	5,030,053	7	11	Hawaii	1,460,137	2
2	Alaska	736,081	1	12	Idaho	1,841,377	2
3	Arizona	7,158,923	9	13	Illinois	12,822,739	17
4	Arkansas	3,013,756	4	14	Indiana	6,790,280	9
5	California	39,576,757	52	15	Iowa	3,192,406	4
6	Colorado	5,782,171	8	16	Kansas	2,940,865	4
7	Connecticut	3,608,298	5	17	Kentucky	4,509,342	6
8	Delaware	990,837	1	18	Louisiana	4,661,468	6
9	Florida	21,570,527	28	19	Maine	1,363,582	2
10	Georgia	10,725,274	14	20	Maryland	6,185,278	8
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Quotas

Definition

- We define a state's standard quota to be the number $q_k = h \cdot \frac{p_k}{p}$.
- This is the state's "fair share" of representatives.
- We define a census's standard divisor to be the number $s = \frac{p}{h}$.
- The standard divisor is the denominator in the standard quota. $$q_k = h \cdot \frac{p_k}{p} = \frac{1}{1/h} \cdot \frac{p_k}{p} = \frac{p_k}{p/h}.$$
- The standard divisor is the number of people each Congressional representative "should" represent.

Quotas

- Ideally would like to set $a_k = q_k = \frac{hp_k}{p}$.
- This would add up to h:

$$
\begin{aligned}
q_1 + q_2 + \cdots + q_n &= h \cdot \frac{p_1}{p} + h \cdot \frac{p_2}{p} + \cdots + h \cdot \frac{p_2}{p} \\
&= \frac{h}{p}(p_1 + p_2 + \cdots + p_n) \\
&= \frac{h}{p} \cdot p = h.
\end{aligned}
$$

The problem

- q_k is usually not a whole number
- Need to pick something else. Round it?

Quotas

Definition

- The lower quota for state k is the standard quota rounded down.
- We write this $\lfloor q_k \rfloor$
- The floor of q_k, or the integer part of q_k, or the greatest integer less than or equal to q_k
- The upper quota for state k is the standard quota rounded up.
- We write this $\lceil q_k \rceil$
- The ceiling of q_k, or the least integer greater than or equal to q_k

Quotas and Apportionment

- Easy to compute q_k
- Can't set $a_k = q_k$ because it's not a whole number
- Can't round them all up: assigns too many seats
- Can't round them all down: doesn't assign enough seats

Discussion Question

How do we decide which states to round up and which to round down?

Discussion Question

Is this the right question?

Hamilton's Method

- Alexander Hamilton, 1792
- The guy in the Broadway show
- Give every state either its lower quota or its upper quota
- Call this a quota method.

Hamilton's Method: Notation

Definition

- Recall the integer part of a real number x is the greatest integer less than or equal to x. We will sometimes notate this $\lfloor x \rfloor$, which we read as the "floor" of x.
- The fractional part of a real number x is the difference between x and its integer part. We can write this as $x - \lfloor x \rfloor$ or sometimes as $\text{frac}(x)$ or $\{x\}$.

Example

- The integer part of 3.14159 is $\lfloor 3.14159 \rfloor = 3$ and the fractional part is $\{3.14159\} = 0.14159$.
- The integer part of 8.126 is $\lfloor 8.126 \rfloor = 8$ and the fractional part is $\{8.126\} = 0.126$.

Hamilton's Method

Hamilton's Idea

- Give every state at least its lower quota
- Allocate remaining seats based on fractional part

Example (2020 Census)

- Maryland: $q_{20} = 8.126$
- Lower quota: $\lfloor q_{20} \rfloor = 8$
- Guaranteed at least 8 seats
- Fractional part is $\{q_{20}\} = .126$
- Small, so we probably don't give it another seat

Hamilton's Method

Hamilton's Idea

- Give every state at least its lower quota
- Allocate remaining seats based on fractional part

Example (2020 Census)

- Kentucky has $q_{17} = 5.924$
- Lower quota: $\lfloor q_{17} \rfloor = 5$
- Guaranteed at least 5 seats
- Fractional part is $\{q_{17}\} = .924$
- Large, so we probably give it another seat
- *Definitely* gets extra seat before Maryland would.

Hamilton's Method

Definition (Hamilton's method)

- As a provisional apportionment, assign each state its lower quota $\lfloor q_k \rfloor$.
- Then assign the seats that remain to the states in decreasing order of the size of the fractional parts of their standard quotas, allocating at most one per state.

Hamilton's Method

Example

Apportion $h = 10$ seats to $n = 3$ states with populations $p_1 = 264, p_2 = 361, p_3 = 375$.

- We get a total population $p = 264 + 361 + 375 = 1000$.
- The standard divisor is $s = p/h = 1000/10 = 100$.
- Want to allocate roughly one seat per hundred people.

$$
\begin{aligned}
q_1 &= \frac{p_1}{s} = \frac{264}{100} = 2.64 \\
q_2 &= \frac{p_2}{s} = \frac{361}{100} = 3.61 \\
q_3 &= \frac{p_3}{s} = \frac{375}{100} = 3.75
\end{aligned}
$$

Hamilton's Method

Example

Apportion $h = 10$ seats to $n = 3$ states with populations $p_1 = 264, p_2 = 361, p_3 = 375$.

- $p = 1000$ and $s = 100$
- $q_1 = 2.64, q_2 = 3.61, q_3 = 3.75$
- Lower quotas: 2, 3, 3.
- How many seats left over? 2.
- Fractional parts: 0.64, 0.61, 0.75
- Largest is 0.75; next is 0.64
- Now we're out of seats
- Final apportionment: $a_1 = 3, a_2 = 3, a_3 = 4$.

Hamilton's Method

Example

Apportion $h = 10$ seats to $n = 3$ states with populations

$p_1 = 264, p_2 = 361, p_3 = 375.$

- Can summarize this work in a table

k	p_k	Standard Quota q_k	Lower Quota	Upper Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	264	2.64	2	3	0.64	3
2	361	3.61	3	4	0.61	3
3	375	3.75	3	4	0.75	4

- We'll be seeing these a lot.

Hamilton's Method

Example

Apportion 100 seats to three states, with populations:

- State A: 4,400,000
- State B: 45,300,000
- State C: 50,300,000

Hamilton's Method: the Alabama Paradox

Example

Allocate ten seats to three states, with populations 1,450,000; 3,400,000; 5,150,000.

- $n = 3$
- $h = 10$

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Hamilton's Method: the Alabama Paradox

Example

Allocate ~~ten~~ eleven seats to three states, with populations

1,450,000; 3,400,000; 5,150,000.

$s = \frac{10{,}000{,}000}{11} \approx 909091$

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.595	1	0.595	1
2	3,400,000	3.740	3	0.740	4
3	5,150,000	5.665	5	0.665	6

Discussion Question

What's wrong with this?

Hamilton's Method: the Alabama Paradox

k	p_k	$h = 10$	$h = 11$
1	1,450,000	2	1
2	3,400,000	3	4
3	5,150,000	5	6

Definition

When adding a house seat would cause a state to lose a representative, we call that the Alabama paradox.

- Could have happened to Alabama in 1880
- Seems unfair!

House Monotonicity

Definition

- An apportionment method is called house monotone if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease.
- Hamilton's method is not house monotone.

Remark

- The word "monotone" implies that something should move only in one direction
- Apportionment should only go up as house size goes up
- Voting outcome should only go up as votes go up