Math 1007: Mathematics and Politics
The George Washington University   Fall 2025

Jay Daigle

Contents

2 Apportionment

> Article I, Section 2 of the Constitution says that the representatives should be apportioned to the states “according to their respective numbers”. Unfortunately, the Constitution does not offer any further guidance about precisely how this should be done, and every 10 years Congress is left to implement an apportionment that meets the spirit of these words. It is not hard to determine each state’s “fair share” of the House of Representatives. . . . The difficulty is that this fair share need not be a whole number, and yet the number of representatives assigned to each state must be a whole number. So some process is needed to round these fair shares to whole numbers. This is the apportionment problem. What process should we use?
>
> At first, there does not seem to be much substance to this question. Can’t we simply choose the whole number that is nearest to the state’s fair share? It is only upon some reflection that one realizes that the problem is more subtle than this. Rounding the fair shares of all states to their nearest whole number may result in assigning too few or too many seats all told. What should we do? It turns out that Alexander Hamilton and Thomas Jefferson began a debate on this topic in 1792. There are a number of sensible approaches to the problem, but each has its flaws.

2.1 Congressional Apportionment and Hamilton’s Method

Suppose we have three states, and are going to allocate 100 Congressional seats to them. The states have the populations

- A: 4,400,000
- B: 45,300,000
- C: 50,300,000

How many seats should each state get?

The total population is 100,000,000, so it seems like each state should get 1 seat per million people. We would give A 4.4 seats, B 45.3 seats, and C 50.3 seats. But we can’t actually allocate fractions of a representative; every state needs a whole number of seats. It seems like A should get 4 seats, B should get 45 seats, and C should get 50 seats, with one seat left over. The question is: who should get it?

Alexander Hamilton suggested we should allocate "extra" seats in order of their fractional parts. So A gets 4 seats and has 0.4 left over; B gets 45 seats and has 0.3 left over; C has 50 seats and 0.3 left over. We give the extra seat to A, and the final lineup would be:

- A 5 seats
- B 45 seats
- C 50 seats.

And that all seems perfectly reasonable. But there are some bugs!

For instance, what if we instead want to allocate 101 seats? This requires some annoying arithmetic, for which we will need a calculator. When we divide 100,000,000 by 101 we get about 990099. (In fact we get $990099.\overline{0099}$.) Dividing each state's population by this number gives a rough-cut allocation of

- A gets $\frac{4{,}400{,}000}{990099} \approx 4.444$ seats;
- B gets $\frac{45{,}300{,}000}{990099} \approx 45.753$ seats;
- C gets $\frac{50{,}300{,}000}{990099} \approx 50.843$ seats.

Now we give C 50 seats, B 45 seats, and A 4 seats, with two left over. Looking at the fractional parts, the extras should go to B and C, giving a final tally of

- A 4 seats
- B 46 seats
- C 51 seats.

But there's something weird here. When we added an extra seat, B and C both gained seats, while A *lost* a seat. And that seems unfortunately backwards.

Now suppose we return to allocating 100 seats, but we take a new census, which give updated population numbers

- A: 4,500,000
- B: 45,200,000
- C: 49,000,000
- Total population: 98,700,000.

This looks like we should allocate one seat for every 987,000 people. Then our rough cut is

- A gets $\frac{4{,}500{,}000}{987000} \approx 4.559$ seats;
- B gets $\frac{45{,}200{,}000}{987000} \approx 45.795$ seats;
- C gets $\frac{49{,}000{,}000}{987000} \approx 49.645$ seats.

Our rough cut is that C gets 49, B gets 45, and A gets 4, with one left over. Then B has the largest fractional part, so our final allocation is

- A gets 4 seats
- B gets 46 seats
- C gets 50 seats.

Relative to our original situation, A has gained population while B and C have both lost it; but A has lost a seat to B.

Finally, let's think about what happens in the original situation if we add a fourth state D with 1,700,000 people, and correspondingly increase the legislature from 100 to 102. We get

- A: 4,400,000
- B: 45,300,000
- C: 50,300,000
- D: 1,700,000
- Total population: 101,700,000.

We should give one representative for every $\frac{101{,}700{,}000}{102} \approx 997{,}059$ people. So our rough allocations are

- A gets $\frac{4{,}400{,}000}{997{,}059} \approx 4.413$ seats;
- B gets $\frac{45{,}300{,}000}{997{,}059} \approx 45.434$ seats;
- C gets $\frac{50{,}300{,}000}{997{,}059} \approx 50.448$ seats.
- D gets $\frac{1{,}700{,}000}{997{,}059} \approx 1.701$ seats.

We give A 4 seats, B 45 seats, C 50 seats, and D 1 seat; this leaves 2 left over. One goes to D, and the second narrowly goes to C, for a final allocation of

- A gets 4 seats
- B gets 45 seats
- C gets 51 seats
- D gets 2 seats.

We added D to the nation, and added 2 seats for D's representatives, which all makes sense; but this also has the effect of moving one seat from A to C, even though nothing about A or C has changed. So there's a lot of weirdness going on here!

The method we've used here is called *Hamilton's method* and we'll study it more soon.

2.1.1 Defining apportionment

We first need to set up the basic notation we'll be using for the rest of section 2. We assume we have n states, where n is a whole number bigger than 1. (In the USA we've had $n = 50$ since 1959.) We need to allocate h Congressional seats among those states, and this also has to be a positive whole number. In the US House of Representatives, we have $h = 435$.

We list the states in some fixed order, and we use p_k to represent the population of the kth state. Then we use the letter p to represent the total population of the nation, so that

$$p = p_1 + p_2 + \cdots + p_n.$$

We call the collection of numbers $h, n, p_1, \ldots, p_n$ a *census*. (Notice we don't have to tell you the value of p since that's determined by the p_k.)

Definition 2.1. An *apportionment method* is a function whose input is a census $h, n, p_1, \ldots, p_n$, and whose output is a collection of positive integers $a_1, a_2, \ldots, a_n$ that add up to

$$a_1 + a_2 + \cdots + a_n = h.$$

We think of these outputs $a_1, \ldots, a_n$ as the number of congressional seats allocated to each state.

In our opening example, we had $n = 3$, $p_1 = 4{,}500{,}000$, $p_2 = 45{,}200{,}000, p_3 = 49{,}000{,}000$, and $h = 100$. Then in our first allocation we got $a_1 = 4, a_2 = 46, a_3 = 50$.

For the rest of this section, we will regularly find ourselves comparing two numbers and wanting to choose the larger one. This raises the question of what happens if the two numbers are the same size, but we're going to ignore that question completely, and just assume we won't have that sort of tie.

So imagine we put all 50 US states in alphabetical order. Then Maryland is the 20th state. (Your textbook says 19th but I've counted six times and I'm pretty sure it's 20th.) In the 2020 census, Maryland has $p_{20} = 6{,}185{,}278$ people, out of a total population of $p = 331{,}108{,}434$. Since we have $h = 435$, you'd want Maryland to have

$$\frac{p_1 9}{p} = \frac{6{,}185{,}278}{331{,}108{,}434} \approx 0.01868 = 1.868\%$$

of the total seats. In our current allocation, Maryland has $a_{20} = 8$ representatives; we compute that

$$\frac{a_{20}}{h} = \frac{8}{435} \approx 0.01839 \approx 1.839\%$$

of the Congressional representatives. We'd like to make those two percentages as close as possible, but there are two limitations.

The first is that they can never be equal, becasue we need to allocate a whole number of seats. The "ideal" allocation where Maryland has 1.868% of all the seats would give it

$$h \cdot \frac{p_{20}}{p} = 435 \cdot \frac{6{,}185{,}278}{331{,}108{,}434} \approx 435 \cdot 0.01868 \approx 8.126$$

Congressional seats, but we can't give 0.126 of a seat. So there's no perfect answer, and giving Maryland 9 seats would be even further off, giving it 2.069% of the seats for just 1.868% of the population.

The second constraint is that we need to allocate to every state, simultaneously. We could decide to be "generous" and round every state's Congressional delegation up. But that would change h. (Based on 2020 Census figures, h would have to increase to 460.) There's nothing intrinsically wrong with changing h; it's not like we couldn't build 25 more seats in the House of Representatives.

But if we change h to 460, then we'd have *different* ideal allocations. For instance, in the 2020 Census, Kentucky has $p_{17} = 4{,}509{,}342$ people. Its ideal allocation is

$$h \cdot \frac{p_{17}}{p} = 435 \cdot \frac{4{,}509{,}342}{331{,}108{,}434} \approx 435 \cdot 0.0136 \approx 5.924.$$

We could round this up to 6, and in fact the 2020 apportionment did so. But if we rounded *every* state up to get $h = 460$, the new ideal allocation would be

$$h \cdot \frac{p_{17}}{p} = 460 \cdot \frac{4{,}509{,}342}{331{,}108{,}434} \approx 460 \cdot 0.0136 \approx 6.265.$$

Do we round the allocation up *again* to 7? It's not clear this process would ever stop.

2.1.2 Quotas

We've repeatedly calculated the number $h \cdot {}^{p_k}/_{p}$, as the "idealized" number of Congressional seats a state should get. We will now call this the stat's *standard quota* $q_k = h \cdot {}^{p_k}/_{p}$. Algebraically we could also write this

$$q_k = \frac{p_k}{p/h}$$

and we call that denominator the *standard divisor* $s = \frac{p}{h}$.

If we could assign $a_k = q_k$ for each k then we would automatically have $a_1 + a_2 + \cdots + a_n = h$, which is what we want. Because

$$\begin{aligned} q_1 + q_2 + \cdots + q_n &= h \cdot \frac{p_1}{p} + h \cdot \frac{p_2}{p} + \cdots + h \cdot \frac{p_2}{p} \\ &= \frac{h}{p}(p_1 + p_2 + \cdots + p_n) \\ &= \frac{h}{p} \cdot p = h. \end{aligned}$$

However we (almost certainly) can't assign $a_k = q_k$ because the quota q_k is probably not a whole number, so we need to pick something else. Two obvious choices are the *lower quota*, which is the standard quota rounded down $\lfloor q_k \rfloor$, or the *upper quota*, which is the standard quota rounded up $\lceil q_k \rceil$.

So one way of looking at our next question is: which states do we round up, and which states do we round down? This can actually be a very important question in terms of actual political influence. In the 2010 Cenus, Montana had a standard quota of 1.40. This was in fact rounded down to give Montana one representative, making Montanans the most underrepresented state residents in the country. The average Congressional district had about 710,000 people, but Montana had 994,416.

In the 2020 census, Montana's standard quota rose to about 1.426, and the allocation method used rounded it up to 2 Congressional seats. These seats each have about 543,000 people, and are the smallest and most overrepresented districts in the US.

2.1.3 Hamilton's Method

Alexander Hamilton in 1792 suggested a method guaranteed to assign each state either its lower quota or its upper quota. We call these methods *quota methods*.

In order to talk about these we need to give some notation for arithmetic.

Definition 2.2. The *integer part* of a real number x is the greatest integer less than or equal to x. We will sometimes notate this $\lfloor x \rfloor$, which we read as the "floor" of x.

The *fractional part* of a real number x is the difference between x and its integer part. We can write this as $x - \lfloor x \rfloor$ or sometimes as $\text{frac}(x)$ or $\{x\}$.

Example 2.3. The integer part of 3.14159 is $\lfloor 3.14159 \rfloor = 3$ and the fractional part is $\{3.14159\} = 0.14159$.

The integer part of 8.126 is $\lfloor 8.126 \rfloor = 8$ and the fractional part is $\{8.126\} = 0.126$.

Note that a state's lower quota is the integer part of its standard quota, $\lfloor q_k \rfloor$. Hamilton thought that every state should be guaranteed to get its lower quota, and that excess seats should be allocated based on the size of the fractional part.

So in the 2020 Census, Maryland's standard quota is $q_{20} = 8.126$. Maryland is guaranteed 8 seats under Hamilton's method, but the fractional part $\{q_{20}\} = 0.126$ is small so we're inclined not to give it another representative. But we saw Kentucky has a standard quota of $q_{17} = 5.924$, with lower quota $\lfloor 5.924 \rfloor = 5$. We should guarantee Kentucky 5 seats, but since the fractional part of its quote $\{q_{17}\} = 0.924$ is quite large, we should be strongly inclined to give it a sixth seat, as indeed we do.

Definition 2.4 (Hamilton's method)**.** As a provisional apportionment, assign each state its lower quota $\lfloor q_k \rfloor$. Then assign the seats that remain to the states in decreasing order of the size of the fractional parts of their standard quotas, allocating at most one per state.

Example 2.5. Apportion $h = 10$ seats to $n = 3$ states with populations $p_1 = 264, p_2 = 361, p_3 = 375$.

We get a total population $p = 264 + 361 + 375 = 1000$. The standard divisor is $s = p/h = 1000/10 = 100$. That means we want to allocate roughly one seat per hundred people.

Our standard quotas are

$$q_1 = \frac{p_1}{s} = \frac{264}{100} = 2.64$$
$$q_2 = \frac{p_2}{s} = \frac{361}{100} = 3.61$$
$$q_3 = \frac{p_3}{s} = \frac{375}{100} = 3.75$$

The lower quotas are 2,3,3, so we allocate those 8 seats, and have 2 seats left over. The fractional parts are $0.64, 0.61, 0.75$. We assign the first seat to state 3 with fractional part

0.75, and the second seat to state 1 with fractional part 0.64. Then we are out of seats to assign, so state 2 sticks with its lower quota. The final apportionment is $a_1 = 3, a_2 = 3, a_3 = 4$, and the process is summarized in figure 2.1.

k	p_k	Standard Quota q_k	Lower Quota	Upper Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	264	2.64	2	3	0.64	3
2	361	3.61	3	4	0.61	3
3	375	3.75	3	4	0.75	4

Figure 2.1: Hamilton's Method in example 2.5

2.1.4 "Paradoxes" in Hamilton's method

Hamilton's method is maybe the most obvious way to solve the apportionment problem. It's so obvious that it seems like this should fully resolve the question. But Hamilton's method does have a number of issues it runs into which aren't ideal. In fact, these flaws are so bad that Hamilton's method is almost completely out of the running as an apportionment method. It was used to some degree in the late 19th century and early 20th, and many of these bad behaviors actually occurred in ways that made the method look much too bad.

Example 2.6 (Alabama Paradox)**.** Suppose we have $n = 3$ states and $h = 10$ seats to allocate to them. If they have populations of $p_1 = 1{,}450{,}000, p_2 = 3{,}400{,}000, p_3 = 5{,}150{,}000$, we get a total population of 10,000,000 and a standard divisor of 1,000,000. Then we get the following table:

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Figure 2.2: The Alabama Paradox Part 1, in example 2.6

So far we have no problems. But now imagine Congress increases the number of seats from 10 to $h = 11$. Which state will gain the additional seat?

Well, now we have a standard divisor of $\frac{10{,}000{,}000}{11} \approx 909091$, and we get the following table:

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.595	1	0.595	1
2	3,400,000	3.740	3	0.740	4
3	5,150,000	5.665	5	0.665	6

Figure 2.3: The Alabama Paradox Part 2, in example 2.6

The lower quotas haven't changed, but the standard quotas and thus the fractional parts have increased. Now the winners are states 2 and 3: state 1 has actually *lost* a representative because Congress added a representative to the House.

This situation is called the *Alabama paradox* because Alabama would have lost a seat to an increase in h during the 1880 reapportionment. And it seems facially unfair that increasing h should cause any state to lose a seat.

Definition 2.7. An apportionment method is called *house monotone* if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease. Thus example 2.6 shows that Hamilton's method is not house monotone.

Why did this problem pop up? Every state's standard quota increased between figures 2.2 and 2.3, and in fact each standard quota increased by exactly 10%. But since state 1 started with $q_1 = 1.45$ and state 3 started with $q_3 = 5.15$, a 10% increase in the latter is much bigger than a 10% increase in the former.

Example 2.8 (Population paradox)**.** We can run into a similar issue when h stays the same but populations increase. Suppose we start again with the same data and same table as before, with $h = 10$:

But now suppose, keeping $h = 10$, we take a new census, and populations have increased in state 1, while decreasing in states 2 and 3. We get $p_1 = 1{,}470{,}000, p_2 = 3{,}380{,}000, p_3 = 4{,}650{,}000$, for a total population of $p = 9{,}500{,}000$. The standard divisor is now 950,000, and we get the following table of quotas:

State 1 has increased its standard quota q_1, while q_2 and q_3 have decreased. But since q_3 in particular has dropped from 5.15 to 4.89, the fractional part is bigger, and now states 2 and 3 both get extra seats, leaving state 1 with $a_1 = 1$.

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Figure 2.4: The Population Paradox Part 1 in example 2.8

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,470,000	1.55	1	0.45	1
2	3,380,000	3.56	3	0.40	4
3	4,650,000	4.89	4	0.15	5

Figure 2.5: The Population Paradox Part 2 in example 2.8

This seems even worse! State 1 *gained* population, while states 2 and 3 lost it. But this leads state 1 to lose a representative.

Definition 2.9. A method is called *population monotone* if a state can never lose a seat when its population increases while no other state's population increases.

In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.

Example 2.8 shows that Hamilton's method is not population monotone.

Finally we need to consider one more situation.

Example 2.10 (Oklahoma paradox)**.** Start again with the same data, where $h = 10$ and $p = 10{,}000{,}000$.

Imagine the union wants to add one more state, with a population of $p_4 = 2{,}600{,}000$. It seems like we'd expect this state to get 3 seats, so we add a corresponding 3 seats to the house for a new $h = 13$, a new population $p = 12{,}600{,}000$, and a new standard divisor of $s = \frac{12{,}600{,}000}{13} \approx 969{,}231$. We get the following table:

We see that state 4 does indeed get $a_4 = 3$ seats. But in the process, state 1 loses a seat, and state 2 gains one.

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Figure 2.6: The Oklahoma Paradox Part 1 in example 2.10

k	p_k	Standard Quota q_k	Lower Quota	Fractional Part $\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.50	1	0.50	1
2	3,400,000	3.51	3	0.51	4
3	5,150,000	5.31	5	0.31	5
4	2,600,000	2.68	2	0.68	3

Figure 2.7: The Oklahoma Paradox Part 2 in example 2.10

This is called the *new states paradox* or the *Oklahoma paradox*, due to the impact of adding Oklahoma as a state in 1907. At the time we were using Hamilton's method of apportionment. When Oklahoma was added it was allocated 5 representatives; but if we had redone the allocation New York would also have had to cede a state to Maine, which was seen as highly undesirable.

Hamilton's method is appealingly simple, but the examples 2.6, 2.8, and 2.10 show that it has major flaws. Is there a way to avoid these problems?

The answer is "sort of", and we'll see a new class of approaches in the next section.

2.2 Divisor Methods

2.2.1 Jefferson's Method

In section 2.1 we treated the house size h as a fixed input to our apportionment function. For almost a hundred years the US has worked that way; the House of Representatives has had 435 members since 1930. But prior to that, the size of Congress varied, slowly increasing over time.

This suggests another approach: we could have some rule for how many representatives each state gets, based on its population, and then let h vary.

Jefferson approached things this way. He started with a theory of how large a Congressional district should be. (In his case, he pointed out the Constitution says the "number of Representatives shall not exceed one for every thirty thousand", which suggested that 30,000 people was about the appropriate size for a district.)

In section 2.1.1 we found the standard divisor $s = p/h$, which was the total population divided by the number of seats to be apportioned. We can think of this as the goal size for a Congressional district. But instead we could start by picking a number d, which we think of as an appropriate size for a district. We call this number a *modified divisor*. We can then divide each state's population by d to obtain its *modified quota* p_k/d, just as we found the quota $q_k = p_k/s$ in the last section.

As before, the modified quota is probably not a whole number, so we can't just give every state p_k/d seats. Jefferson suggested we should round each modified quota down to generate the number of seats, and thus set $a_k = \lfloor p_k/d \rfloor$. This has the advantage of being uniform: everyone gets rounded down.

Poll Question 2.2.1. Why did Jefferson suggest we round down, rather than up?

In Jefferson's original method, we start with d, compute each $a_k = p_k/d$, and then find $h = a_1 + a_2 + \cdots + a_n$. But we can also apply his approach to generate an apportionment function in the sense of definition 2.1, which starts with a fixed value of h.

Definition 2.11 (Jefferson's method)**.** Choose a modified divisor d, compute the modified quotas p_k/d, and round these down to obtain $a_k = \lfloor p_k/d \rfloor$. If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment. Otherwise, choose a new d and try again.

This raises a few of obvious questions.

1. Is there always a d that will work?
2. Is there *more than one* d that will work?
3. If we pick two different ds that both give the same total number of seats, will they give the same apportionment?

The answer to the first question is yes, as long as we ignore the possibility of exact ties in our calculations (which shouldn't happen with large population numbers). We can find the d that works pretty easily. If we get a total number of seats larger than our desired h, that means our d was too small and we need to make it bigger; but if we fewer seats than our desired h, our d is too big and we need to make it smaller. This lets us zero in on the correct d value pretty quickly. (We'll talk about how to do this in practice in section 2.2.2.)

The answer to the second question is also yes: we can pretty much always find multiple d values that will give us the correct number of seats. That makes question 3 especially important; but fortunately it is completely answered by the following proposition.

Proposition 2.12. *Suppose $h, n,$ and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If d and d' are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state k.*

Remark 2.13. We will often be using notation like d' in this section. We read this as "d prime", and use it a lot when we want two different versions of the same number. If we need three, we may write d, d', d''.

There are other solutions to this problem but they're often more annoying. We could write d_1 and d_2, but the as already have subscripts; they could become $a_{1,k}$ and $a_{2,k}$ but that's unpleasant. We could also do something like using Greek letters, making our divisors d and δ and our apportionments a_k and α_k, but then we need to know the Greek alphabet and that still doesn't help when we need three versions.

Proof. Suppose "without loss of generality" that $d \leq d'$. Then, for every state k, the modified quota p_k/d for the first divisor d is at least as big as the modified quota p_k/d' for the second divisor d'. Rounding them down preserves that, so we must have $a_k \geq a'_k$, and that must be true for every state k. That means that

$$a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n.$$

But we know that both apportionments give the same total number of seats, so

$$h = a_1 + \cdots + a_n \geq a_1 + \cdots + a'_n = h$$

and thus the two totals must be the same. □

Example 2.14. Suppose we have $n = 3, h = 10$, and three states with

- $p_1 = 1{,}500{,}000$
- $p_2 = 3{,}200{,}000$
- $p_3 = 5{,}300{,}000$

We need to try some different divisors here.

If we try the standard divisor $s = 1{,}000{,}000$ we get $p_1/s = 1.5$ and thus $a_1 = 1$; $p_2/s = 3.2$ and thus $a_2 = 3$; and $p_3/s = 5.3$ and so $a_3 = 5$. This gives a total number of seats equal to 9. (This shouldn't be a surprise from the work we did last section!)

This makes our house too small, so we need to make d smaller. If we try $d = 900{,}000$ we get modified quotas of $1.\overline{66}$, $3.\overline{55}$, and $5.\overline{88}$, which again would give $a_1 = 1, a_2 = 3, a_3 = 5$ for a total of nine seats.

Making d smaller again we can take $d = 800{,}000$. This gives modified quotas of $1.875, 4, 6.625$ and $a_1 = 1, a_2 = 4, a_3 = 6$ and a total of eleven seats. This is too many, so we need to make d bigger.

Since 800,000 was too small and 900,000 was too big, we might try $d = 850{,}000$. Then we get modified quotas of about $1.76, 3.67, 6.24$ and we get $a_1 = 1, a_2 = 3, a_3 = 6$, and a total of ten seats. This works, so the Jefferson apportionment is $a_1 = 1, a_2 = 3, a_3 = 6$.

		$s = 1{,}000{,}000$		$d = 900{,}000$		$d = 800{,}000$		$d = 850{,}000$	
k	p_k	Quota	Round Down	Quota	Round Down	Quota	Round Down	Quota	Round Down
1	1,500,000	1.50	1	$1.\overline{66}$	1	1.875	1	1.76	1
2	3,200,000	3.20	3	$3.\overline{55}$	3	4	4	3.67	3
3	5,300,000	5.30	5	$5.\overline{88}$	5	6.625	6	6.24	6
Total	10,000,000		9		9		11		10

Figure 2.8: The Jefferson method in example 2.14

Note that this is different from the Hamilton apportionment, which would start with our (unmodified) quotas of $1.5, 3.2$, and 5.3 to generate preliminary allocations of 1,3,5, and then allocate the extra seat to state 1, which has the largest fractional part. So the Hamilton apportionment would be $a_1 = 2, a_2 = 3, a_3 = 5$.

2.2.2 Critical Divisors

We can calculate Jefferson's method with that sort of trial and error approach, but there's a more direct way.

The breakpoints in the number d happen when one of the states would get one additional representative. So for instance, with $d = 850{,}000$, state 3 has a quota of 6.24 which rounds down to 6. Another way of thinking about that sentence is that $\frac{5{,}300{,}000}{6} \geq 850{,}000$, but $\frac{5{,}300{,}000}{7} < 850{,}000$. A little work should convince you that $\frac{5{,}300{,}000}{5} \geq 850{,}000$ and $\frac{5{,}300{,}000}{8} < 850{,}000$. So there are exactly six positive integers m so that $\frac{p_3}{m} \geq 850{,}000$, which are 1, 2, 3, 4, 5, 6. And that's the number of seats that state 3 was allocated!

Definition 2.15. We call a number of the form $\frac{p_k}{m}$ for a positive integer k a *(Jefferson) critical divisor* for the state k.

This gives us a better way of thinking about hunting for a good modified divisor. When we choose a modified divisor d, each seat will get a number of seats equal to the number of critical divisors that are greater than or equal to d. So we want to pick a d such that, when we consider all states together, there are exactly h critical divisors greater than or equal to d.

Example 2.16. Let's use this method to look at example 2.14 again. We had populations $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$, and $p_3 = 5{,}300{,}000$. We can make a table of the critical divisors:

d	State 1	State 2	State 3
1	1,500,000/1 = 1,500,000	3,200,000/1 = 3,200,000	5,300,000/1 = 5,300,000
2	1,500,000/2 = 750,000	3,200,000/2 = 1,600,000	5,300,000/2 = 2,650,000
3	1,500,000/3 = 500,000	3,200,000/3 = 1,066,667	5,300,000/3 = 1,766,667
4	1,500,000/4 = 375,000	3,200,000/4 = 800,000	5,300,000/4 = 1,325,000
5	1,500,000/5 = 300,000	3,200,000/5 = 640,000	5,300,000/5 = 1,060,000
6	1,500,000/6 = 250,000	3,200,000/6 = 533,333	5,300,000/6 = 883,333
7	1,500,000/7 = 214,857	3,200,000/7 = 457,143	5,300,000/7 = 757,143

Figure 2.9: The Jefferson method with critical divisors from example 2.16

Counting down, we get: 5,300,000, 3,200,000, 2,650,000, 1,766,667, 1,600,000, 1,500,000, 1,325,000, 1,066,667, 1,060,000, and 883,333 makes ten. Then the eleventh number down is 800,000, so we can pick any number between 883,333 and 800,000. 850,000 is of course one of them, and that's why that divisor worked.

Now that we picked $d = 850{,}000$, we can also read that off our table. We see one number bigger than 850,000 in the first column, three in the second, and six in the third. And that gives us $a_1 = 1, a_2 = 3, a_3 = 6$.

This critical divisors approach has a couple advantages. One is that we don't have to do a trial-and-error search for d; we just have a straightforward calculation.

We can actually speed this up a bit more. We know the correct Jefferson divisor will always be smaller than the standard divisor $s = p/h$. So we can start in with our critical divisors there. For instance, if we look at our same example 2.14 one more time, we see that the standard divisor s would allocate nine seats. We can calculate the *next* critical divisor for each state, and we know we need to move past exactly one of them.

We see in figure 2.10 that if the standard divisor allocates a_k seats, we have a critical divisor at p_k/a_k, and the *next* critical divisor for that state will happen at $\frac{p_k}{a_k+1}$. So we can compute $\frac{p_k}{a_k+1}$ for each state, and allocate our next seat to whichever state has the biggest (next) critical divisor.

		$s = 1{,}000{,}000$			$d = 850{,}000$	
k	p_k	Standard Quota	Lower Quota a_k	$\frac{p_k}{a_k+1}$	Modified Quota	Jefferson allocation
1	1,500,000	1.50	1	750,000	1.76	1
2	3,200,000	3.20	3	800,000	3.76	3
3	5,300,000	5.30	5	883,333	6.24	6
Total	10,000,000		9			10

Figure 2.10: The Jefferson method with critical divisors, starting with the standard divisor

But note that if we want to allocate another seat, we may have to do the whole process over. If we allocate a seat to state 3, we will need to compute another critical divisor for state 3. In figure 2.10, if we want to allocate an eleventh seat, the next critical divisor for state 1 is still 750,000, and the next critical divisor for state 2 is still 800,000. But we'll also need to check the next critical divisor for state 3. This is

$$\frac{5{,}300{,}000}{6+1} = \frac{5{,}300{,}000}{7} = 757{,}143.$$

This is less than 800,000, so the eleventh seat goes to state 2. (But note it's bigger than 750,000; if we allocate a twelfth seat that would go to state 3 again, not to state 1.)

2.2.3 Results on Jefferson's Method

Jefferson's method favors large states, relative to Hamilton's method.

Example 2.17. Take $n = 2, h = 10, p_1 = 1{,}800{,}000$, and $p_2 = 8{,}200{,}000$. The standard divisor is $s = p/h = 1{,}000{,}000$. Jefferson's method finds a modified divisor of $d = 910{,}000$. Hamilton's method gives state 1 a second seat; Jefferson's method gives nine seats to state 2.

k	p_k	Standard Quota q_k	Hamilton Apportionment	Modified Quota $d = 910{,}000$	Jefferson Apportionment
1	1,800,000	1.8	2	1.98	1
2	8,200,000	8.2	8	9.02	9

These results can get even more extreme, in a fairly surprising way.

Example 2.18. Suppose we have $n = 4, h = 10$, with populations given in figure 2.11.

k	p_k	Standard Quota q_k	Hamilton Apportionment	Modified Quota $d = 800{,}000$	Jefferson Apportionment
1	1,500,000	1.5	2	1.88	1
2	1,400,000	1.4	1	1.75	9
3	1,300,000	1.3	1	1.62	1
4	5,800,000	5.8	6	7.25	7

Figure 2.11: Jefferson's method violates quota in example 2.18

This time Hamilton's method rounds the fourth state's allocation up from $q_4 = 5.8$ to 6. But Jefferson's method allocates even more representatives, giving the state 7 in total.

Definition 2.19. We say it's a *quota violation* if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.

An apportionment method satisfies the *quota rule* if it assigns every state either its lower quota or its upper quota.

This seems like a reasonable rule that we would want an apportionment method to satisfy, but Jefferson's method does not. So we're going to have to talk about it.

Sometimes it's useful to split this rule up into two pieces.

Definition 2.20. An apportionment method satisfies the *upper quota rule* if it never assigns a state more than its upper quota.

A violation of this rules is an *upper quota violation*.

Definition 2.21. An apportionment method satisfies the *lower quota rule* if it never assigns a state less than its lower quota.

A violation of this rules is an *lower quota violation*.

Proposition 2.22. *Jefferson's method satisfies the lower quota rule.*

Proof. If we give every state a provisional apportionment equal to its lower quota, that's the Jefferson apportionment using the standard quota as a modified divisor. This will apportion fewer than h seats. Since we want to apportion h seats, we have to lower the modified divisor, which can only give states more seats, not fewer. Thus each state gets at least its lower quota. □

Proposition 2.23. *Jefferson's method satisfies the house monotonicity criterion.*

Modified divisor perspective. An increase in h will lead to a smaller modified divisor d. Decreasing d gives a higher modified quota p_k/d for each state k. Rounding down a *larger* number will never give a *smaller* number, so reducing d can't ever cause a state to lose a seat. □

Critical divisor perspective. Suppose we list all the critical divisors in decreasing order. The critical divisor p_k/m is associated with state k. If we choose the first h divisors in the list, the seats are assigned to the states that are associated to each divisor. If we increase h to $h+1$, then we will choose the first $h+1$ divisors instead, which will include the first h divisors (and then one more). So all the seats we originally allocated will still be allocated to their original states, and then we will allocate one more. So each state will still get at least as many seats as they did when we allocated h seats.

□

2.2.4 Other divisor methods

Jefferson's method is an example of a *divisor method*, in which the populations of the states are divided by modified divisors to obtain modified quotas, which are then rounded to whole numbers.

The way divisor methods differ is in the rounding method. Jefferson's method rounds modified quotas down to the nearest whole number, but there are other options.

Definition 2.24 (Adams's method)**.** Choose a modified divisor d. Compute the modified quotas p_k/d, and round each of these numbers *up* to obtain a_k. If $a_1 + a_2 + \cdots + a_n = h$ then we have the Adams apportionment.

Jefferson's method has a bias toward large states (like Virginia!). Adams's method has a bias toward small states (like much of New England). Consider two situations which each have a total population of $p = 10{,}000{,}000$ and a house size of $h = 10$.

Example 2.25. First, we give state 1 $p_1 = 1{,}800{,}000$ people and give state 2 $p_2 = 8{,}200{,}000$ people. Hamilton's method gives two seats to state 1. In Jefferson's method, we round down the standard quota to get 1 and 8, and then we compute the critical divisors

$$\frac{p_1}{a_1+1} = \frac{1{,}800{,}000}{2} = 900{,}000$$
$$\frac{p_2}{a_2+1} = \frac{8{,}200{,}000}{9} = 911{,}111$$

so state 2 gains another seat at $d = 911{,}111$ and state 1 gains another seat at $d = 900{,}000$. We can take any divisor in between to get the Jefferson apportionment; we use 910,000.

In Adams's method we round the standard quota up to get 2 and 9, which is too many. We compute critical divisors and we get

$$\frac{p_1}{a_1} = \frac{1{,}800{,}000}{1} = 1{,}800{,}000$$
$$\frac{p_2}{a_2} = \frac{8{,}200{,}000}{8} = 1{,}025{,}000$$

State 2 loses a seat when the divisor hits $d = 1{,}025{,}000$ and state 1 loses a seat when the divisor hits $d = 1{,}800{,}000$, so we can pick any divisor between those numbers. (Actually, that's not quite true: state 2 will lose *another* seat at $d = 1{,}171{,}429$ so we need to pick a $1{,}025{,}000 \leq d < 1{,}171{,}429$. We chose 1,100,000.

		$d = 1{,}000{,}000$		$d = 910{,}000$		$d = 1{,}100{,}000$	
k	p_k	Standard Quota	Hamilton a_k	Jefferson Quota	Jefferson a_k	Adams Quota	Adams a_k
1	1,800,000	1.8	2	1.98	1	1.64	2
2	8,200,000	8.2	8	9.02	9	7.45	8

Example 2.26. Now instead we give state 1 $p_1 = 1{,}200{,}000$ people and give state 2 $p_2 = 8{,}800{,}000$ people. Hamilton's method gives one seat to state 1 and two to state 2. In

Jefferson's method, we round down the standard quota to get 1 and 8, and then we compute the critical divisors

$$\frac{p_1}{a_1+1} = \frac{1{,}200{,}000}{2} = 600{,}000$$
$$\frac{p_2}{a_2+1} = \frac{8{,}800{,}000}{9} = 977778$$

so state 2 gains another seat at $d = 977{,}778$ and state 1 gains another seat at $d = 600{,}000$. We need to take a divisor between those to get the Jefferson apportionment; in fact we need one bigger than $8{,}800{,}000/10 = 880{,}000$. In the table below we pick $d = 900{,}000$.

In Adams's method we round the standard quota up to get 2 and 9, which is too many. We compute critical divisors and we get

$$\frac{p_1}{a_1} = \frac{1{,}200{,}000}{1} = 1{,}200{,}000$$
$$\frac{p_2}{a_2} = \frac{8{,}800{,}000}{8} = 1{,}100{,}000$$

State 2 loses a seat when the divisor hits $d = 1{,}100{,}000$ and state 1 loses a seat when the divisor hits $d = 1{,}200{,}000$, so we can pick any divisor between those numbers. We choose 1,150,000.

		$d = 1{,}000{,}000$		$d = 900{,}000$		$d = 1{,}150{,}000$	
k	p_k	Standard Quota	Hamilton a_k	Jefferson Quota	Jefferson a_k	Adams Quota	Adams a_k
1	1,200,000	1.2	1	1.33	1	1.04	2
2	8,800,000	8.8	9	9.78	9	7.65	8

Proposition 2.27. *Adams's method violates the lower quota rule.*

Proof. Exercise □

Adams's method also automatically meets the Constitutional requirement that each state gets at least one Congressional seat, which Hamilton and Jefferson do not.

We saw that Jefferson's method benefits large states by rounding down, while Adams's method benefits small states by rounding up. A reasonable response is to split the difference, and round "normally"—where fractional parts of 0.5 or greater are rounded up, while fractional parts less than 0.5 are rounded down.

Definition 2.28 (Webster's method)**.** Choose a modified divisor d. Compute the modified quotas p_k/d, and round each of these numbers to the nearest whole number, whether up or down, to obtain a_k. If $a_1 + a_2 + \cdots + a_n = h$ then we have the Webster apportionment.

Example 2.29. We can compute apportionments for all of these methods.

		$d = 1{,}000{,}000$		$d = 850{,}000$		
k	p_k	Standard Quota	Hamilton a_k	Jefferson critical divisor	Modified Jefferson Quota	Jefferson a_k
1	3,300,000	3.3	3	825,000	3.88	3
2	5,100,000	5.1	5	850,000	6	6
3	1,600,000	1.6	2	800,000	1.88	1

		$d = 1{,}100{,}000$			$d = 1{,}000{,}000$		
k	p_k	Adams critical divisor	Modified Adams Quota	Adams a_k	Webster critical divisor	Modified Webster Quota	Webster a_k
1	3,300,000	1,100,000	3	3	942,857	3.3	3
2	5,100,000	1,020,000	4.64	5	927,273	5.1	5
3	1,600,000	1,600,000	1.45	2	1,066,667	1.6	2

We can generalize this even further. And we're going to have to if we want to understand the method the US actually uses today.

2.2.5 Other rounding methods

Definition 2.30. A *rounding function* is a function that takes in a real number, outputs an integer, and has the following two properties:

1. If x is an integer, then $f(x) = x$.
2. If $x > y$, then $f(x) \geq f(y)$.

That is, every integer rounds to itself, and a bigger number will never round to a result less than a smaller number rounds to.

We've seen three rounding functions so far.

- The floor $f(x) = \lfloor x \rfloor$ rounds to the greatest integer less than or equal to x. This was used in Jefferson's method.

- The ceiling function $f(x) = \lceil x \rceil$ rounds to the least integer greater than or equal to x. This was used in Adams's method.
- The "regular" rounding where $f(x)$ is the integer nearest to x, used in Webster's method.

We sometimes call this last method *arithmetic rounding*. This is because the cutoff is the "arithmetic mean", where the average of m and $m+1$ is $\frac{m+(m+1)}{2} = m + \frac{1}{2}$.

Instead we can use as the cutoff the so-called *geometric mean*, which is $\sqrt{m(m+1)}$. That is, instead of adding two numbers and then dividing by 2, we multiply the numbers and take the square root. Then our rounding function will give $f(x) = m$, where m satisfies $\sqrt{m(m-1)} \leq x < \sqrt{m(m+1)}$.

This geometric rounding might seem a little weird. It is great for measuring things like average growth rates, which behave multiplicatively. But the main reason it's worth discussing is that we use it in the US today.

Definition 2.31 (Hill's method)**.** Choose a modified divisor d. Compute the modified quotas p_k/d, and round each of these numbers geometrically to obtain a_k. If $a_1 + a_2 + \cdots + a_n = h$ then we have the Hill apportionment.

Joseph Hill was the chief statistician at the Census Bureau from 1909 to 1921. He suggested this method, and in 1941, Congress officially made Hill's method the permanent apportionment method in the US, which it has been ever since.

We can also consider one more rounding function.

Definition 2.32. The *harmonic mean* of two numbers is the reciprocal of the average of their reciprocals

$$\frac{1}{\frac{\frac{1}{x}+\frac{1}{y}}{2}} = \frac{2}{\frac{1}{x}+\frac{1}{y}} = \frac{2}{\frac{y+x}{xy}} = \frac{2xy}{x+y}.$$

If we want to take the harmonic mean of two adjacent numbers, we get $\dfrac{2m(m+1)}{2m+1}$.

The function $f(x) = m$ where m satisfies the condition

$$\frac{2m(m-1)}{2m-1} \leq x < \frac{2m(m+1)}{2m+1}$$

is the *harmonic rounding* function.

Definition 2.33. Choose a modified divisor d. Compute the modified quotas p_k/d, and round each of these numbers harmonically to obtain a_k. If $a_1 + a_2 + \cdots + a_n = h$ then we have the Dean apportionment.

This method was considered in 1830, but was never used.

Because these rounding functions can be hard to work with, it's useful to have a table of cutoffs. This is table 8.7 in your textbook, which is reproduced here:

Figure 2.12: Cutoffs for rounding small numbers according to various methods

	Rounding Function and Method				
	Rounding Up	Harmonic Rounding	Geometric Rounding	Arithmetic Rounding	Rounding Down
	Adams	Dean	Hill	Webster	Jefferson
0–1	0	0	0	0.5	1
1–2	1	1.333	1.414	1.5	2
2–3	2	2.400	2.449	2.5	3
3–4	3	3.429	3.464	3.5	4
4–5	4	4.444	4.472	4.5	5
5–6	5	5.455	5.477	5.5	6
6–7	6	6.462	6.481	6.5	7
7–8	7	7.467	7.484	7.5	8

So for instance, the number 1.42 would be rounded down to 1 by arithmetic rounding (and of course by Jefferson rounding down), but it would be rounded up by geometric rounding and harmonic rounding. The number 5.46 would be rounded up by harmonic rounding, but down by geometric rounding and arithmetic rounding.

Just like with Jefferson's method, we can compute all the divisor apportionments with a critical divisor approach. For the Jefferson method, we computed numbers of the form p_k/m, since we were looking for breakpoints where some state would get a new representative. In effect, we wanted to know when $p_k/d = m$ was an integer, and then we rearranged to get the divisor $d = p_k/m$.

In other methods, these break points don't occur when p_k/d is an integer, but when instead when p_k/d crosses one of the rounding thresholds that we saw in figure 2.12.

For instance, in the Hill method with geometric rounding, our apportionments change when we cross a cutoff of the form $\sqrt{m(m+1)}$. So we want to find out when $p_k/d = \sqrt{m(m+1)}$, and thus rearranging, we compute all the divisors $d = \frac{p_k}{\sqrt{m(m+1)}}$. We call these

Hill critical divisors; they are the values of d at which a state acquires an additional seat.

You might notice a bit of a glitch here: when $m = 0$ we wind up dividing by zero. We're just going to think of this as an infinitely large number, corresponding to the fact that under Hill's method, each state is guaranteed at least one representative. Then we write the rest of our numbers in order from largest to least, and take the h largest (including the infinite ones, one for each state). An appropriate value for the Hill divisor is any number between the hth largest element and the $h + 1$st largest element in this list.

It's useful to have summarized what critical divisors look like for each method.

Figure 2.13: Summary of critical divisor formulae

Method	Critical divisor for state k
Adams	$\frac{p_k}{a_k}$
Dean	$\frac{p_k(2(a_k+1))}{2a_k(a_k+1)}$
Hill	$\frac{p_k}{\sqrt{a_k(a_k+1)}}$
Webster	$\frac{p_k}{(a_k+1/2)}$
Jefferson	$\frac{p_k}{a_k+1}$

Example 2.34. Consider a problem with $n = 3, h = 10$, and populations of 1,385,000; 2,390,00; and 6,225,000. The lower quotas are 1,2, and 6, and the Hamilton apportionments will be 1, 3, and 6. If we round arithmetically for Webster's method, all of these will round down and we get 1, 3, and 6 again. We need to lower d to get an apportionment. We can compute the Webster critical divisors:

$$\frac{p_1}{a_1 + 1/2} = \frac{1{,}385{,}000}{1.5} = 923{,}333$$
$$\frac{p_2}{a_2 + 1/2} = \frac{2{,}390{,}000}{2.5} = 956{,}000$$
$$\frac{p_3}{a_3 + 1/2} = \frac{6{,}225{,}000}{6.5} = 957{,}692.$$

We allocate the next seat at 957,692 so we need to pick a d smaller than that, but bigger than 956,000. In the table below we use $d = 957{,}000$. This gives us a final seat allocation $a_1 = 1, a_2 = 2, a_3 = 7$.

If we round geometrically for Hill's method, all three standard quotas also round down. That means we need to lower d again. We compute the Hill critical divisors

$$\frac{p_1}{\sqrt{a_1(a_1+1)}} = \frac{1{,}385{,}000}{\sqrt{1\cdot 2}} = \frac{1{,}385{,}000}{1.414} = 979{,}343$$
$$\frac{p_2}{a_2(a_2+1)} = \frac{2{,}390{,}000}{\sqrt{2\cdot 3}} = \frac{2{,}390{,}000}{2{,}449} = 975{,}713$$
$$\frac{p_3}{a_3(a_3+1)} = \frac{6{,}225{,}000}{\sqrt{6\cdot 7}} = \frac{6{,}225{,}000}{6.481} = 960{,}538.$$

So the next delegate is allocated at $d = 979{,}343$, to state 1. We need a d smaller than that, but bigger than 975,713. In the chart below we use $d = 977{,}000$.

		$d = 1{,}000{,}000$		$d = 957{,}000$		$d = 977{,}000$	
k	p_k	Standard Quota	Hamilton a_k	Webster Quota	Webster a_k	Hill Quota	Hill a_k
1	1,385,000	1.385	1	1.447	1	1.418	2
2	2,390,000	2.390	3	2.497	2	2.446	2
3	6,225,000	6.225	6	6.505	7	6.372	6

We could also just do the calculation by making tables of critical divisors. For Adams we would get:

m	State 1	State 2	State 3
0	$1{,}385{,}000/0 = \infty$	$2{,}390{,}000/0 = \infty$	$6{,}225{,}000/0 = \infty$
1	$1{,}385{,}000/1 = 1{,}385{,}000$	$2{,}390{,}000/1 = 2{,}390{,}000$	$6{,}225{,}000/1 = 6{,}225{,}000$
2	$1{,}385{,}000/2 = 692{,}500$	$2{,}390{,}000/2 = 1{,}195{,}000$	$6{,}225{,}000/2 = 3{,}112{,}500$
3	$1{,}385{,}000/3 = 461{,}667$	$2{,}390{,}000/3 = 796{,}667$	$6{,}225{,}000/3 = 2{,}075{,}000$
4	$1{,}385{,}000/4 = 346{,}250$	$2{,}390{,}000/4 = 597{,}500$	$6{,}225{,}000/4 = 1{,}556{,}250$
5	$1{,}385{,}000/5 = 277{,}000$	$2{,}390{,}000/5 = 478{,}00$	$6{,}225{,}000/5 = 1{,}245{,}000$
6	$1{,}385{,}000/6 = 230{,}833$	$2{,}390{,}000/6 = 398{,}333$	$6{,}225{,}000/6 = 1{,}037{,}500$

The ten largest "numbers" (including infinity three times) are in blue on the table; that gives an allocation of $a_1 = 2, a_2 = 2, a_3 = 6$. This corresponds to any divisor larger than 1,195,000 but less than 1,245,000; indeed we can pick $d = 1{,}200{,}000$ and get lower quotas of 1.154, 1.991, and 5.188, which round up to 2, 2, and 6.

Taking the same approach for Hill's method gives us the table:

m	State 1	State 2	State 3
0	$1{,}385{,}000/0 = \infty$	$2{,}390{,}000/0 = \infty$	$6{,}225{,}000/0 = \infty$
1	$1{,}385{,}000/\sqrt{2} = 979{,}343$	$2{,}390{,}000/\sqrt{2} = 1{,}689{,}985$	$6{,}225{,}000/\sqrt{2} = 4{,}401{,}740$
2	$1{,}385{,}000/\sqrt{6} = 565{,}424$	$2{,}390{,}000/\sqrt{6} = 975{,}713$	$6{,}225{,}000/\sqrt{6} = 2{,}541{,}346$
3	$1{,}385{,}000/\sqrt{12} = 399{,}815$	$2{,}390{,}000/\sqrt{12} = 689{,}934$	$6{,}225{,}000/\sqrt{12} = 1{,}797{,}003$
4	$1{,}385{,}000/\sqrt{20} = 309{,}695$	$2{,}390{,}000/\sqrt{20} = 689{,}934$	$6{,}225{,}000/\sqrt{20} = 1{,}391{,}952$
5	$1{,}385{,}000/\sqrt{30} = 252{,}865$	$2{,}390{,}000/\sqrt{30} = 436{,}352$	$6{,}225{,}000/\sqrt{30} = 1{,}136{,}524$
6	$1{,}385{,}000/\sqrt{42} = 213{,}710$	$2{,}390{,}000/\sqrt{42} = 368{,}785$	$6{,}225{,}000/\sqrt{42} = 960{,}538$

The tenth number on this list is 979,343, and the eleventh is 975,713, so we can pick any number between them. In the table above we used 977,000.

2.3 Evaluating Apportionment Methods

We want a method to treat all states the same, only taking their populations into account.

Definition 2.35. An apportionment method is *neutral* if permuting the populations of states permutes the resulting numbers of seats in the same way.

This doesn't prevent a bias against large states, or small states; Adams's and Jefferson's methods are both neutral in this sense. But it prohibits a bias against Western states, or states that start with a vowel. (Arguably our current method is not neutral, if you include Washington D.C. on your list of states to be apportioned.)

Definition 2.36. An apportionment method is *proportional* if it produces the same result for two censuses with the same house size, and the same relative populations p_k/p.

The idea here is that we shouldn't care about the absolute numbers of people, just their relative share of the population. We say the *population distribution* is the list $p_1/p, p_2/p, \ldots, p_n/p$, which tells you what fraction of the total population each state has. A proportional apportionment method can be computed from just the population distribution, without knowing the absolute numbers of people.

The core intuition here is that if *every* state doubles its population, this shouldn't affect the apportionment at all.

Proposition 2.37. *Hamilton's method, and every divisor method, is proportional.*

Proof. Hamilton's method depends only on the standard quotas $q_k = p_k/s$. But the standard divisor $s = p/h$, so we can rewrite this as $p_k = h\frac{p_k}{p}$, so the standard quota depends only on the house size h and the population distribution p_k/p.

For divisor methods, we are dividing p_k by a divisor d. If all populations increase by a factor of c, we can increase the divisor d by that same factor and get the same modified quotas, with the same rounded result.

Alternatively, if we think in terms of critical divisors, we are writing the critical divisors in order from greatest to least. These critical divisors look like $p_k/f(m)$ for some function of the whole numbers m. If we instead compute $\frac{p_k/p}{f(m)}$ those numbers will be in exactly the same order and give the same result; so the output depends only on the population distribution. □

There's another obvious sense of fairness: smaller states should definitely not get *more* seats.

Definition 2.38. An apportionment method is *order-preserving* if, whenever $a_i > a_j$, then $p_i > p_j$.

Note this just says that a state with more seats must have a larger population. The opposite is not true; a state with a larger population may not have more seats. It can't have *fewer* seats, by this rule, but it's possible for two states with different populations to have the same number of seats. (Indeed, that's very hard to avoid!)

We *could* call this property "population monotone", but we're going to save that for the stronger idea first stated in definition 2.9.

We can also recall the ideas from definitions 2.19 and 2.20:

Definition. We say it's a *quota violation* if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.

An apportionment method satisfies the *quota rule* if it assigns every state either its lower quota or its upper quota.

An apportionment method satisfies the *upper quota rule* if it never assigns a state more than its upper quota.

A violation of this rules is an *upper quota violation*.

Hamilton's method obviously satisfies the quota rule. We saw that Jefferson's method violates the upper quota rule, and similarly Adams's method violates the lower quota rule. We'll see in section 2.3.5 that *any* divisor method has to have quota violations.

2.3.1 House Monotonicity

Recall definition 2.7:

Definition. An apportionment method is called *house monotone* if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease. Thus example 2.6 shows that Hamilton's method is not house monotone.

The Alabama paradox in 1880 is a violation of house monotonicity, and shows that Hamilton's method is not house monotone. Had h increased from 299 to 300, Alabama's apportionment would have dropped from 8 seats to 7. See also example 2.6 where we saw an example of Hamilton's method failing house monotonicity.

Divisor methods do better. We saw in proposition 2.23 that Jefferson's method is house monotone. That's just a special case of the following result:

Proposition 2.39. *All divisor methods are house monotone.*

Proof. Consider any divisor method applied to states with populations $p_1, p_2, \ldots, p_n$, and suppose a divisor d gives an apportionment of h seats.

If we want to increase h, we will need to decrease d, which will increase the modified quota p_k/d for each k. Rounding a larger number can never give a smaller result (by definition 2.30). So no state can ever get a smaller apportionment from a larger h. □

2.3.2 Population Monotonicity

Recall definition 2.9:

Definition. A method is called *population monotone* if a state can never lose a seat when its population increases while no other state's population increases.

In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.

We know that Hamilton's method doesn't satisfy population monotonicity, from example 2.8. However, divisor methods do better here:

Proposition 2.40. *All divisor methods satisfy population monotonicity.*

Proof. Suppose $a_i' < a_i$ and $a_j' > a_j$. By definition 2.30 of rounding functions, this must mean that the modified divisor p_i/d of state i decreased, and the modified divisor of p_j/d increased. That gives us the two inequalities

$$p_i'/d' < p_i/d \qquad\qquad p_j'/d' > p_j/d.$$

We can rearrange these inequalities to get

$$p_i' < p_i \frac{d'}{d} \qquad\qquad p_j' > p_j \frac{d'}{d}.$$

Now we think about two possibilities. First, let's suppose that $d' \leq d$ and so $d'/d \leq 1$. That implies that

$$p_i' < p_i \frac{d'}{d} \leq p_i$$

and thus $p_i' < p_i$, which satisfies the definition of population monotonicity.

If that's not true, then we must have $d' > d$ and so $d'/d > 1$. That would imply that

$$p_j' > p_j \frac{d'}{d} > p_j$$

and thus $p_j' > p_j$, also satisfying the definition of population monotonicity.

□

Corollary 2.41. *Hamilton's method is not a divisor method.*

This corollary is more interesting that it seems. Obviously we didn't *define* Hamilton's method as a divisor method. But conceivably there could be some weird rounding function that would always give the same result as the Hamilton algorithm. But in fact that's not the case; any divisor method will be population monotone, and Hamilton's method is not, so there's no possible rounding function that will give those results.

We can leverage our ideas here to make proposition 2.39 kind of superfluous..

Proposition 2.42. *Any method that is population monotone is also house monotone.*

Proof. Suppose a method is population monotone, and consider a situation in which our house size changes from h to $h' = h + 1$. And suppose that no populations change, which we can write as $p_i' = p_i$ and $p_j' = p_j$.

Increasing the house size means that at least one state will gain a seat. So assume (without loss of generality) that state j gains a seat, and thus $a_j' > a_j$.

Now imagine that some state loses a seat, meaning that $a_i' < a_i$. By population monotonicity, we must have either $p_i' < p_i$, or $p_j' > p_j$. But we know that $p_i' = p_i$ and $p_j' = p_j$, so that's not possible.

Therefore no state can lose a seat, and so the method is house monotone. □

An interesting note is that this proof works because the definition of population monotonicity doesn't have an all-else-equal clause; the property holds even if h changes. All of that work is loaded into the premise that one states gains seats and another loses seats.

Proposition 2.43. *Any method that is population monotone and neutral must be order-preserving.*

Proof. Suppose a method is neutral and population monotone, and suppose we get a census with $p_j > p_i$. We can construct another census by swapping the populations of states i and j—that is, we write $p'_i = p_j$ and $p'_j = p_i$, and then $p'_k = p_k$ for all other states k.

This results in a (hypothetical) increase in population for state i, and a decrease in population for state j. Since $p'_i \geq p_i$ and $p'_j \leq p_j$, it follows that either $a'_i \geq a_i$, or $a'_j \leq a_j$. (Or both!)

But by neutrality, swapping the populations should swap the apportionments, so $a'_i = a_j$ and $a'_j = a_i$. Thus we can make at least one of the following two arguments:

$$a_j = a'_i \geq a_i$$
$$a_j \geq a'_j = a_i.$$

In either case, we see that $a_j \geq a_i$.

Thus if $p_j > p_i$, we must have $a_j \geq a_i$, which is the definition of order preserving. □

2.3.3 Relative population monotonicity

There are actually two ways of thinking about population change. If a state goes from an original population p_k to a later population p'_k, the most obvious way to measure the change is with the *absolute change* $\Delta p_k = p'_k - p_k$. We can also think of this as the arithmetic population change. It's the amount by which the population has grown.

Remark 2.44. Δ is the Greek letter capital Delta, which is a capital D. It's often to refer to a change in a quantity over time; you can think of it as standing for "distance". You may recognize it from algebra, where we sometimes write the slope of a line as $\frac{\Delta y}{\Delta x}$, the change in y divided by the change in x.

But this isn't actually how we talk about population growth most of the time. Generally we talk about growth in percentage terms. This makes sense, because gaining 10,000 people makes a much bigger difference for a state with 500,000 people than to a state with 20,000,000 people.

Instead, we often want to talk about the *relative change*

$$\frac{\Delta p_k}{p_k} = \frac{p'_k - p_k}{p_k}.$$

This measures how big the growth is relative to the starting population.

k	1	2	3
p_k	10,000	10,000	100,000
p'_k	11,000	20,000	110,000
Δp_k	1,000	10,000	10,000
$\Delta p_k / p_k$	0.1	1.0	0.1
%	10%	100%	10%

Example 2.45. Suppose we have the following table of populations:

State 2 and State 3 have the same absolute population growth; each gains 10,000 people between the two censuses. But that change is much more dramatic for state 2 than state 3, because state 2 started with many fewer people. State 1 and state 3 have the same *relative* population growth, of 10%; state 2, rather, grows 100% between the two censuses.

Definition 2.46. An apportionment method is *relative population monotone* if, whenever we consider states with positive population and $a'_i < a_i$ and $a'_j > a_j$, then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.

This is essentially the same as definition 2.9 that we studied in section 2.3.2, but instead it focuses on relative changes. This makes it better at comparing states of very different populations. It also makes the criterion simpler, because we don't need to worry about whether the states are growing or shrinking; we can just make one statement.

We need to consider the case where a state has zero population because we want to talk about introducing new states from one census to another; we can think of this as the state having zero population in the first census. But we can't talk about relative population growth from a baseline of zero population, since that would involve dividing by zero.

This property is stronger than regular population monotonicity. In particular, if a method is relative population monotone, then it's population monotone.

Proposition 2.47. *If an apportionment method is relative population monotone, then it is population monotone.*

Proof. Suppose $a'_i < a_i$ and $a'_j > a_j$, and that all populations are positive. By relative population monotonicity, we conclude that $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$. We can conclude, in particular, that either $\Delta p_j / p_j$ is positive, or $\Delta p_i / p_i$ is negative. (Or both!)

If $\Delta p_j / p_j$ is positive, that means that Δp_j is positive, and thus $p'_j < p_j$. If $\Delta p_i / p_i$ is negative, that means that Δp_i is negative, and thus $p'_i < p_i$. And that's the definition of population monotonicity. □

But the property isn't *too* strong. We can achieve it with reasonable methods.

Proposition 2.48. *All divisor methods are relative population monotone.*

Proof. Suppose we care computing apportionments of two censuses using a divisor method, and suppose that $a_i' < a_i$ and $a_j' > a_j$. We know that the modified quota p_i/d must have decreased and the modified quota of p_j/d must have increased. Another way of writing that is:

$$\frac{p_i'}{d'} < \frac{p_i}{d} \qquad \frac{p_j'}{d'} > \frac{p_j}{d}$$

$$\text{thus} \qquad \frac{p_i'}{p_i} < \frac{d'}{d} \qquad \frac{p_j'}{p_j} > \frac{d'}{d}$$

We can link these two inequalities together, since they both have the d'/d term.

$$\begin{aligned}\frac{p_i'}{p_i} &< \frac{p_j'}{p_j} \\ \frac{p_i'}{p_i} - 1 &< \frac{p_j'}{p_j} - 1 \\ \frac{p_i'}{p_i} - \frac{p_i}{p_i} &< \frac{p_j'}{p_j} - \frac{p_j}{p_j} \\ \frac{p_i' - p_i}{p_i} &< \frac{p_j' - p_j}{p_j}.\end{aligned}$$

And that's just the statement that $\frac{\Delta p_i}{p_i} < \frac{\Delta p_j}{p_j}$, which is what we needed to prove. □

And in fact relative population monotone isn't all that much stronger than population monotone at all. In particular, if we assume proportionality, they're the same.

Proposition 2.49. *If an apportionment method is proportional and population monotone, then it's relative population monotone.*

Proof. Suppose an apportionment method is proportional, but not relative population monotone. That means we have some pair of states i and j where $a_i' < a_i$ and $a_j' > a_j$, but $\frac{\Delta p_j}{p_j} \leq \frac{\Delta p_i}{p_i}$.

For notational convenience we set

$$r = 1 + \frac{\Delta p_i}{p_i} \qquad s = 1 + \frac{\Delta p_j}{p_j}$$

and observe that $0 < s$ since the relative growth rate can't be less than -100%, and $s \leq r$ by our hypothesis.

Now we compute

$$rp_i = \left(1 + \frac{\Delta p_i}{p_i}\right) = p_i + \Delta p_i = p_i + \left(p'_i - p_i\right) = p'_i$$
$$sp_j = \left(1 + \frac{\Delta p_j}{p_j}\right) = p_j + \Delta p_j = p_j + \left(p'_j - p_j\right) = p'_j$$

Now suppose we have a third census (which we'll denote with double-prime letters, like p''), where for every state k we set $p''_k = p'_k/r$. This is a pure rescaling of the second census, where every state's population is scaled by a factor of r. By proportionality, this can't change the apportionment, so we have $a''_k = a'_k$ for every state k. But we can compute

$$p''_i = p'_i/r = p_i$$
$$p''_j = p'_j/r = \frac{s}{r}p'_j \leq p_j.$$

If we compare our original census to this third census, we must have $a''_i = a'_i < a_i$ and $a''_j = a'_j > a_j$; but we have $p''_i = p_i$ and $p''_j \leq p_j$, meaning neither $p''_i < p_i$ nor $p''_j > p_j$. So this apportionment system is also not population monotone.

□

Since we're generally only going to consider proportional methods, we can use population monotonicity and relative population monotonicity interchangeably; in the future we won't really specify.

2.3.4 The New States Paradox

An interesting specific situation is when a new state is introduced entirely to the union; this has happened many times in the history of the US (most recently with Hawai'i in 1959). We can interpret this as a situation where a state's population increases from 0 to a positive number. (Even the Adams method doesn't assign a seat to a state with a population of zero!)

Definition 2.50. Suppose state k is joining the union as a new state, and thus $p_k = 0$ and $p'_k > 0$. Suppose there are other states i and j whose populations are unchanged. We say a *new states paradox* or *Oklahoma paradox* occurs if $a'_i < a_i$ and $a'_j > a_j$.

Remark 2.51. There's nothing weird about $a'_i < a_i$, or $a'_j > a_j$, in isolation. Maybe adding the new state means some old states have to lose representation; that will certainly happen if we don't increase h. And if we do increase h to accommodate the new state, then maybe other states will get increased representation too.

But if both happen at once, that means that adding the new state has caused other states to transfer seats among themselves; that's the odd bit.

We don't have a ton of results to prove here, though. If an apportionment method is population monotone, then in particular it can't suffer from the new states paradox. Thus by proposition 2.40, no divisor method can suffer from the new states paradox. On the other hand, example 2.10 shows that Hamilton's method is vulnerable to the new states paradox. (This also furnishes another proof that Hamilton's method is not a divisor method.)

2.3.5 The Impossibility Theorem

Let's return to one of our earliest criteria, the quota criterion. Hamilton's method satisfies the quota rule; we've seen that Jefferson's method does not. (We will see that no divisor method can satisfy the quota rule, in fact.) There are other quota methods, such as Lowndes's method:

Definition 2.52 (Lowndes's method)**.** As a provisional apportionment, assign every state its lower quota. Then assign the remaining seats to the states, at most one per state, in decreasing order of $\frac{\{q_k\}}{\lfloor q_k \rfloor}$, the ratio of the fractional part of the standard quota to the lower quota.

This has the same basic approach as Hamilton's method: we give every state its lower quota, then use the fractional part of the standard quota to decide which states to round up to the upper quota. But it gives an extra bonus to smaller states, since $\frac{\{q_k\}}{\lfloor q_k \rfloor}$ will be bigger when $\lfloor q_k \rfloor$ is smaller.

The underlying logic is that the extra representative gives more "extra representation" in the smaller state; cutting a seat from a smaller state will create bigger districts than cutting a seat from a larger state will.

For example, if we have one state with quota $q_i = 2.3$ and another state with quota $q_j = 7.9$, then Hamilton's method would give an extra representative to state j rather than state i. But if we deny state i its extra district, that makes each district $\frac{.3}{2} = 15\%$ bigger than the desired standard divisor; if we instead deny state j, that makes each district $\frac{.9}{7} \approx 12.86\%$ bigger than the desired size. So Lowndes argues that it's better to give the extra district to state i.

Exercise 2.53. *Lowndes's method is not house monotone or population monotone, and is vulnerable to the new states paradox.*

So far we've seen quota methods, which fail various monotonicity criteria; and we've seen divisor methods, which we haven't shown can avoid quota violations. (Adams's method avoids upper quota violations, and Jefferson's method avoids lower quota violations, but that's more because they have strong biases in the other direction than because they particularly respect quotas.)

In fact, there's no way to completely solve both these problems at once.

Theorem 2.54 (Balinski and Young)**.** *No apportionment rule that is neutral and population monotone can satisfy the quota rule.*

Proof. We can prove this by constructing a specific example where there's no neutral way to apportion seats that satisfies both population monotonicity and the quota rule. So consider a pair of censuses where we want to allocate $h = 10$ seats:

$$p_1 = 69{,}000 \qquad p_1' = 68{,}000$$
$$p_2 = 5{,}200 \qquad p_2' = 5{,}500$$
$$p_3 = 5{,}000 \qquad p_3' = 5{,}600$$
$$p_4 = 19{,}900 \qquad p_4' = 5{,}700.$$

In the "before" situation, we have a total population of 100,000, so the standard divisor is $s = 10{,}000$ and the standard quotas are

$$q_1 = 6.99 \qquad q_2 = 0.52 \qquad q_3 = 0.50 \qquad q_4 = 1.99.$$

Since the method satisfies the quota rule, we know that state 1 has 7 seats or less, and state 4 has 2 seats or less, so states 2 and 3 have to get at least one seat between them. Because this method is population monotone and neutral, proposition 2.43 shows it must be order-preserving; so state 2 must receive at least one seat.

Now let's consider the after census. The total population is 84,800 so the standard divisor is 8,480. We can compute the standard quotas for this after census, and we get

$$q_1' = 8.02 \qquad q_2' = 0.65 \qquad q_3' = 0.66 \qquad q_4' = 0.67.$$

Again by the quota rule, state 1 has to get at least 8 seats, leaving at most two seats for the other three states. By the order-preserving property, we can't give a seat to state 2 if states 3 and 4 don't have one each, and since there are only two seats to go around, so state 2 cannot get any seats at all.

So if we have a neutral, population monotone method that satisfies the quota rule, it must give state 1 at most 7 seats in the before situation, and at least 8 in the after situation; it must give state 2 at least one seat in the before situation, and cannot give state 2 any in the after situation. Thus state 1 will gain seats, and state 2 will lose seats.

But the population of state 1 had declined while the population of state 2 has increased. That's a violation of population monotonicity, so we cannot have such a method.

□

Corollary 2.55. *No divisor method satisfies the quota rule.*

This shows that even though both the quota rule and population monotonicity are intuitively appealing, but we cannot have them both; we have to choose. (In the US today we use Hill's method, which is a divisor method and thus violates the quota rule.)

But theorem 2.54 doesn't mention *house* monotonicity. And it turns out it is in fact possible to get a method that satisfies the quota rule and is still house monotone.

2.4 Balinski and Young Apportionment

Jefferson's method is very prone to quota violations, but it's straightforward to calculate and seems like it has a fair amount of constitutional support. Can we tweak it to also satisfy the quota rule?

Theorem 2.54 says we can't get everything we want. If it satisfies the quota rule, it can't possibly be population monotone. But it turns out we can maintain the house monotonicity and still keep the quota rule. And we can do this fairly straightforwardly, by just instituting a rule that we never allow an upper quota violation.

We will define this apportionment method iteratively, or inductively. We're going to assign seats one at a time, in order; so that the way we assign 10 seats is to assign the first 9 seats, and then assign one more.

Recall we can look at Jefferson's method itself that way. We start for $h = 0$, in which case we obviously apportion 0 seats to each state, $a_k = 0$. (In formal "mathematical induction" we call this the base case.)

If we've already apportioned h seats, and have $a_1, a_2, \ldots, a_n$, we can always apportion the next seat. We compute the critical divisors $\frac{p_k}{a_k+1}$, which represent how large a Congressional district in state k would be if we assign it one more seat; the larger this number is, the less overrepresented state k would be if we give them another seat. Thus the $h+1$st seat goes to the state with the largest critical divisor. We saw this worked out in example 2.16.

This method is straightforward, and obviously house monotone. But it is unfortunately prone to upper quota violations; in most reasonable situations large states will get more than their upper quotas. For instance, with 2020 census data, California has a standard quota of 51.99, giving it an upper quota of 52 and a lower quota of 51. Jefferson's method would allocate it 54 seats. Jefferson's method would also give New York and Texas one seat more than their upper quotas; it would give Vermont and Wyoming no seats at all.

Definition 2.56 (Balinksi and Young method)**.** We define the method of Balinski and Young inductively.

If $h = 0$, then set $a_k = 0$ for every k.

Suppose we have an apportionment for some fixed h, given by $a_1, a_2, \ldots, a_n$ such that $a_1 + \cdots + a_n = h$.

For each state k, compute the quotient $\frac{p_k}{a_k+1}$ and call this the strength of the kth state's claim for the next seat. We "want" to give the next seat to the state with the strongest claim, but we don't want to have any upper quota violations.

So we say a state is *eligible* if $a_k + 1 \leq \left\lceil (h+1)\frac{p_k}{p} \right\rceil$, so that giving the state another seat would not give an upper quota violation.

Then we assign the $h + 1$st seat to the eligible state with the strongest claim.

Poll Question 2.4.1*.* Why do we use Jefferson rather than Adams or Hill or Webster as the base for this method?

This method will sort of "obviously" avoid upper quota violations, since we simply refuse to allocate a seat when it would cause an upper quota violation. But that maybe leads to a question of what happens if no state is eligible. Fortunately that can't happen.

Proposition 2.57. *At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.*

Proof. The basic idea of this proof is that for a state to be ineligible, we have to have given it a lot of seats, relative to the total number of seats h. But we can't give every state a lot of seats relative to h, because h is the total number of seats we can allocate. So at least one state must have space.

Consider a census with populations $p_1, p_2, \ldots, p_n$, and suppose that Balinski and Young have apportioned $a_1, a_2, \ldots, a_n$ seats for a total of h seats to the various states.

After we apportion the $h+1$st state, the standard divisor will be $s = \frac{p}{h+1}$, so the standard quota for each state will be $\frac{p_k}{s} = (h+1)\frac{p_k}{p}$. A state will be ineligible if giving it one more seat will put it over this upper quota; thus it's ineligible if $a_k + 1 > \left\lceil (h+1)\frac{p_k}{p} \right\rceil$.

But since those are both whole numbers, the only way we can have $a_k + 1$ *bigger* than the upper quota is if a_k is *at least as big* as the upper quota. (If we have fractions that wouldn't be true; we can have $3 < 3.5$ but $3 + 1 > 3.5$. But for whole numbers, if $a_k + 1$ is bigger than some whole number than a_k must be at least the same size.) So if a state is ineligible, we must have

$$a_k \geq \left\lceil (h+1)\frac{p_k}{p} \right\rceil \geq (h+1)\frac{p_k}{p}.$$

Now if *every* state is ineligible, we have the following series of inequalities:

$$\begin{aligned} a_1 &\geq (h+1)\frac{p_1}{p} \\ a_2 &\geq (h+1)\frac{p_2}{p} \\ &\vdots \\ a_n &\geq (h+1)\frac{p_n}{p}. \end{aligned}$$

If we add all these inequalities, we get

$$\begin{aligned} a_1 + \cdots + a_n &\geq (h+1)\frac{p_1}{p} + (h+1)\frac{p_2}{p} + \cdots + (h+1)\frac{p_n}{p} \\ &= (h+1)\left(p_1 + p_2 + \cdots + p_n\right)\frac{1}{p} \\ &= (h+1)(p)\frac{1}{p} = h+1. \end{aligned}$$

That is, once we add up all these allocations, we see that $a_1 + \cdots + a_n \geq h + 1$, we have to have already allocated at least $h + 1$ seats. But we're at the step where we have allocated exactly h seats, so that can't be true. So that means at least one state must have

$$a_n < (h+1)\frac{p_n}{p}$$

and be eligible to receive the next seat.

□

Let's work through an example comparing Jefferson's method to Balinski and Young's method. Suppose we have states A, B, C with populations $p_1 = 7, p_2 = 22, p_3 = 71$, for a total $p = 100$. Suppose we want to attain a house size of $h = 15$. We can think of Jefferson's method as working iteratively like this, as seen in figure 2.14

Now let's see what this would look like in the method of Balinski and Young. The basic approach will be the same, but at each step we remove ineligibles, and avoid awarding seats to any state that isn't eligible for another seat. We see this process in figure 2.15

h	Jefferson Critical Divisor			Jefferson Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3
0				0	0	0
1	7	22	**71**	0	0	1
2	7	22	**35.5**	0	0	2
3	7	22	**23.67**	0	0	3
4	7	**22**	17.75	0	1	3
5	7	11	**17.75**	0	1	4
6	7	11	**14.2**	0	1	5
7	7	11	**11.83**	0	1	6
8	7	**11**	10.14	0	2	6
9	7	7.33	**10.14**	0	2	7
10	7	7.33	**8.875**	0	2	8
11	7	7.33	**7.89**	0	2	9
12	7	**7.33**	7.1	0	3	9
13	7	5.5	**7.1**	0	3	10
14	**7**	5.5	6.45	1	3	10
15	3.5	5.5	**6.45**	1	3	11

Figure 2.14: Jefferson's method worked iteratively

2.4.1 The Quota Rule for Balinski and Young

It's clear that the Balinski-Young method is house monotone, because of the way we produce it iteratively: the seats are given away one at a time, and we never back up and change an earlier apportionment. It's also clear that it satisfies the upper quota rule, because we simply never assign a seat that would violate the upper quota rule.

The trick is to show that it also satisfies the lower quota rule. The Jefferson method, of course, satisfies the lower quota rule, but that's not realy for any deep reason. We need to check that the Balinksi and Young method doesn't ever accidentally ruin that.

Proposition 2.58. *The Balinski and Young method satisfies the lower quota rule.*

h	Jefferson Critical Divisor			Jefferson Apportionment			Standard Quotas			Balinski and Young Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3	q_1	q_2	q_3	a_1	a_2	a_3
0				0	0	0						
1	7	22	**71**	0	0	1	0.07	0.22	0.71	0	0	1
2	7	22	**35.5**	0	0	2	0.14	0.44	1.42	0	0	2
3	7	22	**23.67**	0	0	3	0.21	0.66	2.13	0	0	3
4	7	**22**	7.75	0	1	3	0.28	0.88	.84	0	1	3
5	7	11	**17.75**	0	1	4	0.35	1.1	3.55	0	1	4
6	7	11	**14.2**	0	1	5	0.42	1.32	4.26	0	1	5
7	7	**11**	~~11.83~~	0	1	6	0.49	1.54	~~4.97~~	0	2	5
8	7	7.33	**11.83**	0	2	6	0.56	1.76	5.68	0	2	6
9	7	7.33	**10.14**	0	2	7	0.63	1.98	6.39	0	2	7
10	7	7.33	**8.88**	0	2	8	0.7	2.2	7.1	0	2	8
11	7	**7.33**	~~7.89~~	0	2	9	0.77	2.42	~~7.81~~	0	3	8
12	7	5.5	**7.89**	0	3	9	0.84	2.64	8.52	0	3	9
13	7	5.5	**7.1**	0	3	10	0.91	2.86	9.23	0	3	10
14	**7**	5.5	~~6.45~~	1	3	10	0.98	3.08	~~9.94~~	1	3	10
15	3.5	5.5	**6.45**	1	3	11	1.05	3.3	10.65	1	3	11

Figure 2.15: Apportionment by the Balinski and Young method

Proof. □

Corollary 2.59. *The Balinski and Young method is not population monotone.*

Proof. Since the Balinski and Young method satisfies the quota rule, Theorem 2.54 (proven by Balinski and Young!) shows it can't be population monotone. □

2.5 Why choose different rounding functions?

In sections 2.2.4 and 2.2.5 we covered a collection of rounding methods, which all give almost, but not quite, the same results. What's the point of having all that variety?

We talked about this briefly at the time, but each rounding method is "the best" at a slightly different sort of thing. We can view each rounding function as finding the optimal solution to a specific question; they differ in what question they answer.

2.5.1 Degree of Representation and Webster's Method

Definition 2.60. The *degree of representation* of a state k is the number $\frac{a_k}{p_k}$, which measures the fraction of a congressional seat each individual citizen is allocated.

In the 2020 census, Maryland had 6,185,278 people and got allocated 8 congressional seats. This means that each citizen of Maryland is represented by $\frac{8}{6,185,278} \approx .000,001,293$ of a Congressperson.

Because every state gets a whole number of representatives, we can't give every state exactly the same degree of representation. But Webster's method, the divisor method with arithmetic rounding does the best it can.

Proposition 2.61. *Webster's method is the unique apportionment method with the following property: the difference between the degrees of representation of any two states cannot be decreased by transferring a seat from the better represented state to the worse represented state.*

2.5.2 District Size and Dean's Method

We also can't make it so that districts have the same number of people in each state. But Dean's method, the divisor method with harmonic rounding, does the best it can.

Proposition 2.62. *Dean's method is the unique apportionment method with the following property: the difference between the sizes of districts in any two states cannot be made smaller by transferring a seat from the smaller-district state to the bigger-district state.*

2.5.3 Average district size and Hill's Method

There is more that one way to compare how close two numbers are. One is to subtract them from each other and see how big the result is—or in other words, how far it is from 0. But this doesn't work well when our numbers have radically different scales. It might not really capture what we care about to say that 1 is closer to 1,000,000 than 1,000,000 is to 2,000,000.

Another way of measuring the distance between two numbers is to divide one by the other, and see how close we are to 1. By this method, 100 and 200 are closer than 10 and 30 are, and 1,000,000 is much closer to 2,000,000 than it is to 1.

It makes sense to say that we want the districts in different states to be as close in size as possible. If we measure closeness by subtraction—arithmetically—then Dean's method does this best. But if we measure closeness geometrically, by multiplication, then Hill's method, the divisor method with geometric rounding, does the best.

It also makes sense to say we want the states to have degrees of representation as close as possible. Once again, we saw that Webster's method does that, if we measure closeness arithmetically. But if we measure closeness geometrically, once again Hill's method does the best we can.

Proposition 2.63. *Hill's method is the unique apportionment method with the following property: the ratio between the average sizes of districts in any two states (expressed as number greater than 1) cannot be made smaller by transferring a seat from the smaller-district state to the bigger-district state.*

Further, it is also the unique apportionment method with the property: the ratio between the degrees of representation in any two states cannot be made smaller by transferring a seat from the better-represented state to the worse-represented state.