Math 1231: Single-Variable Calculus I

The George Washington University   Fall 2025

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Contents

1 Functions and Limits

1.1 Quick Review Facts

Functions

Recall that a *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input. Here we remember some facts about common functions.

Polynomials: You should remember the quadratic formula, which says that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also useful to recall that

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$
- $(a^2 + ab + b^2)(a - b) = a^3 - b^3$.

Rational functions are the ratio of two polynomials.

Trigonometric functions: In this course we will *always* use radians, because they are unitless and thus easier to track (especially when using the chain rule). Useful facts include:

- The most important trigonometric identity, and really the only one you probably need to remember, is $\cos^2(x) + \sin^2(x) = 1$.
- From this you can derive the fact that $1 + \tan^2(x) = \sec^2(x)$.
- $\sin(-x) = -\sin(x)$. We call functions like this "odd".
- $\cos(-x) = \cos(x)$. We call functions like this "even."
- $\sin(x + \pi/2) = \sin(\pi/2 - x) = \cos(x)$
- A fact that we will probably use exactly twice is the sum of angles formula for sine: $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- Similarly, $\cos(x + y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

Set and interval notation

We write $\{x : \text{condition}\}$ to represent the set of all numbers x that satisfy some condition. We will sometimes write $\mathbb{R}$ to refer to all the real numbers. We will also refer to various intervals:

$(a,b) = \{x : a < x < b\}$	open interval	$[a,b] = \{x : a \leq x \leq b\}$	closed interval
$[a,b) = \{x : a \leq x < b\}$	half-open interval	$(a,b] = \{x : a < x \leq b\}$	half-open interval

1.2 Approximation

Let's start with an easy question:

Question 1.1. What is the square root of four?

Everyone can probably tell me that the answer is "two". So now let's do a harder one:

Question 1.2. What is the square root of five?

Without a calculator, you probably can't tell me the answer. But you should be able to make a pretty good guess. Five close to four; so $\sqrt{5}$ should be close to two.

We call this sort of estimate a *zeroth-order approximation*. In a zeroth-order approximation, we only get to use one piece of information: the value of our function at a specific number. Then we use that information to estimate its value at nearby numbers.

We can only do so good a job with that limited amount of information, but we can still do a surprising amount.

Example 1.3. The high temperatures in Washington DC on August 25-27 2022 were 88, 91, 88. It's not surprising that the high temperature on August 28 was 90.

Example 1.4. The high temperatures on January 1-2 were 59 and 63. Can we estimate the high on January 3?

The obvious guess is that it's roughly 60. But it turns out the actual answer is 30. Often the weather stays similar from day to day, but not always; we can't approach this question the same way we did the square root function, because some times the weather is just erratic.

This example shows that we can't always do what we did with $\sqrt{5}$. Some functions jump around too much for this sort of approximation thing to work; values of similar inputs don't have similar outputs.

And sometimes this is pretty important!

Example 1.5. Used cars that have been driven more are worth less money when sold. A 2012 study by Nicola Lacetera, Devin G. Pope, and Justin Sydnor (Heuristic Thinking and Limited Attention in the Car Market) collected data on average car price by mileage. The found (among other things) the following approximate data:

Mileage	Average Price
36000	\$11500
37000	\$11400
38000	\$11350
39000	\$11250

Based on this, what would you expect the average price of a car with 40000 miles to be?

Based just on this data it seems like you'd maybe expect a price of something like \$11200 or \$11150. But in fact the price they found was \$11000. And we see a pattern like this if we zoom out and consider all the data:

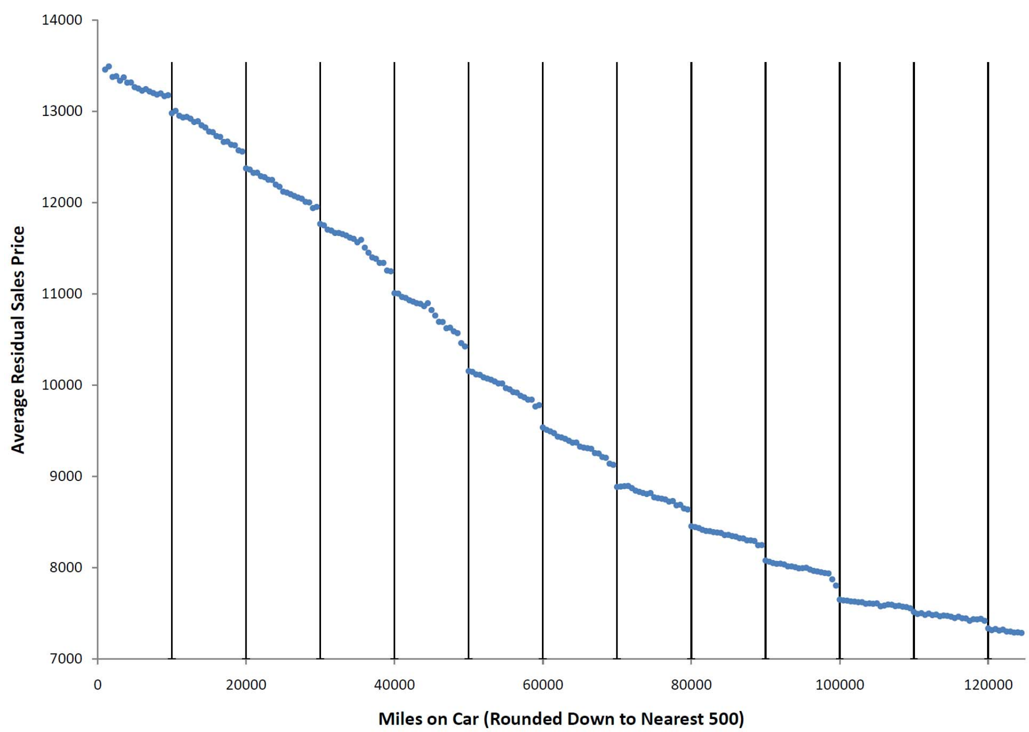


Figure 1.1: Average price of a used car at given mileages

There's a noticeable step-down every time the *first* digit of the mileage changes, because that's more noticeable to people than a change in the second or third digit. The difference between 39000 and 40000 *looks* bigger that the difference between 38000 and 3900, so there's an unexpectedly large drop in price.

On the other hand, this is all a matter of degree. The temperature dropping thirty degrees overnight is unusual, and dropping sixty is even more so; in real-world contexts we can almost always approximate things a little bit. So maybe a better question is: *how well* can we approximate a given function?

Example 1.6. Suppose I want to run a current of 5 amps, to within half of an amp, through a wire with a resistance of 2 ohms. How much voltage do I need to apply?

The formula we need is Ohm's Law, which says that $I = V/R$; in this case, we want to solve $5 = V/2$ so we get $V = 10$.

But what about the error? We want to be correct to within half an ampere, so we really want I between 4.5 and 5.5. There are a couple of ways we could write this down.

One is to say $4.5 < V/2 < 5.5$. Then we can multiply through by 2, and get $9 < V < 11$; so we need between 9 and 11 volts, or V in $(9, 11)$, or $V = 10 \pm 1$. If we want the error in our current to be less than .5, the error in our voltage has to be less than 1.

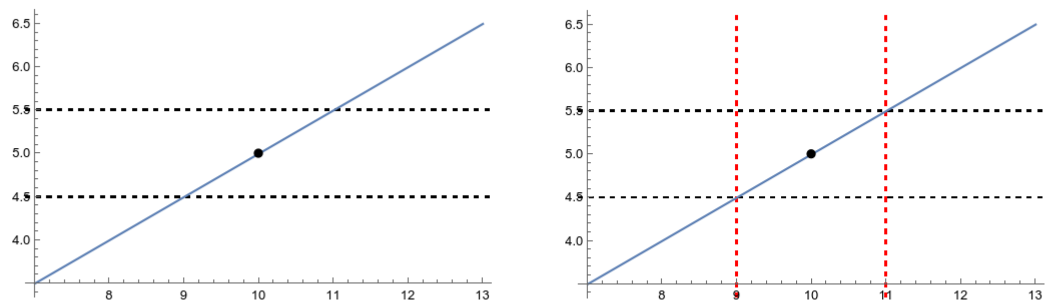


Figure 1.2: Left: We want our output to stay between the black dashed lines. Right: If our input stays between the red dashed lines, we'll hit our error threshold.

Maybe the more sophisticated way to write it is to talk about the *error*. We want the difference between our current I and our target 5 to be less than .5. We could just try writing

$$I - 5 < .5$$

but that's not quite right; it's still bad if I is too small, not just if it's too big. So instead we use the absolute value function:

$$|I - 5| < .5.$$

Since $I = V/2$, we get

$$\begin{aligned} |V/2 - 5| &< .5 \\ -.5 < V/2 - 5 &< .5 \\ 4.5 < V/2 &< 5.5 \\ 9 < V &< 11 \end{aligned}$$

which gives us our original answer.

We sometimes use the Greek letter ε, which is basically the Greek lower-case "e", to represent the error in our output. So in this past example we can say that we had $\varepsilon = .5$.

What if we change the error margin? If we want $\varepsilon = .1$, we do the same setup, but instead we have

$$\begin{aligned} |V/2 - 5| &< \varepsilon = .1 \\ -.1 < V/2 - 5 &< .1 \\ 4.9 < V/2 &< 5.1 \\ 9.8 < V &< 10.2. \end{aligned}$$

So our voltage needs to be $10 \pm .2$, and the allowable error in our input voltage is .2. Just like we use the Greek letter ε for the desired error in our output, we sometimes use the Greek letter δ for the error in input. So we see here that if we want $\varepsilon = .1$ we need $\delta = .2$.

Here's a similar problem but with a more complicated function.

Example 1.7. Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

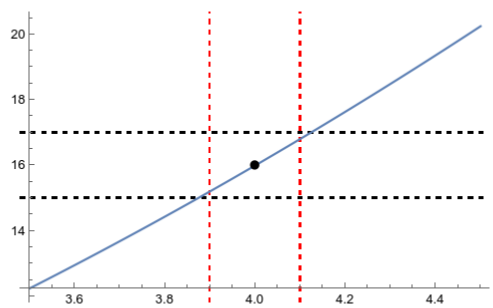
If the side length is s then the area is s^2, so we want $s^2 = 16$ or $s = 4$. (Why can't we have $s = -4$? The number -4 is also a solution to that equation, but it doesn't reflect something physically possible so we can ignore it.)

But what do we need to do to stay within our error margin $\varepsilon = 1$? Obviously, we want s to be between $\sqrt{15}$ and $\sqrt{17}$, but that's not helpful if we don't know what $\sqrt{17}$ is. Instead we're going to estimate again.

We want $|s^2 - 16| < \varepsilon = 1$, and factoring the left hand side gives $|s - 4| \cdot |s + 4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s + 4$ might be bigger than 8, but it's certainly less than 10. Then we just need to solve

$$\begin{aligned} 10|s-4| &< 1 \\ |s-4| &< .1 \\ -.1 < s-4 &< .1 \\ 3.9 < s &< 4.1. \end{aligned}$$

And indeed we can compute $3.9^2 = 15.21$ and $4.1^2 = 16.81$, both of which are within one meter of 16. We can say that we want s in the interval $(3.9, 4.1)$, or that we want s to be $4 \pm .1$. Thus we need $\delta = .1$ for our $\varepsilon = 1$.



Remark 1.8. The answer we got in example 1.7 isn't the optimal or exact bound. $4.12^2 = 16.9744 < 17$ so $\delta = .12$ still works; but this is a reasonably easy-to-compute error margin that we know will work.

In fact, the true bound doesn't have to be symmetric. If we plug in more numbers, we see that $3.875^2 = 15.0156$, which is within one square meter of 16. But $4.125^2 = 17.0156$, which gives us an error bigger than 1. We generally want to pick a δ that works in *both* directions, because we don't know whether we're going to overshoot or undershoot our targts.

Example 1.9 (recitation)**.** How does this change if we want an area of 25 square meters instead? We know we need $s \approx 5$. We want $|s^2 - 25| < 1$, which gives $|s-5| \cdot |s+5| < 1$. And here we *cannot* assume that $|s+5| < 10$, because s might be a little bigger than 10. But we can assume it's smaller than 11, so we want $s = 5 \pm 1/11$, or s in $(54/11, 56/11) \approx (4.909, 5.091)$.

And again we have $(54/11)^2 \approx 24.0992$ and $(56/11)^2 \approx 25.9174$ so $\delta = 1/11$ is in fact an acceptable amount of error in the input.

(And similarly to the last example: $4.9^2 = 24.01$ keeps us within our error margin; but $5.1^2 = 26.01$ does not. So if we want to be safe not matter which direction our error is, $\delta = .1$ is too big.)

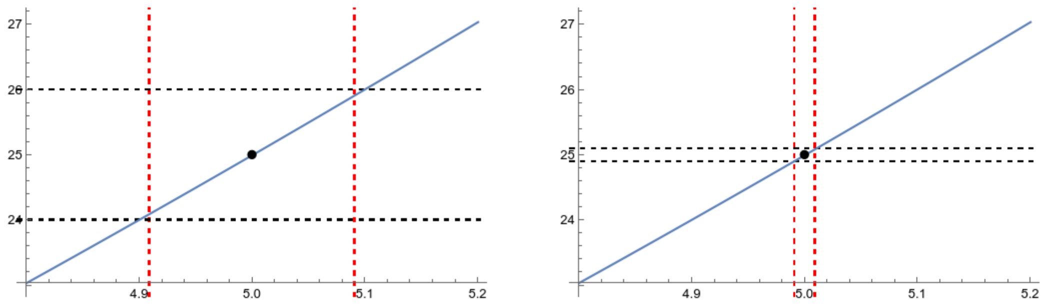


Figure 1.3: Left: $\varepsilon = 1$. Right: $\varepsilon = .1$.

Example 1.10 (recitation)**.** What if we want a smaller error, say $\varepsilon = .1$ square meters? Then we run the same calculations, and we want $|s-5| \cdot |s+5| < .1$, and thus $|s-5| < .1/11 = \frac{1}{110}$. So we want $s = 5 \pm \frac{1}{110}$, and $\delta = 1/110$.

You might be noticing at this point that if we change ε but leave everything else the same, we can generally find the new δ pretty quickly by reusing our old work.

Example 1.11 (recitation)**.** Now let's return to the square root function we started the section with. We know that $\sqrt{4} = 2$. If we want to allow an output error of $\varepsilon = 1$, how large of an input error can we tolerate? We know that $\sqrt{1} = 1$ and $\sqrt{9} = 3$, so we need to stay in the interval $(1, 9)$. Since we're aiming for 4, this means we can get away with undershooting by 3, or overshooting by 5. So we have to take $\delta = 3$.

What if we want $\varepsilon = .5$? Well, that means we want $1.5 < \sqrt{x} < 2.5$; squaring the equation gives $2.25 < x < 6.25$. Thus we can undershoot by 1.75 or overshoot by 2.25; we have to take $\delta = 1.75$.

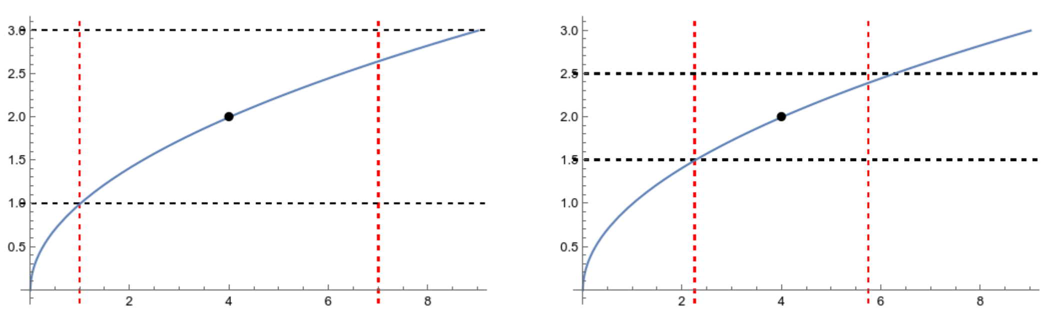


Figure 1.4: Left: $\varepsilon = 1$. Right: $\varepsilon = .5$.

Here's an example that's more complicated still—in fact, so complicated we're going to cheat:

Example 1.12 (Bonus)**.** We want to dilute an acid by a factor of ten, so that we produce a solution that's ten percent acid and 90 percent water. If we have one liter of acid, how much water do we need to add to get within one percentage point of our desired concentration?

Ideally we'd add nine liters of water, to get exactly 10% acid. But what's the error margin there, to land between 9% and 11%?

Our concentration will be $f(x) = \frac{1}{1+x}$ where x is the number of liters of water we add. We could, if we wanted, do some algebra:

$$\begin{aligned} \left| \frac{1}{1+x} - \frac{1}{10} \right| &< .01 \\ \left| \frac{10-(1+x)}{10(1+x)} \right| &< .01 \\ |9-x| &< .1(1+x) \end{aligned}$$

and since $x \approx 9$ we can assume $1+x$ is close to 10; say it's between 9 and 11. Then we need

$$\begin{aligned} |9-x| &< .9 \\ -.9 < 9-x &< .9 \\ -9.9 < -x &< -8.1 \\ 8.1 < x &< 9.9. \end{aligned}$$

And we can check that if $x = 9.9$ we get a concentration of $\dfrac{1}{1+9.9} \approx .09174$ and if $x = 8.1$ we get a concentration of $\dfrac{1}{1+8.1} \approx .10989$.

But honestly, that algebra is kind of nasty. And this function isn't even that complicated, right? It gets worse. So if we really need to know the answer, we can be lazy.

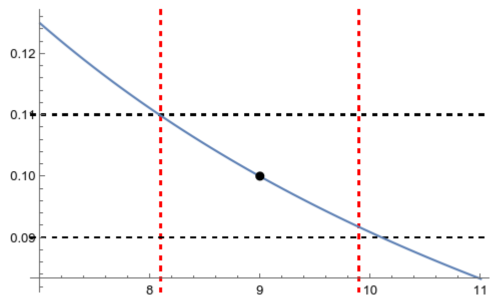
$f(9) = .1$	$f(8.9) \approx .10101$	$f(8.8) \approx .10204$
$f(8.7) \approx .10309$	$f(8.6) \approx .10417$	$f(8.5) \approx .10526$
$f(8.4) \approx .10538$	$f(8.3) \approx .10753$	$f(8.2) \approx .10870$
$f(8.1) \approx .10989$	$f(8.0) \approx .11111$	

And in the other direction

$f(9.1) \approx .09901$	$f(9.2) \approx .09804$	$f(9.3) \approx .09709$
$f(9.4) \approx .09615$	$f(9.5) \approx .09524$	$f(9.6) \approx .09434$
$f(9.7) \approx .09346$	$f(9.8) \approx .09259$	$f(9.9) \approx .09174$
$f(10.0) \approx .09091$	$f(10.1) \approx .09009$	$f(10.2) \approx .08929.$

so we can get away with anywhere between 8.1 and 10.1 liters of water.

Then we'd have to pick $\delta = .9$, if we want to be safe in both directions.



Example 1.13. The *Heaviside function* is used to describe the behavior of a lightswitch. Before you flip the switch, no current is flowing through the circuit; when you flip the switch, current instantly jumps to 1 amp.

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

Now we're going to ask some kind-of-dumb questions.

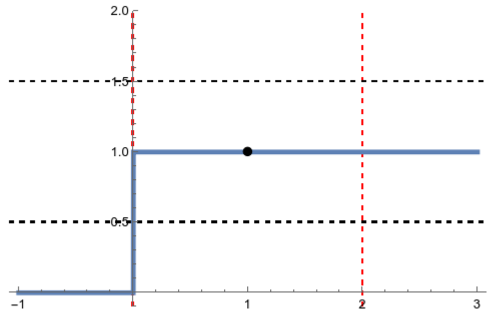
Suppose we want a current of 1 amp, plus or minus 2 amps (so $\varepsilon = 2$). What range of inputs will land us within the target range? Here we want a current between -1 amp and 3 amps, so literally any input will work.

Now let's reduce ε. What if we want a current of $1 \pm .5$? That means we want a current between .5 and 1.5. Any value $t \geq 0$ will work. But no negative value will work.

If we reduce ε still further, nothing changes. If we want a current of $1 \pm .000001$, we can still take any $t \geq 0$ and no $t < 0$.

But with just those questions, we can't really pick a δ, right? Because I didn't give you a center point. So now suppose we can take $t = 1 \pm \delta$, and we want $H(t) = 1 \pm .5$. What's the biggest δ we can possibly pick? Well, we need $t > 0$ to get $H(t) > .5$, so the biggest δ can be is 1, giving us t in $(0, 2)$.

On the other hand, what if we want $t = 0 \pm \delta$? From the definition of H, we know that $H(0) = 1$ exactly. But how far away from 0 can we get, and still land $H(t)$ in the $1 \pm .5$ target zone we're aiming for? A little thought, and maybe sketching some pictures, should convince you that we *cannot* do this. If we're aiming for $t = 0$ we cannot tolerate any error at all.



1.3 Limits

In the last subsection we talked about error margins: we have some function, and some amount of error we can accept in the output, and then we ask how much error we're allowed to make in the input. And we saw that if we picked a smaller error margin ε, we would get a smaller input tolerance δ.

But we also saw that in many of the functions we looked at, we had no problem hitting a smaller ε margin as long as we could make δ smaller at the same time.

Example 1.14. In example 1.6 we had current as a function of voltage: we got the formula $I = V/2$. And we worked out that if we want an output of $5 \pm 1/2$, we need an input of 10 ± 1.

But algebraically, we said that we want $|V/2 - 5| < \varepsilon$, and multiplying through by 2 we get $|V - 10| < 2\varepsilon$, which implies that $V = 10 \pm 2\varepsilon$ will be precise enough to serve our purposes. But we can do that whole argument without saying what ε is in the first place!

If $\varepsilon = .5$ then we get $\delta = 1$. But if $\varepsilon = .1$ we get $\delta = .2$. And if $\varepsilon = .000001$ then $\delta = .000002$. No matter *how* precise we need our output, if we make our input precise enough we can match it.

Example 1.15. In example 1.7 we were looking for the area of a square as a function of its side length. We wanted an area of 16 square meters, so we found that our error would be

$|s^2 - 16|$; after doing some tricky algebra, we saw that the condition we wanted was

$$\begin{aligned} 10|s-4| &< \varepsilon \\ |s-4| &< \varepsilon/10. \end{aligned}$$

At the time we said we wanted an error margin of 1, so we got a δ of 1/10 and wanted $s = 4 \pm 1/10$. But we can see that if we wanted to hit $\varepsilon = 1/100$ we'd just need to take $\delta = 1/1000$; we can get the area as precise as we want, as long as we can control the side length precisely.

When we have a value that we can approximate as precisely as we want, we call that a *limit*. And we can turn all the discussion we've done so far into a technical and scary-looking definition:

Definition 1.16. Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x \to a} f(x) = L,$$

if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

There's a lot of notation here, but this is just putting all the work we've been doing together. We have some target output L, and some target input a. And we have a margin of acceptable error around L, given by the number ε: we want $f(x) = L \pm \varepsilon$, or in other words we want our error $|f(x) - L| < \varepsilon$. If we can *always* hit that margin of error just by making δ small, then we say there's a limit.

Example 1.17. If $f(x) = x/2$, prove $\lim_{x \to 10} f(x) = 5$.

This is exactly what we saw in our current and voltage argument in examples 1.6 and 1.14.

Let $\varepsilon > 0$, and set $\delta = 2\varepsilon$. Then if $0 < |x - 10| < \delta$, we have

$$\begin{aligned} |f(x) - 5| = |x/2 - 5| &= \frac{1}{2}|x - 10| \\ &< \frac{1}{2}\delta = \frac{1}{2}(2\varepsilon) = \varepsilon. \end{aligned}$$

Thus $|f(x) - 5| < \varepsilon$, and so $\lim_{x \to 10} f(x) = 5$.

Example 1.18. If $f(x) = 3x$ then prove $\lim_{x\to 1} f(x) = 3$.

Let $\varepsilon > 0$ and set $\delta = \underline{\varepsilon/3}$. Then if $0 < |x-1| < \delta$ then

$$|f(x) - 3| = |3x - 3| = 3|x-1| < 3\delta = \varepsilon.$$

Example 1.19. If $f(x) = x^2$ then prove $\lim_{x\to 0} f(x) = 0$.

Let $\varepsilon > 0$ and set $\delta = \underline{\sqrt{\varepsilon}}$. Then if $|x - 0| < \delta$, then

$$|f(x) - 0| = |x^2| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

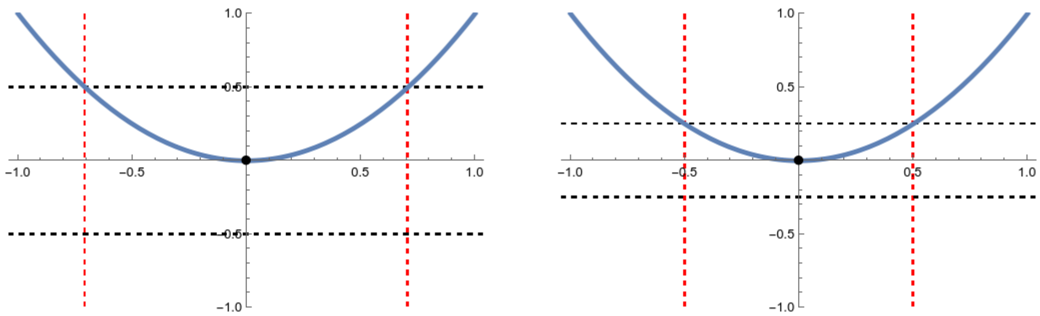


Figure 1.5: Left: $\varepsilon = 1/2$ and $\delta = 1/\sqrt{2}$. Right: $\varepsilon = 1/4$ and $\delta = 1/2$.

What does it look like when we *don't* have a limit?

Example 1.20. If $H(t)$ is the Heaviside function from example 1.13, can we compute $\lim_{t\to 0} H(t)$?

From the work we've done already, we've already seen that although $H(0) = 1$, the limit $\lim_{t\to 0} H(t) \neq 1$. If we take $\varepsilon = .5$, we're asking all our inputs to lie between .5 and 1.5. But as long as our input is $0 \pm \delta$, then no matter how small we make δ we're still allowing negative numbers as inputs, so our outputs will include 0.

It's a little harder and more annoying to really rigorously prove that there's *no* limit, because we somehow need to talk about every possible limit at once. But just like $0 \pm \delta$ always includes negative numbers, it also always includes positive numbers, so 1 will always be an output we get. But if we ask for any number that *isn't* 1, then if ε is really small then 1 won't be between $L - \varepsilon$ and $L +$
varepsilon. We can't make epsilon as small as we like, so we don't have a limit.

However, the Heaviside function clearly behaves well if look only at one side or the other of it. And just as we could talk about continuity to one side or the other, we can talk about *one-sided limits*.

Definition 1.21. Suppose a is a real number, and f is a function which is defined for all $x < a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the left is L," and we write

$$\lim_{x \to a^-} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but less than) a.

Suppose a is a real number, and f is a function which is defined for all $x > a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the right is L," and we write

$$\lim_{x \to a^+} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but greater than) a.

Under this definition, we see that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$.

The most subtle aspect of the definition is that we don't actually care what happens when we get our input exactly right. We're only asking if we can *approximate* the desired output, not if we can get it exactly. This makes limits incredibly useful for talking about functions that are undefined at individual points.

Example 1.22. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x \to 1} f(x) = 2$.

This is harder to see at first, until we recall or notice that this function is mostly the same as $x + 1$.

Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then if $0 < |x - 1| < \delta$, we have

$$\begin{aligned} |f(x) - 2| &= \left| \frac{x^2 - 1}{x - 1} - 2 \right| \\ &= |x + 1 - 2| && \text{since } x \neq 1 \\ &= |x - 1| < \delta = \varepsilon. \end{aligned}$$

This formal definition of limits is useful for a lot of technical work, and also for when we're trying to control the error in our output when we don't have precise control of our inputs. But it's often useful to think of it a bit more informally.

Definition 1.23 (informal)**.** Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

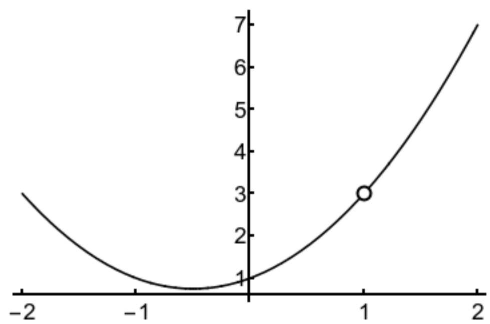
if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a, 0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Notice that we're trying to use knowing $f(x)$ to tell us what happens near a. So we specifically ignore the value of $f(a)$ even if we already know it.

Example 1.24. Let's consider the function $f(x) = \frac{x^3-1}{x-1}$. We can see the graph below. Notice that the function isn't defined at $a = 1$, so $f(1)$ is meaningless and we can't compute it.

But f is defined for all x near 1, so we can compute the limit. Looking at the graph and estimating suggests that when x gets close to 1, then $f(x)$ gets close to 3, and so we can say that $\lim_{x\to 1} f(x) = 3$.



Informally, we can estimate limits by eyeballing the graph. Formally, we can justify this limit claim by writing out a full $\varepsilon - \delta$ proof, but that's tedious and annoying. We'd like a middle route, which allows us to compute limits algebraically without having to set up a full proof; and we can do that using two core principles. The first is what I call the Almost Identical Functions property.

Lemma 1.25 (Almost Identical Functions)**.** *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.*

This tells us that two functions have the same limit at a if they have the same values near a. This makes sense, because the limit only depends on the values near a.

How does this help us? Ideally, we take a complicated function and replace it with a simpler function.

Example 1.26. Above, we looked at the function $f(x) = \frac{x^3-1}{x-1}$. You may know that we can factor the numerator; thus we in fact have $f(x) = \frac{(x-1)(x^2+x+1)}{x-1}$.

At this point you probably want to cancel the $x-1$ term on the top and the bottom. But in fact that would change the function! For $f(1)$ isn't defined. But the function $g(x) = x^2+x+1$ is perfectly well-defined at $a = 1$. Thus $f(1) \neq g(1)$, and so f and g can't be the same function.

However, they do give the same value if we plug in any number other than 1. If $y \neq 1$ then $y - 1 \neq 0$, so we have

$$f(y) = \frac{(y-1)(y^2+y+1)}{y-1} = y^2 + y + 1 = g(y).$$

Thus f and g aren't the same, but they are *almost* the same. So lemma 1.62 tells us that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$.

However, this doesn't fully answer our question. We've replaced a complicated function $f(x) = \frac{x^3-1}{x-1}$ with a simpler function $g(x) = x^2 + x + 1$, but we still haven't figured out what to do with that function. (We could still write a $\varepsilon - \delta$ proof, and we still don't want to.)

But $g(x) = x^2 + x + 1$ is a straightforward function. We can just plug numbers into it and get some sort of answer. And we would *hope* that $\lim_{x\to 1} g(x)$, which is an attempt to approximate $g(1)$, will actually give us the same answer as $g(1)$. But will it?

1.4 Continuity and Computing Limits

We began class with the observation that approximation works really well for some functions (like $\sqrt{x}$), and much less well for other functions (like the high temperature as a function of the day). We like working with the well-approximatable functions, so we give them a name: we call them *continuous*.

Informally, we say a function f is continuous at a number a if $f(x)$ is a good approximation of $f(a)$ as long as x is close to a. Formally:

Definition 1.27. We say that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

The definition of continuity says that $\lim_{x\to a} f(x) = f(a)$. This secretly actually requires three distinct things to happen:

(a) The function is defined at a; that is, a is in the domain of f.

(b) $\lim_{x\to a} f(x)$ exists.

(c) The two numbers are the same.

There are a few different ways for a function to be discontinuous at a point:

(a) A function f has a *removable discontinuity* at a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

(b) A function f has a *jump discontinuity* at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are unequal.

(c) A function f has a *infinite discontinuity* if f takes on aribtrarily large or small values near a. We'll talk about this more soon.

(d) It's also possible for the one-sided limits to not exist, but this doesn't have a special name. We'll see this with $\sin(1/x)$ when we study trigonometric functions in section 1.5. In this class, I'll just call a function like this *really bad*. But we'll mostly avoid talking about them.

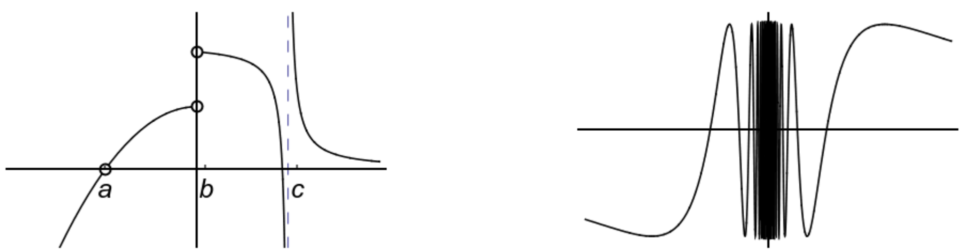


Figure 1.6: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: a "bad" discontinuity in the function $\sin(1/x)$.

Some functions get even worse than that. My two favorite discontinuous functions are:

$$T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \qquad \chi(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Example 1.28. The Heaviside function of example 1.13 is not continuous, since there's a jump at 0.

It is continuous from the right at 0, since $\lim_{x \to 0^+} H(x) = 1 = H(0)$. This function is not continuous from the left, since $\lim_{x \to 0^-} H(x) = 0 \neq H(0)$.

In fact, in some sense "most functions" aren't at all continuous. If you found away to choose $f(x)$ completely at random for each real number x, you would get a spectacularly discontinuous function. But you would never actually be able to describe it sensibly.

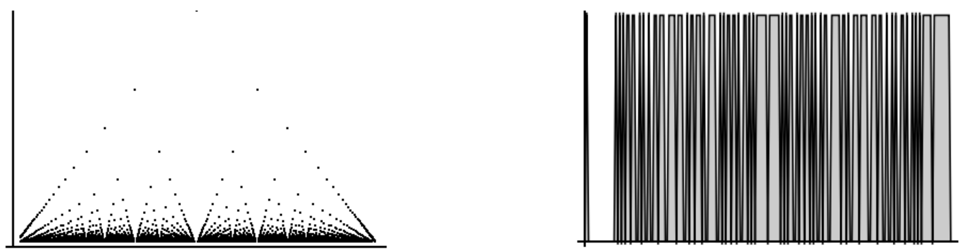


Figure 1.7: Left: $T(x)$ is really discontinuous. Right: $\chi(x)$ is really really discontinuous

In contrast, if you can draw a function reasonably, it pretty much has to be (mostly) continuous. A common informal definition is that a continuous function is one whose we can draw without lifting our pencil from the paper. Once we make this precise, this is another way to think about continuous functions. And we make it precise via the Intermediate Value Theorem

Theorem 1.29 (Intermediate Value Theorem)**.** *Suppose f is continuous (and defined!) on the closed interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$. Then there is a c in (a, b) with $f(c) = y$.*

Example 1.30. Suppose $f(x)$ is a continuous function with $f(0) = 3, f(2) = 7$. Then by the Intermediate Value Theorem there is a number c in $(0, 2)$ with $f(c) = 5$.

Example 1.31. Let $g(x) = x^3 - x + 1$. Use the Intermediate Value Theorem to show that there is a number c such that $g(c) = 4$.

To use the intermediate value theorem, we need to check that our function is continuous, and then find one input whose output is less than 4, and another whose output is greater than 4. g is a polynomial and thus continuous. Testing a few values, we see $g(0) = 1, g(1) = 1, g(2) = 7$. Since $g(1) = 1 < 4 < 7 = g(2)$, by the Intermediate Value Theorem ther is a c in $(1, 2)$ with $g(c) = 4$.

Example 1.32. Show that there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

We know that sin is a continuous function, and that $\sin(0) = 0$ and $\sin(\pi/2) = 1$. Since $0 < 1/3 < 1$, by the Intermediate Value Theorem there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

Remark 1.33. The converse of this theorem is not true. It is possible to have a function that satisfies the conclusions of the Intermediate Value Theorem, but is not continuous; these functions are called Darboux Functions.

For example, let $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then f satisfies the conclusion of the intermediate value theorem: it's continuous except at zero, so the theorem works on any interval that doesn't contain zero. Any interval containing zero contains every value in $[-1, 1]$, so if $a < 0 < b$ and y is between $f(a)$ and $f(b)$, then $-1 \leq y \leq 1$ and so there is a c in (a, b) such that $f(c) = y$. Thus f is Darboux.

Historically, the main reason we didn't take this as the definition of continuous, instead of the limit definition that we actually use, is that we didn't want to treat functions like this as "continuous".

1.4.1 Limits of Continuous Funtions

Most of the functions that we can easily describe, or draw graphs of, are continuous most of the time.

Fact 1.34. *Any reasonable function given by a reasonable single formula is continuous at any number for which it is defined.*

In particular, any function composed of algebraic operations, polynomials, exponents, and trigonometric functions is continuous at every number in its domain.

If a function is continuous at every number in its domain, we just say that it is continuous. Note, importantly, that a continuous function doesn't have to be continuous at every real number.

Example 1.35. The function

$$f(x) = \frac{x^3 - 5x + 1}{(x-1)(x-2)(x-3)}$$

is "reasonable", so it is continuous. This means that it is continuous exactly on its domain, which is $\{x : x \neq 1, 2, 3\}$.

Example 1.36. Where is $\sqrt{1 + x^3}$ continuous?

Answer: Root functions are continuous on their domains. $1 + x^3 \geq 0$ when $x \geq -1$ so the function is continuous on its domain, $[-1, +\infty)$.

Remark 1.37. Sometimes we might also talk about functions that are "continuous from the right" at a. This means that $f(a)$ is a good approximation of $f(x)$ if x is close to a and also bigger than—and thus to the right of—a.

And if we know a function is continuous, it is *very easy* to compute a limit.

Example 1.38. (a) The function $f(x) = 3x$ is continuous at 1, so $\lim_{x\to 1} f(x) = f(1) = 3$.

(b) The function $f(x) = x^2$ is continuous at 0, so $\lim_{x\to 0} f(x) = f(0) = 0$.

(c) The function $f(x) = \frac{x^2-1}{x-1}$ is definitely not continuous at 1, because it's not defined there. But we can use almost identical functions:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2.$$

This might seem like the whole limit thing has no point; most functions are continuous, and if a function is continuous then we can compute limits just by plugging in values. But there is one very important type of question where limits are doing real work.

Example 1.39. What is $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$?

We use a trick called multiplication by the conjugate, which takes advantage of the fact that $(a+b)(a-b) = a^2 - b^2$. This trick is used *very often* so you should get comfortable with it.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+x}-9}{x} \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} \\
&= \lim_{x\to 0} \frac{(9+x)-3}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+3)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\lim_{x\to 0} \sqrt{9+x}+3} = \frac{1}{6}.
\end{aligned}$$

We can also use these continuity arguments to calculate one-sided limits.

Example 1.40. What is $\lim_{x\to 1^-} f(x)$ if $f(x) = \begin{cases} x^2+2 & x > 1 \\ x-3 & x < 1 \end{cases}$?

Answer: -2.

Remark 1.41. At a jump discontinuity, a function will often be continuous from one side but not the other. This is not necessarily the case, though: consider the function

$$f(x) = \begin{cases} 2 & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

Limits exist from the right and the left, but the function is not continuous from either side.

1.4.2 Function Extensions

Recall we like continuous functions because we can use their values at one point to approximate the values they should have at nearby points. And we observed that this is really unhelpful at any point where the function isn't defined. So if we have a function that's continuous everywhere it's defined, we'd like to replace it with a function that is continuous—and defined—everywhere.

Definition 1.42. We say that g is an *extension* of f if the domain of g contains the domain of f, and $g(x) = f(x)$ whenever $f(x)$ is defined.

In general, we can only extend a function to be continuous at all real numbers if the only discontinuities were removable. This is why we call discontinuities like that "removable".

Example 1.43. Let $f(x) = \frac{x^2-1}{x-1}$. Can we define a function g that agrees with f on its domain, and is continuous at all reals?

f is continuous everywhere on its domain, and is undefined at $x = 1$. We can see that $g(x) = x + 1$ will give the same value as f everywhere on f's domain, and it is continuous since it is a polynomial. Thus g is a continuous extension of f to all reals.

Alternatively, we could compute that $\lim_{x\to 1} f(x) = 2$. Then we define

$$h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1. \end{cases}$$

The function $h(x)$ is defined at all reals, and since it is continuous at 1 by our computation, it is continuous everywhere. It also must extend f since it is just defined to be f everywhere in the domain of f. So h is a continuous extension of f to all reals.

Importantly, g and h are actually the same function, since they give the same output for every input. There is at most one continuous extension of any given function; but there are multiple ways to describe that extension.

Example 1.44. The function $f(x) = 1/x$ is continuous on its domain, but we cannot extend it to a function continuous at all reals, because the limit at 0 does not exist.

Example 1.45. Let $f(x) = \frac{x^2-4x+3}{x-3}$. Can we extend f to a function continuous at all reals?

Answer: f is continuous at all reals except $x = 3$. But the function $g(x) = x - 1$ is the same everywhere except for 3, and is continuous at 3.

Example 1.46. Let

$$g(x) = \begin{cases} x^2 + 1 & x > 2 \\ 9 - 2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

Answer: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 9 - 2x = 5$, and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 1 = 5$, so the limit at 2 exists. Thus we can extend g to

$$g_f(x) = \begin{cases} x^2 + 1 & x \geq 2 \\ 9 - 2x & x \leq 2 \end{cases}$$

which is continuous at all reals.

Example 1.47. What is $\lim_{x \to 1^-} f(x)$ if $f(x) = \begin{cases} x^2 + 2 & x > 1 \\ x - 3 & x < 1 \end{cases}$?

Answer: -2.

1.5 Trigonometry and the Squeeze Theorem

We now want to look at limits of trigonometric functions. Fortunately, they behave *mostly* how we want them to.

Proposition 1.48. *If a is a real number, then* $\lim_{x \to a} \sin(x) = \sin(a)$ *and* $\lim_{x \to a} \cos(x) = \cos(a)$.

In fact, since trigonometric functions are just ways of combining sine and cosine, essentially all trigonometric functions behave this way where they are defined.

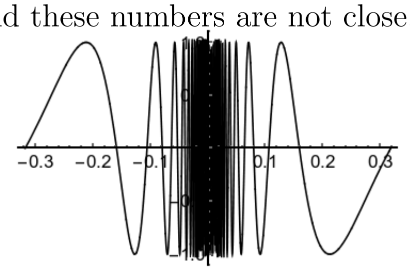
Example 1.49. $\lim_{x \to \pi} \cos(x) = -1$.

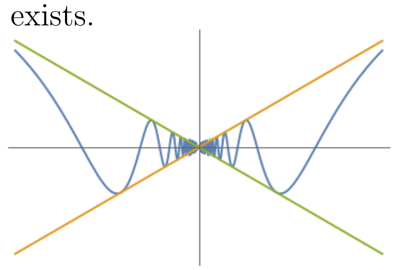
$\lim_{x \to \pi} \tan(x) = 0$.

But where the functions are not defined, sometimes very odd things can happen. We've seen a graph of $\sin(1/x)$ before, in section 1.4. We said that the function wasn't continuous at 0. In fact, no limit exists there.

Suppose a limit does exist at zero; specifically, let's suppose that $\lim_{x \to 0} \sin(1/x) = L$. Then if x is close to 0, it must be the case that $\sin(1/x)$ is close to L.

But however close we want x to be to 0, we can find a $x_1 = \frac{1}{(2n+1/2)\pi}$, and then $\sin(1/x_1) = \sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$. But we can also find an $x_2 = \frac{1}{(2n+3/2)\pi}$ so that $\sin(1/x_2) = \sin(2n\pi + 3\pi/2) = \sin(3\pi/2) = -1$. So L must be really close to 1 and really close to -1, and these numbers are not close. So no limit exists.





Left: graph of $\sin(1/x)$, Right: graph of $x\sin(1/x)$

In contrast, from the graph it appears that $\lim_{x\to 0} x\sin(1/x)$ does exist. We can't possibly prove this by replacing $x\sin(1/x)$ with an almost identical function and plugging values in: the function is gross and complicated, and any almost identical function will also be gross and complicated.

But we can easily see that $\lim_{x\to 0} x = 0$. This doesn't mean that $\lim_{x\to 0} xf(x) = 0$ for any $f(x)$; if $f(x)$ gets really big then it can "cancel out" the x term getting very small. (A good example of this is $\lim_{x\to 0} x\frac{1}{x}$, which is of course 1).

But if we can prove that the second term, which in this case is $\sin(1/x)$, does *not* get really big, then the entire limit will have to go to zero. We make this intuition precise with the following important theorem:

Theorem 1.50 (Squeeze Theorem). *If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

To use the Squeeze Theorem, we need to do two things:

(a) Find a lower bound and an upper bound for the function we're interested in; and

(b) show that their limits are equal.

We usually do this by factoring the function we care about into two pieces, where one goes to zero and the other is bounded, and thus doesn't get infinitely big.

In this case, we know that $-1 \leq \sin(a) \leq 1$ for any real number a, so in particular $-1 \leq \sin(1/x) \leq 1$. We "want" to multiply both sides of the equation by x to get $-x \leq x\sin(1/x) \leq x$, but this actually doesn't quite work! If x is negative this is in fact backwards, as we can see on the graph below:

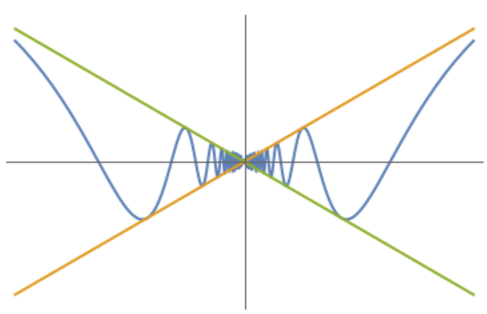
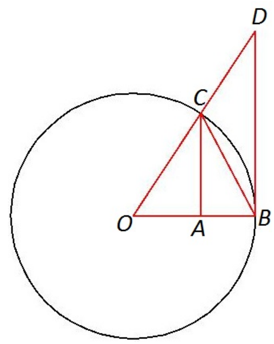


Figure 1.8: $x\sin(1/x)$ graphed with x and $-x$. Notice how they cross over in the middle.

There is one more important limit involving sin:

Proposition 1.51 (Small Angle Approximation)**.**

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



Proof. The proof of this isn't anything I'd expect you to be able to recreate, but it's interesting and fun and points to useful facts about trigonometry.

We'll assume θ is small and positive; this all still works if θ is small and negative, with different signs. We'll start by drawing the unit circle, and then including some extra lines to construct triangles.

Let θ be the measure of angle AOC in our diagram. Observe that $\sin(\theta)$ is precisely the length of the line segment AC by definition, and the line segment OC has length one because this is a unit circle. So triangle BOC has area $\frac{\sin(\theta)}{2}$.

Now we want to find the area of the wedge BOC. This will be larger than the triangle, since it includes the whole triangle plus an extra sliver. We know that the area of the entire circle is π. But the wedge is $\frac{\theta}{2\pi}$ of the circle, since the circle measures 2π radians; so the area of the wedge is $\pi \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$.

Since the triangle is contained in the wedge, we have $\frac{\sin(\theta)}{2} \leq \frac{\theta}{2}$ and thus $\frac{\sin(\theta)}{\theta} \leq 1$. That gives us one inequality involving $\frac{\sin(\theta)}{\theta}$, and now we need another.

We want to find the area of the large triangle BOD. But we know that $\tan(\theta)$ is the ratio of the opposite side of this triangle to the adjacent, so $\tan(\theta) = \frac{\text{BD}}{\text{OB}}$. Since OB has length 1, the length of DB is $\tan(\theta)$, and the area of this triangle is $\frac{\tan(\theta)}{2}$.

Since the wedge BOC is contained in this triangle BOD, we know that it has a smaller area, so $\frac{\theta}{2} \leq \frac{\tan(\theta)}{2}$. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, after rearranging we see that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta}$.

Thus $\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$. But $\lim_{\theta \to 0} \cos \theta = 1$, and $\lim_{\theta \to 0} 1 = 1$, so by the squeeze theorem we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

□

Remark 1.52. This means that the function

$$f(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is a continuous extension of $\sin(x)/x$ to all reals.

It also tells us that when x is a small number, $\sin(x) \approx x$. This is a fact we'll return to a couple times throughout the course.

The small angle approximation is useful on its own, but we can also use it as a new limiting rule, which allows us to compute limits of functions that mix algebra and trigonometry.

Example 1.53. Suppose we want to compute $\lim_{x\to 0} \frac{\sin(2x)}{2x}$. If we take $\theta = 2x$, then this is $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

Example 1.54. What is $\lim_{x\to 3} \frac{\sin(x-3)}{x-3}$?

This is a small angle approximation again, since we can take $\theta = x - 3$, which is approaching zero. Thus the limit is 1.

In general, we can use the small angle approximation plus a bit of algebra to work out all sorts of these computations.

Example 1.55. What is $\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x}$?

We can write

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x} &= \lim_{x\to 0} \frac{\left(\sin(4x)/4x \cdot 4x\right)\left(\sin(6x)/6x \cdot 6x\right)}{\left(\sin(2x)/2x \cdot 2x\right) \cdot x} \\
&= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 6x}{6x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{24x^2}{2x^2} \\
&= 1 \cdot 1 \cdot 1 \cdot 12 = 12.
\end{aligned}$$

Here we are simply pairing off the $\sin(\theta)$'s with θs, for $\theta = 4x, 6x, 2x$, and then collecting the remainder into the last term.

Example 1.56 (Bonus)**.** What is $\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)}$?

When we see a tangent in a problem, it is often helpful to rewrite it in terms of sin and cos. We can then collect terms:

$$\begin{aligned}
\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)} &= \lim_{x\to 0} \frac{x\sin(2x)}{\sin(3x)/\cos(3x)} \\
&= \lim_{x\to 0} \frac{3x}{\sin(3x)} \cdot \frac{\sin(2x)\cos(3x)}{3} = 1 \cdot \frac{0}{3} = 0.
\end{aligned}$$

Example 1.57 (Recitation)**.** What is $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$?

We have a sin(0) on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} = \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

Example 1.58. What is $\lim_{x\to 0} \frac{1-\cos x}{x}$?

We can see that the limits of the top and the bottom are both 0, so this is an indeterminate form. We can't use the small angle approximation directly because there is no sin here at all. But we can fix that by multiplying by the conjugate.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$
$$= \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0.$$

1.6 Infinite Limits

A few times in the past couple sections we've talked about vertical asymptotes, or functions going to infinity. In this section we want to look at exactly what that means. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 1.59. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

1.6.1 Limits To Infinity

Definition 1.60. We write

$$\lim_{x\to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x\to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x\to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Remark 1.61. Important note: If the limit of a function is infinity, the limit *does not exist.* This is utterly terrible English but I didn't make it up so I can't fix it. All the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Lemma 1.62. *Let $f(x), g(x)$ be functions defined near a, such that $\lim_{x \to a} f(x) = c \neq 0$ and $\lim_{x \to a} g(x) = 0$. Then*

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \pm\infty.$$

Further, assuming $c > 0$ then the limit is $+\infty$ if and only if $g(x) \geq 0$ near a, and the limit is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$ then the opposite is true.

Remark 1.63. If the limit of the numerator is zero, then this lemma is *not useful.* That is one of the "indeterminate forms" which requires more analysis before we can compute the limit completely.

Example 1.64. What is $\lim_{x \to 3} \frac{-1}{\sqrt{x-3}}$? We see the top goes to 1 and the bottom goes to 0, so the limit is $\pm\infty$. Since the denominator is always positive and the numerator is negative, the limit is $-\infty$.

We have to be careful while working these problems: the limit laws that work for finite limits don't always work here, since the limit laws assume that the limits exist, and these do not. In particular, adding and subtracting infinity *does not work.* In recitation we'll experiment a bit with the ways that normal limit laws fail. Instead, we need to arrange the function into a form where we can use lemma 1.62.

It's helpful organize our thinking about these situations in terms of the "indeterminate forms", which are: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty \pm \infty, 1^\infty, \infty^0$. Notice that none of these are actual numbers, and they can never be the correct answer to pretty much any question.

More importantly, indeterminate forms don't even tell us what the answer should be; if plugging in gives you one of those forms, the true limit could potentially be pretty much anything. We have to do more work to get our functional expression into a determinate form. As a general rule, we use algebraic manipulations to get a form of $\frac{0}{0}$, then factor out and cancel $(x - a)$ until either the numerator or the denominator is no longer 0.

In contrast, neither $\frac{0}{1}$ nor $\frac{1}{0}$ is an indeterminate form. $\frac{0}{1}$ is just a number, equal to 0. $\frac{1}{0}$ is not a number and is never the correct answer to a question; if you ever tell me anything is equal to $\frac{1}{0}$, in basically any context, you're not getting full credit. But it's also not indeterminate. The whole point of 1.62 is that this must be $\pm\infty$.

Similarly, $\frac{0}{\infty}$ and $\frac{\infty}{0}$ are also not numbers but not indeterminate. The first suggests the limit is 0; the second suggests the limit is $\pm\infty$.

Remark 1.65. The form $\infty \cdot \infty$ mostly works fine, and gives you another ∞ whose sign depends on the signs of the ∞s you're multiplying. But again, $\infty \cdot \infty$ is never the actual answer to any actual question.

Example 1.66. What is $\lim_{x \to 3^+} \frac{1}{(x-3)^3}$? This is *not* an indeterminate form. the limit of the top is 1, and the limit of the bottom is 0, so the limit is $\pm\infty$. But when $x > 3$ the denominator is ≥ 0, so the limit is in fact $+\infty$. Conversely $\lim_{x \to 3^-} \frac{1}{(x-3)^3} = -\infty$ since when $x < 3$ we have $(x-3)^3 \leq 0$.

$\lim_{x \to -1^+} \frac{1}{(x+1)^4} = +\infty$. And $\lim_{x \to -1^-} \frac{1}{(x+1)^4} = +\infty$. Thus $\lim_{x \to -1} \frac{1}{(x+1)^4} = +\infty$.

Example 1.67. What is $\lim_{x \to -2} \frac{1}{x+2} + \frac{2}{x(x+2)}$? This looks like $\infty + \infty$ so it is an indeterminate form, and we have to be careful. We have

$$\begin{aligned}\lim_{x \to -2} \frac{1}{x+2} + \frac{2}{x(x+2)} &= \lim_{x \to -2} \frac{x}{x(x+2)} + \frac{2}{x(x+2)} \\ &= \lim_{x \to -2} \frac{x+2}{x(x+2)} = \lim_{x \to -2} \frac{1}{x} = \frac{-1}{2}.\end{aligned}$$

1.6.2 Limits at infinity

A related concept is the idea of limits "at" infinity, which answers the question "what happens to $f(x)$ when x gets very big?"

Example 1.68.

In principle, we want to do the same thing we did for finite limits, where we find an almost identical function that's continuous, and then plug in a value of x. But we can't actually "plug in" infinity, because it's not a number, so instead we use the following rule:

Fact 1.69. $\lim_{x \to \pm\infty} \frac{1}{x} = 0$.

This combined with tools we already have is enough to do pretty much any calculation we might want

Example 1.70. If we want to calculuate $\lim_{x \to +\infty} \frac{1}{\sqrt{x}}$, we see that

$$\lim_{x \to +\infty} \frac{1}{\sqrt{x}} = \sqrt{\lim_{x \to +\infty} \frac{1}{x}} = \sqrt{0} = 0.$$

Example 1.71. What is $\lim_{x\to+\infty} \frac{x}{x^2+1}$?

This problem illustrates the primary technique we'll use to solve infinite limits problems. It's difficult to deal with problems that have variables in the numerator and denominator, so we want to get rid of at least one. Thus we will divide out by xs on the top and the bottom until one has none left:

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x/x}{x^2/x + 1/x} = \lim_{x\to+\infty} \frac{1}{x+\frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Example 1.72 (recitation/bonus)**.** Some more examples of this technique:

$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$
$$\lim_{x\to-\infty} \frac{x}{3x+1} = \lim_{x\to-\infty} \frac{1}{3+\frac{1}{x}} = \frac{1}{3}.$$

Example 1.73. What is $\lim_{x\to+\infty} \dfrac{x^{3/2}}{\sqrt{9x^3+1}}$?

This one is a bit tricky. We want to get the xs out of the numerator, so we divide the top and bottom by $x^{3/2}$.

$$\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9x^3+1}/x^{3/2}}.$$

Then we can observe that $x^{3/2} = \sqrt{x^3}$, and so we have

$$\lim_{x\to+\infty} \frac{1}{\sqrt{9x^3+1}/\sqrt{x^3}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9+1/x^3}} = \frac{1}{\sqrt{9+0}} = \frac{1}{3}.$$

Example 1.74. Sometimes it's a bit harder to see how this works. For instance, what is $\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}$? It's not obvious, but since we can say that $x = \sqrt{x^2}$ we can use the same technique:

$$\begin{aligned}\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to+\infty} \frac{x/x}{\sqrt{x^2+1}/x} \\ &= \lim_{x\to+\infty} \frac{1}{\sqrt{x^2/x^2+1/x^2}} \\ &= \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.\end{aligned}$$

But something subtle happened there that isn't obvious. It becomes important in problems that tweak things slightly.

Example 1.75. What is $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$?

We can do the same thing, but we have to be *very careful*. We know that $\sqrt{x^2}$ is always positive, so if $x < 0$ then $\sqrt{x^2} \neq x$! Instead, $x = -\sqrt{x^2}$. Thus we have

$$\begin{aligned} \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/x} \\ &= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/(-\sqrt{x^2})} \\ &= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1. \end{aligned}$$

When we encounter new functions, one of the ways we will often want to characterize them is by computing their limits at $\pm\infty$. Sometimes these limits do not exist.

Example 1.76. $\lim_{x\to+\infty} \sin(x)$ does not exist, since the function oscillates rather than settling down to one limit value.

$\lim_{x\to+\infty} x\sin(x)$ also does not exist; this function oscillates more and more wildly as x increases.

But $\lim_{x\to+\infty} \frac{1}{x}\sin(x)$ does in fact exist. We can prove this with the squeeze theorem: we can see that $\frac{-1}{x} \leq \frac{1}{x}\sin(x) \leq \frac{1}{x}$, and we know that $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$. So by the Squeeze Theorem, $\lim_{x\to+\infty} \frac{1}{x}\sin(x) = 0$.

Another technique that will also often appear in these limits is combining a sum or difference into one fraction. If we have a sum of two terms that both have infinite limits, we need to combine or factor them into one term to see what is happening.

Example 1.77. What is $\lim_{x\to-\infty} x - x^3$?

Each term goes to $-\infty$, so this is a difference of infinities and thus indeterminate. But we can factor: $\lim_{x\to-\infty} x(1-x^2)$. The first term goes to $-\infty$ and the second term also goes to $-\infty$, so we expect that their product will go to $+\infty$. Thus $\lim_{x\to-\infty} x - x^3 = +\infty$.

To be precise, I should compute:

$$\lim_{x\to-\infty} x - x^3 = \lim_{x\to-\infty} \frac{x-x^3}{1} = \lim_{x\to-\infty} \frac{1/x^2 - 1}{1/x^3}.$$

We see the limit of the top is -1 and the limit of the bottom is 0, so the limit of the whole is $\pm\infty$. In fact the bottom will always be negative (since $x \to -\infty$), and thus the limit is $+\infty$.

Example 1.78. What is $\lim_{x\to+\infty} \sqrt{x^2+1} - x$?

We might want to try to use limit laws here, but we would get $+\infty - +\infty$ which is not defined (and is one of the classic indeterminate forms). Instead we need to combine our expressions into one big fraction.

$$\begin{aligned}
\lim_{x \to +\infty} \sqrt{x^2+1} - x &= \lim_{x \to +\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\
&= \lim_{x \to +\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1}+x} = \lim_{x \to +\infty} \frac{1}{\sqrt{x^2+1}+x} \\
&= \lim_{x \to +\infty} \frac{1/x}{\sqrt{1+1/x^2}+1} = 0.
\end{aligned}$$

This tells us that as x increases, x and $\sqrt{x^2+1}$ get as close together as we wish.

You may have noticed the appearance of our old friend, multiplication by the conjugate. We will often use that technique in this sort of problem.

2 Derivatives

2.1 Linear Approximation

In the last section we talked about continuous functions as functions that we could approximate. We know that $\sqrt{5}$ is about 2, and 3.1^3 is about 27. In this section we want to be a bit more precise than that. Most of you told me not only that $\sqrt{5}$ is "about 2", but it's a bit *more* than 2. We want to find a way to estimate that bit more.

We need to use a more complicated formula. But we want to keep the amount of complexity under control. So we want to use a simple function to approximate $f(x)$. The simplest possible function is a constant function; and that's exactly what we used last section. (3.1^3 is about 27, and 3.01^3 is about 27, and 3.2^3 is about 27.) If a is a fixed number then $f(a)$ is a constant, and thus $f(x) \approx f(a)$ approximates f with a constant function.

The next most complex function, as we usually think of it, is a linear function. So we want to approximate f with a linear function. There are a few ways we can write the equation for a line, depending on what information we already know:

$$
\begin{aligned}
y &= mx + b && \text{Slope-Intercept Formula} \\
y - y_0 &= m(x - x_0) && \text{Point-Slope Formula} \\
y - y_0 &= \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) && \text{Two Points Formula}
\end{aligned}
$$

The most common and popular is the slope-intercept formula, which is great for *computing* things; but to write down the equation, you need to know the slope m, and also the y-intercept b. For our approximations we won't generally know this.

The two points formula also isn't terribly useful for us. We know one point: since we're approximating a function f near a, we know it goes through the point $(a, f(a))$. But if we knew the value at other points, we wouldn't need to approximate! (The approximation $f(x) - f(a) \approx \frac{f(x)-f(a)}{(x-a)}(x - a)$ is true, but is kind of vacuous and tautological; it doesn't actually help us).

But the point-slope formula can get us somewhere. We already have a point, so we just need to find the slope. We'll see how to do that soon, but for now we'll just give the slope a name: if we're taking a linear approximation to a function $f(x)$ near a point a, then we will denote the slope $f'(a)$. This tells us, essentially, how much we care about the distance between x and a. When this is small, then $f(x)$ is close to $f(a)$; when $f'(a)$ is large, then $f(x)$ moves away from $f(a)$ pretty quickly.

The equation for our linear approximation is

$$f(x) \approx f'(a)(x-a) + f(a) \tag{1}$$

This is the most important formula in the entire course; essentially everything we do for the next two months will refer back to this approximation in some way.

Example 2.1. We earlier said that $\sqrt{5} \approx \sqrt{4} = 2$. We can see that in fact $\sqrt{5}$ should be a little bigger than 2. But how much better?

A linear approximation would tell us that $\sqrt{5} \approx 2 + f'(2)(5-4)$. That is, we know that $\sqrt{5}$ is a bit bigger than two—and it's a bit bigger by the amount of this mysterious $f'(2)$ slope. We'll see how to compute this later, but for right now I'll tell you that $f'(2) = \frac{1}{4}$. Then we get that $\sqrt{5} \approx 2 + \frac{1}{4}(5-4) = 9/4 = 2.25$.

From this we can make other estimates. For instance, we have that $\sqrt{4.5} \approx 2 + \frac{1}{4}(4.5-4) = 17/8$, and $\sqrt{6} \approx 2 + \frac{1}{4}(6-4) = 5/2$.

We can go in the other direction as well. We estimate that $\sqrt{3} \approx 2 + \frac{1}{4}(3-4) = 7/4$. And $\sqrt{2} \approx 2 + \frac{1}{4}(2-4) = 3/2$.

But notice: this gives us $\sqrt{1} \approx 2 + \frac{1}{4}(1-4) = 5/4$, which we know is wrong. And $\sqrt{9} \approx 2 + \frac{1}{4}(9-4) = 13/4$, which is also wrong. For that matter, we get $\sqrt{100} \approx 2 + \frac{1}{4}(100-4) = 26$, which is really wrong. What's going on here?

We can even graph the formula for this approximation: we have

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

and if we graph $\sqrt{x}$ and $2 + \frac{1}{4}(x-4)$ on the same graph, we get figure 2.1: We may notice

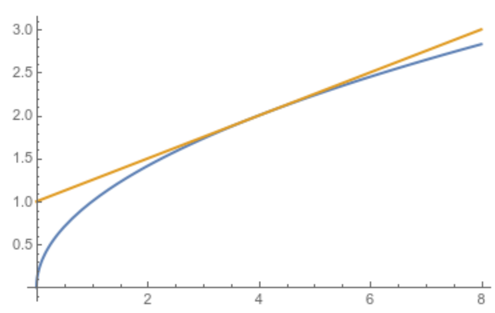


Figure 2.1: $\sqrt{x}$ (blue) and its linear approximation at 4 (yellow)

that the graph of our linear approximation is *tangent* to the graph of our function. We'll revisit this idea in section 2.8

A linear approximation is good when x is close to $a = 4$. As x gets further away from a, then our estimate for $f(x)$ gets further from $f(a)$; but in general we would also expect our estimate to get further from the correct answer. These techniques work best when x is very close to a.

(We're not yet ready to be precise about what "very close" means here).

Example 2.2. We've dressed this up in fancy language, but we engage in this sort of reasoning all the time. Suppose you are driving at 30 miles per hour. After an hour, you expect to have gone about thirty miles. After six minutes, you expect to have gone about three miles.

This is just a linear approximation. If $f(t)$ is our position as a function of time, our approximation is that we're moving 30 miles per hour, or half a mile per minute. Then we have $f(t) \approx 0 + \frac{1}{2}(t - 0)$, and if we plug in $t = 6$ we have $f(6) \approx 0 + \frac{1}{2}(6 - 0) = 3$.

We'll revisit this idea in section 2.7.

2.2 The Derivative

We understand that we want to do linear approximation now. But without a way to actually find the slope $f'(a)$, it isn't terribly helpful.

So let's look at our formula from equation (3) again. We want to understand $f'(a)$, so we'll solve the equation for that:

$$\begin{aligned} f(x) &\approx f'(a)(x-a) + f(a) \\ f(x) - f(a) &\approx f'(a)(x-a) \\ \frac{f(x)-f(a)}{x-a} &\approx f'(a). \end{aligned}$$

Thus we get a new formula. This formula should also make sense to us. The slope $f'(a)$ tells us how different $f(x)$ is from $f(a)$, based on how x is different from a. This new, rearranged formula tells us that $f'(a)$ approximates the ratio of the change in $f(x)$ to the change in x, which we sometimes write as $\frac{\Delta f}{\Delta x}$. Thus it should tell us how much a change in the input value affects the output value—which is exactly the question we need to answer to write a linear approximation.

But we've also seen this formula somewhere else. In the two points formula for a line, the slope is $\frac{y_1 - y_0}{x_1 - x_0}$. if $y_1 = f(x_1) = f(x)$ and $y_0 = f(x_0) = f(a)$, then this is just the approximation we have for $f'(a)$. Thus we're saying that $f'(a)$ is approximately the slope of the line through the point $(a, f(a))$ that we know, and the point $(x, f(x))$ that we want. We'll explore this angle more in lab.

On its own, this still isn't helpful: we have an approximate formula for $f'(a)$, but it requires us to already know $f(x)$, which is what we started out wanting to compute. But one more step makes this actually useful.

Definition 2.3. Let f be a function defined near and at a point a. We say the *derivative* of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The second formula is just a change of variables from the first, setting $h = x - a$. It's not substantively any different, but it's sometimes easier to compute with.

We will also sometimes write $\frac{df}{dx}(a)$ for the derivative of f at a. The is called "Leibniz notation", as opposed to the "Newtonian notation" of $f'(a)$.

Thus the derivative is given by taking our approximate formula for $f'(a)$, and taking the limit as x and a get closer together. Our linear approximation is better when x and a are closer; so as x approaches a, the approximation becomes perfect, and we get an exact equation.

Remark 2.4. Note that we need *two* pieces of information here. You hand me a function f *and* a point a, and I tell you the derivative of f at a. We'll adopt different perspectives from time to time later on in the course.

Example 2.5. Let $f(x) = x^2 + 1$. Then

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + 1 - 2^2 - 1}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = 4,$$

and more generally, for any number a we have

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = 2a.$$

Example 2.6. Let $f(x) = \sqrt{x}$. Then given a number a, we have

$$f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h \to 0} \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Note that f is defined at 0, and we have $f(0) = 0$. But by this computation we have $f'(0) = \frac{1}{2 \cdot 0}$ which is undefined. This isn't an artifact of the way we computed it; the limit in fact does not exist. Further, this isn't just becasue 0 is on the edge of the domain of f, as we shall see:

Example 2.7. Let $g(x) = \sqrt[3]{x}$. Then we can compute $g'(0)$ and we get

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

The cube root function g has no defined derivative at 0, even though the function is defined there. This brings us to a discussion of ways for a function to fail to be differentiable at a point. (There's always the catchall category of "the limit just doesn't exist," which we won't really discuss because there's not much to say about it).

(a) **Vertical Tangent Line**

Our first example of $g(x) = \sqrt[3]{x}$ is not differentiable at 0, and the limit

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = +\infty.$$

Graphically, the line tangent to g at 0 is completely vertical; the function is "increasing infinitely fast" at 0.

(b) **Corner**

Any function with a sharp corner at a point doesn't have a well-defined rate of change at that point; the change is instantaneous. For instance, if we let $a(x) = |x|$ be the absolute value function, then

$$a'(x) = \lim_{h \to 0} \frac{a(x+h) - a(x)}{h}.$$

To study piecewise functions we usually break them up and study each piece separately. If $x > 0$, then $a(x) = x$ and $a(x+h) = x + h$ for small h. We have

$$a'(x) = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} 1 = 1.$$

Conversely, if $x < 0$ then $a(x) = -x$ and $a(x+h) = -x - h$, and

$$a'(x) = \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} -1 = -1.$$

But if $x = 0$ then the left and right limits don't agree again: the right limit is 1 and the left limit is -1, so the limit does not exist. Thus we have

$$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0. \end{cases}$$

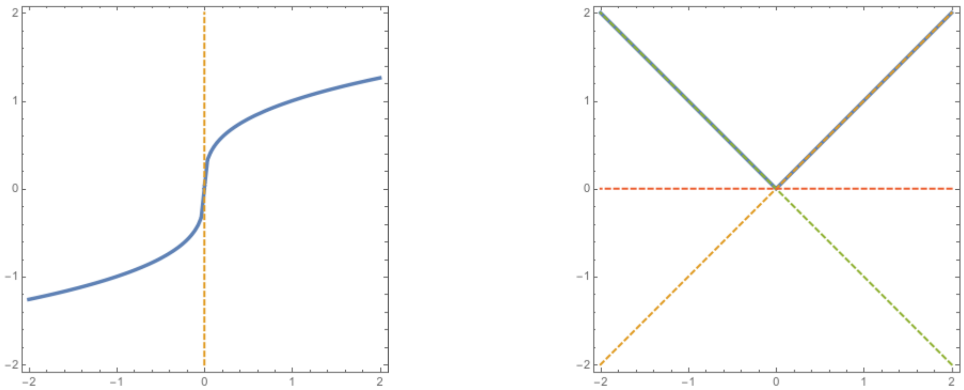


Figure 2.2: A vertical tangent line and a corner

(c) **Cusp**

Sometimes a function has a "cusp" at a point. This is a point where the tangent line is vertical, but depending on the side from which you approach, you can get a tangent line that goes up incredibly fast or one that goes down incredibly fast.

Consider the funtion $f(x) = \sqrt[3]{x^2}$. We have

$$f'(0) = \lim_{h \to 0} \frac{\sqrt[3]{h^2} - \sqrt[3]{0}}{h} = \lim_{h \to 0} \frac{h^{2/3}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

This is different from the $\sqrt[3]{x}$ example because the limit is $\pm\infty$ rather than just $+\infty$.

(d) **Discontinuity**

Any function that is not continuous at a point cannot be differentiable at that point. In particular, if f is differentiable at a, then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

converges. But the bottom goes to zero, so the top must also go to zero, and we have

$$\lim_{x \to a} f(x) = f(a),$$

which is precisely what it means to be continuous.

Conceptually, if the function isn't continuous, it isn't changing smoothly and so doesn't have a "speed" of change. Graphically, a function that has a disconnect in it doesn't have a clear tangent line.

An example here is the Heaviside function $H(x)$. We have

$$\lim_{h \to 0^+} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$

but

$$\lim_{h\to 0^-} \frac{H(h) - H(0)}{h} = \lim_{h\to 0^-} \frac{-1}{h} = +\infty.$$

Since the one-sided limits aren't equal, the limit does not exist.

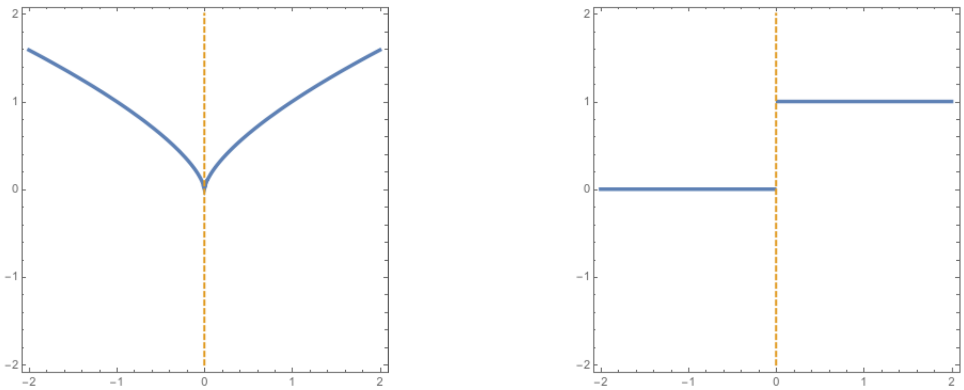


Figure 2.3: A cusp and a discontinuous function

The Derivative as a Function When we defined the derivative, we said it was a *number*: in definition 2.3 we defined the derivative *at* a to be the slope of the linear approximation *at* a. But this means we have a relationship where at each number a, we get a number $f'(a)$, and that means we have a function! Thus f' is a function and we can study it the way we did earlier functions.

Definition 2.8. The *derivative of a function* f is the function that takes in an input x and outputs

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Remark 2.9. On the first day of class, we said that functions don't have to take in and output numbers, it's just that the ones we're studying involve numbers. But if we want to get very abstract, "the derivative" is a function $\frac{d}{dx}$ which takes in a function f and gives back a new function f'. This is why we sometimes write $\frac{d}{dx} f(x) = f'(x)$.

Example 2.10. (a) If $f(x) = x^2 + 1$, we computed that $f'(x) = 2x$. The domain of f is all reals, and so is the domain of $f'(x)$.

(b) If $g(x) = \sqrt{x}$ then $g'(x) = \frac{1}{2\sqrt{x}}$. The domain of g is all reals ≥ 0, and the domain of g' is all reals > 0.

(c) We saw above that if $a(x) = |x|$, then the function $a'(x)$ is given by the piecewise formula

$$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0 \end{cases} = \frac{|x|}{x}.$$

The domain of a is all reals and the domain of a' is all reals except 0.

Given a function, we can compute the derivative. Since f' is a function we can ask about the derivative of the function f' at a point a.

Definition 2.11. Let f be a function which is differentiable at and near a point a. The *second derivative of f at a* is the derivative of the function $f'(x)$ at a, which is

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(x)}{h} = \frac{d^2 f}{dx^2}(a).$$

This is again a limit and may or may not exist.

Remark 2.12. The Leibniz notation for a second derivative is $\frac{d^2f}{dx^2}$ and not $\frac{df^2}{dx^2}$. Conceptually, you can think of $\frac{d}{dx}$ as a function whose input is the function f and whose output is the derivative function f'. The second derivative results from applying this function twice, and thus is $\left(\frac{d}{dx}\right)^2 = \frac{d^2}{dx^2}$.

Example 2.13. What is the second derivative of $f(x) = x^3$ at $a = 2$?

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3h^2 + h^3}{h} = \lim_{h \to 0} 3x^2 + 3h + h^2 = 3x^2.$$

$$\begin{aligned} f''(2) &= \lim_{h \to 0} \frac{f'(2+h) - f'(2)}{h} = \lim_{h \to 0} \frac{3(2+h)^2 - 3 \cdot 2^2}{h} = \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 12}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} = \lim_{h \to 0} 12 + 3h = 12. \end{aligned}$$

We won't say much more about the second derivative now, but we'll discuss it extensively in section 3.

2.3 Computing Derivatives

By now we're getting pretty tired of computing those examples over and over. In this section we'll come up with some techniques to make computation of derivatives easier.

(a) **Constants** If c is a constant and $f(x) = c$ then $f'(x) = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Conceptually, a constant function never changes, so the rate of change is 0.

Geometrically, a constant function is a horizontal line; thus we think of the slope everywhere as being 0.

Example 2.14. $(3^{3^{3^3}})' = 0$.

(b) If $f(x) = x$, then $f'(x) = 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} 1 = 1.$$

Conceptually, if we have the "identity" function, then whenever we change the input then the output should change by exactly the same amount. Thus the rate of change is 1.

Geometrically, this is a line with slope 1.

(c) If c is a constant and g is a function and $f(x) = c \cdot g(x)$, then $f'(x) = c(g'(x))$.

$$f'(x) = \lim_{h\to 0} \frac{cg(x+h) - cg(x)}{h} = c \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = c \cdot g'(x).$$

Conceptually, if changing x by a bit changes $g(x)$ by a certain amount, then it will change $2g(x)$ by twice that amount–multiplying by a scalar should just change the rate of change by the same amount everywhere.

Geometrically, multiplying by a constant is just stretching vertically–and all the slopes will be stretched by that same amount.

Example 2.15. If $f(x) = 5x$ then $f'(x) = (5 \cdot x)' = 5 \cdot x' = 5$.

(d) If f and g are functions then $(f+g)'(x) = f'(x) + g'(x)$.

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).\end{aligned}$$

Conceptually, if changing the input by a bit changes f by a certain amount and g by a different amount, then it changes $f+g$ by the sum of those two amounts–figure out how much it changes each part and then add them together to find out how much it changes the whole.

Geometrically, if we add two functions together it's just like stacking them on top of one another, so the slope at any point will be the sum of the slopes.

Example 2.16. Let $f(x) = 3x - 7$. Then $f'(x) = (3x)' - 7' = 3(x') - 0 = 3$.

This rule is really important but so far we can't do much with it–we don't have quite enough rules yet.

(e) (Power Rule) If $f(x) = x^n$ where n is a positive integer, then $f'(x) = nx^{n-1}$. In fact, if $g(x) = x^r$ and r is any real number, then $g'(x) = rx^{r-1}$. We'll only prove this for integers, using the difference-of-nth-powers rule.

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{z^n - x^n}{z - x} = \lim_{z \to x} \frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x} \\ &= \lim_{z \to x} z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1} = x^{n-1} + \cdots + x^{n-1} = nx^{n-1}. \end{aligned}$$

Now that we have this, we can compute all sorts of derivatives.

Example 2.17.
- $(x^2 + 1)' = 2x + 0 = 2x$.
- $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.
- $(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.
- $(3\sqrt{x} + x^5 - 7)' = \frac{3}{2\sqrt{x}} + 5x^4 + 0$.

(f) (Product Rule) If f and g are functions then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Conceptually, we sort of know this already; if we add a bit on to f and a bit on to g, then we get $(f + f_h)(g + g_h) = fg + fg_h + gf_h + g_hf_h$, and in the limit we can treat g_hf_h as being zero. So this is the same as multiplying the bit we add to g with f, and multiplying the bit we add to f with g, and then adding the two.

Example 2.18. $((3x-2)(x-1))' = (3x^2 - 5x + 2)' = 6x - 5$.

Alternatively, $((3x-2)(x-1)' = (3x-2)'(x-1) + (3x-2)(x-1)' = 3 \cdot (x-1) + 1 \cdot (3x-2) = 6x - 5$.

This rule isn't terribly important as long as we're only working with rational functions. Once we include anything else, like trig functions, it is critical.

Proof.

$$\begin{aligned}
(f \cdot g)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) + g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

□

(g) (Quotient Rule): If f and g are functions then

$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

I don't know of any good intuition for this, or explanation of why it should sound true. This is an example of how sometimes you need to just do a calculation, and then trust it.

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left(\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} + \lim_{h\to 0} \frac{f(x)g(x) - f(x)g(x+h)}{h}\right) \\
&= \frac{1}{g(x)^2} \left(g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

Example 2.19.
- $\left(\frac{x-1}{x^3}\right)' = (x^{-2} - x^{-3})' = -2x^{-3} + 3x^{-4}$.

  Alternatively,
  $$\left(\frac{x-1}{x^3}\right)' = \frac{(x-1)'x^3 - (x-1)3x^2}{x^6} = \frac{x^3 - 3x^3 + 3x^2}{x^6} = -2x^{-3} + 3x^{-4}.$$
- $$\begin{aligned}\left(\frac{2+3x}{3-5x}\right)' &= \frac{(2+3x)'(3-5x) - (2+3x)(3-5x)'}{(3-5x)^2} \\ &= \frac{9 - 15x + 10 + 15x}{(3-5x)^2} = \frac{19}{(3-5x)^2}.\end{aligned}$$

2.4 Trigonometric derivatives

We cannot neglect the trigonometric functions—no matter how much we might wish to on occasion. All of the rules for trigonometric derivatives rely on what are known as the *angle addition formulas*:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Note: you probably won't ever need to know these formulas again in this class. But I will need them for another page or so of these notes.

Using this we can compute

(a)
$$\begin{aligned}(\sin(x))' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\ &= \left(\lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}\right) + \left(\lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h}\right) \\ &= \cos(x)\lim_{h\to 0}\frac{\sin h}{h} + \sin(x)\lim_{h\to 0}\frac{\cos h - 1}{h} \\ &= \cos(x) + \sin(x)\lim_{h\to 0}\frac{\cos^2(h) - 1}{h(\cos(h)+1)} \\ &= \cos(x) - \sin(x)\lim_{h\to 0}\frac{\sin^2(h)}{h(\cos(h)+1)} \\ &= \cos(x) - \sin(x)\left(\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1}\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\ &= \cos(x) - \sin(x)\cdot 0 \cdot 1 = \cos(x).\end{aligned}$$

(b) A similar argument shows that $(\cos(x))' = -\sin(x)$.

Further using the product and quotient rules, we observe that

- $$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

- $$(\cot(x))' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

- $$(\sec(x))' = \left(\frac{1}{\cos x}\right)' = \frac{0 + \sin x}{\cos^2(x)} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec(x)\tan(x)$$

- $$(\csc(x))' = \left(\frac{1}{\sin x}\right)' = \frac{0 - \cos(x)}{\sin^2(x)} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\csc(x)\cot(x).$$

Remember that as long as you know the derivatives of sin and cos you can always compute these four derivatives whenever you need them.

Example 2.20. (a) If $f(t) = 3\sin t + \cos t$, then $f'(t) = 3\cos t - \sin t$.

(b) Find the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.

(c) IF $g(\theta) = \theta\sin\theta + \frac{\cos\theta}{\theta}$, then

$$g'(\theta) = (\sin\theta + \theta\cos\theta) + \frac{-\theta\sin\theta - \cos\theta}{\theta^2}.$$

(d) If $h(x) = \frac{x}{2-\tan x}$, then

$$h'(x) = \frac{(2 - \tan x) + x\sec^2 x}{(2 - \tan x)^2}.$$

(e) We can also compute second derivatives. $\sin'' x = -\sin x$. $\cos'' x = -\cos x$.

$$\tan'' x = (\sec x \sec x)' = \sec x \tan x \sec x + \sec x \tan x \sec x = 2\sec^2 x \tan x.$$

2.5 The Chain Rule

To start with an example, suppose $g(x) = (\sin x)^2$. Then

$$g'(x) = ((\sin x)(\sin x))' = \cos x \sin x + \cos x \sin x = 2 \sin x \cos x.$$

Remembering that $(x^2)' = 2x$, we notice that this looks suggestive. It also leads us to ask what happens when we build up functions by composition, that is, plugging one function into another, as we have here.

If we want to freely build complex functions from simple ones, we need to be able to combine them in chains. Remember that we define the function $f \circ g$ by $(f \circ g)(x) = f(g(x))$; we take our input x, plug it into g, and then take the output $g(x)$ and plug it into f.

We can see how this is useful in two different ways. First, as we saw earlier, it lets us build up functions.

(a) $(x + 1)^2 = (f \circ g)(x)$ where $g(x) = x + 1$ and $f(x) = x^2$.

(b) $(x^2 + 1)^2 = (f \circ g)(x)$ where $g(x) = x^2 + 1$ and $f(x) = x^2$.

(c) $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$.

Second, sometimes composition of functions really is the best way to describe what's going on, especially when you have a "causal chain" where one process causes a second which causes a third. For instance, suppose you're driving up a mountain at 2 km/hr, and the temperature drops 6.5° C per kilometer of altitude. You can think about your temperature as a function of your height, which is itself a function of the time; then the numbers I gave you are the rates of change, or derivatives, of each function.

It's not that hard to convince yourself that you'll get colder by about 13° C per hour. Does this work in general?

Proposition 2.21 (Chain Rule). *Suppose f and g are functions, such that g is differentiable at a and f is differentiable at $g(a)$. Then $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.*

Proof.

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\
&= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} \\
&= \left(\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \right) \left(\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \right) \\
&= f'(g(a)) \cdot g'(a).
\end{aligned}$$

□

Remark 2.22. (a) When we write $f'(g(x))$, we mean the function f' evaluated at the point $g(x)$, or in other words, the derivative of f at the point $g(x)$.

(b) It can be helpful as a way of remembering the chain rule that

$$\frac{d(f \circ g)}{dx} = \frac{d(f \circ g)}{dg} \cdot \frac{dg}{x}.$$

Don't take this too seriously as actively meaning anything, since it only sort of does, but it's quite helpful for the memory.

Example 2.23. (a) $(x+1)^2 = (f \circ g)(x)$ where $g(x) = x+1$ and $f(x) = x^2$. Then $f'(x) = 2x$ and $g'(x) = 1$, so

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 1 = 2(x+1) \cdot 1 = 2x + 2.$$

Sanity check:

$$(f \circ g)'(x) = (x^2 + 2x + 1)' = 2x + 2.$$

(b) $(x^2+1)^2 = (f \circ g)(x)$ where $g(x) = x^2 + 1$ and $f(x) = x^2$. Then $f' = 2x$, $g' = 2x$, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 2x = 2(x^2+1) \cdot 2x = 4x^3 + 4x.$$

Sanity check:

$$(f \circ g)'(x) = (x^4 + 2x^2 + 1)' = 4x^3 + 4x.$$

(c) $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$. Then $f'(x) = 2x$, $g'(x) = \cos x$, and we have

$$(f \circ g)'(x) = 2(g(x)) \cdot \cos x = 2(\sin x)\cos x.$$

(d) $\cos(3x) = (f \circ g)(x)$ where $f(x) = \cos(x)$ and $g(x) = 3x$. Then $f'(x) = -\sin(x)$ and $g'(x) = 3$ and

$$(f \circ g)'(x) = -\sin(3x) \cdot 3.$$

(e) $\sin(x^2) = (f \circ g)(x)$ where $f(x) = \sin(x)$ and $g(x) = x^2$. Then $f'(x) = \cos x, g'(x) = 2x$, and

$$(f \circ g)'(x) = \cos(g(x)) \cdot 2x = 2x\cos(x^2).$$

(f) If $f(x)$ is any function, then we can write $(f(x))^r$ as $(g \circ f)(x)$ where $g(x) = x^r$. Then

$$\frac{d}{dx}(f(x)^r = (g \circ f)'(x) = r(f(x))^{r-1} \cdot f'(x).$$

(g) The derivative of $\sec(5x)$ is $\sec(5x)\tan(5x)5$.

(h) What is the derivative of $\frac{1}{\sqrt[3]{x^4-12x+1}}$? We can view this as $(x^4 - 12x + 1)^{-1/3}$, and using the chain rule, we have

$$\frac{d}{dx}\frac{1}{\sqrt[3]{x^4 - 12x + 1}} = \frac{-1}{3}(x^4 - 12x + 1)^{-4/3} \cdot (4x^3 - 12).$$

(i) What is the derivative of $\sec^2(x)$? By the chain rule this is $2 \cdot \sec(x) \cdot \sec'(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$.

(j) What is the derivative of $\sec^4(x)$? We get $4\sec^3(x)\sec'(x) = 4\sec^3(x)\sec(x)\tan(x) = 4\sec^4(x)\tan(x)$.

(k) Sometimes we have to nest the chain rule. What is the derivative of $\sqrt{x^3 + \sqrt{x^2 + 1}}$? We can pull this apart slowly.

$$\begin{aligned}
\frac{d}{dx}\sqrt{x^3 + \sqrt{x^2+1}} &= \frac{1}{2}(x^3 + \sqrt{x^2+1})^{-1/2} \cdot \left(\frac{d}{dx}\left(x^3 + \sqrt{x^2+1}\right)\right) \\
&= \frac{1}{2\sqrt{x^3 + \sqrt{x^2+1}}}\left(3x^2 + \frac{1}{2}(x^2+1)^{-1/2} \cdot \left(\frac{d}{dx}x^2 + 1\right)\right) \\
&= \frac{3x^2 + \frac{2x}{2\sqrt{x^2+1}}}{2\sqrt{x^3 + \sqrt{x^2+1}}}
\end{aligned}$$

As we have just seen the chain rule can stack, or chain together. As functions get more complicated we will have to use multiple applications of the product rule, quotient rule, and chain rule to pull our derivative apart.

Example 2.24. Find

$$\frac{d}{dx}\sec(x^2 + \sqrt{x^3+1}).$$

$$\begin{aligned}
\frac{d}{dx}\sec(x^2 + \sqrt{x^3+1}) &= \sec(x^2 + \sqrt{x^3+1}) \cdot \tan(x^2 + \sqrt{x^3+1}) \cdot (x^2 + \sqrt{x^3+1})' \\
&= \sec(x^2 + \sqrt{x^3+1}) \cdot \tan(x^2 + \sqrt{x^3+1}) \cdot (2x + \frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2)
\end{aligned}$$

Example 2.25 (recitation)**.** Find

$$\frac{d}{dx}\frac{\sin(x^2) + \sin^2(x)}{x^2+1}$$

$$\begin{aligned}
\frac{d}{dx}\frac{\sin(x^2) + \sin^2(x)}{x^2+1} &= \frac{(\sin(x^2) + \sin^2(x))'(x^2+1) - 2x(\sin(x^2) + \sin^2(x))}{(x^2+1)^2} \\
&= \frac{(\cos(x^2) \cdot 2x + 2\sin(x)\cos(x))(x^2+1) - 2x(\sin(x^2) + \sin^2(x))}{(x^2+1)^2}.
\end{aligned}$$

We can keep going with increasingly complicated problems, basically until we get bored. These are really good practice for making sure you understand how the rules fit together.

Example 2.26. Find

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}}$$

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)' \\ &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2-2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}\end{aligned}$$

Example 2.27 (Bonus)**.** Calculate

$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$

2.6 Linear Approximation

In section 2.1 we defined the derivative in terms of approximation. We took an *algebraic* approach where we wanted to approximate a function with a line, and found a number $f'(a)$ that made the line $y = f'(a)(x-a)+f(a)$ approximate the function f as well as possible.

In this section we want to return to this idea, now that we know how to compute derivatives. Then in section 2.7 we'll see how we can use this to model physical, economic, and other practical phenomena. Finally in section 2.8 we'll take a *geometric* perspective, where we see how we can use derivatives to understand geometric pictures and graphs of functions.

We know that if we have a function $f(x)$ and know what it looks like at a point a, we can use the derivative to give a linear approximation

$$f(x) \approx f(a) + f'(a)(x-a).$$

Example 2.28. We can find an estimate of 2.1^5.

To a "zeroth approximation", we might say that $2.1^5 \approx 2^5 = 32$; that's the approach we took in section 4. We can now use the derivative to refine that estimate. We take $f(x) = x^5$ and $a = 2$. Then $f'(x) = 5x^4$, so we have $f(2) = 32, f'(2) = 80$, and

$$f(2.1) \approx 80(2.1-2)+32 = 40.$$

The exact answer is 40.841, so this estimate is pretty good!

What if we approximate $(2.5)^5$ using $a = 2$. What if we pproximate 3^5? We have

$$(2.5)^5 \approx 80 \cdot (2.5 - 2) + 32 = 72$$
$$3^5 \approx 80 \cdot (3 - 2) + 32 = 112.$$

The true answers are 97.6563 and 243. These estimates are not especially good. This is because 3 is actually not very close to 2—especially proportionately. Of course, it's not that hard to compute 3^5 directly.

These methods are best when $x - a$ is very small relative to everything else. We often use them in the real world for $x - a < .1$ or so.

Example 2.29. Let's approximate $\sqrt[3]{28}$ and $\sqrt[4]{82}$.

We take $a = 27$ and $a = 81$ respectively.

$$\sqrt[3]{28} \approx \frac{1}{3}(27)^{-2/3}(28 - 27) + 3 = \frac{1}{27} + 3 \approx 3.03704$$
$$\sqrt[4]{82} \approx \frac{1}{4}(81)^{-3/4}(82 - 81) + 3 = \frac{1}{108} + 3 \approx 3.00926.$$

The true answers are approximately 3.03659 and 3.00922 respectively.

Now we'll approximate 28^3 and 82^4 using the same base points

We have

$$28^3 \approx 3(27)^2(28 - 27) + 27^3 = 21870$$
$$82^4 \approx 4(81)^3(82 - 81) + 81^4 = 45172485$$

In contrast the true answers are 21952 and 45172485.

These approximations aren't *terrible* but they aren't very good either. Since the derivative is changing quickly here (the second derivatives are $6 \cdot 27$ and $12 \cdot 81^2$ respectively), the approximation won't be very good.

Example 2.30. If you take $a = 0$ and $f(x) = x^{10}$, we can use a linear approximation to approximate $f(2)$. We have $f'(x) = 10x^9$, so we have $f'(0) = 0$, and thus

$$f(2) \approx 0(2 - 0) + 0 = 0.$$

Since the true answer is 1024, this is not very good. What if we use $a = 1$ instead? If we take $a = 1$, we have

$$f(2) \approx 10(2 - 1) + 1 = 11.$$

This is a little better, but still not good. In essence, the derivative is changing so quickly that the tangent line approximation is not very good over those distances. Later, in section 4.1, we'll talk a little bit about how we can handle this situation better.

There are a few specific linear approximation *formulas* that come up really frequently in other applications, enough to get their own names. I want to take a moment to look at each of them.

Example 2.31 (Binomial Approximation)**.** As a warmup, let's approximate $(1.01)^{10}$. Our function is $f(x) = x^{10}$ and our $a = 1$. So $f(a) = 1$ and $f'(a) = 10a^9 = 10$. Then we have

$$f(1.01) \approx 10(1.01 - 1) + 1 = 1.1.$$

The true answer is about 1.10462.

Now let's approximate $(1.01)^\alpha$ where $\alpha \neq 0$ is some constant. (The letter α is a Greek lower-case "a". I'm using it here instead of the friendlier n because it's fairly standard for the formula we're developing.)

We have $f(x) = x^\alpha$, so $f'(x) = \alpha x^{\alpha-1}$. We again have $f(1) = 1$ and $f'(1) = \alpha(1)^{\alpha-1} = \alpha$, so

$$f(1.01) \approx \alpha(1.01 - 1) + 1 = 1 + \alpha/100.$$

Now let's get the fully general useful formula: approximate $(1 + x)^\alpha$ where x is some small number and $\alpha \neq 0$ is a constant. (This rule is the called the "binomial approximation" and is often useful in physics and engineering).

We still take $f(x) = x^\alpha$ and $a = 1$. But we compute

$$f(1 + x) \approx 1 + \alpha(1 + x - 1) = 1 + \alpha x.$$

It is probably more helpful in the long run to think about $f(x) = (1+x)^\alpha$, though. Then we have $f'(x) = \alpha(1 + x)^{\alpha-1}$, and we get

$$f(x) \approx 1 + \alpha x.$$

Example 2.32 (Small Angle Approximation)**.** Let's find a formula to approximate $\sin(x)$ when x is small. You might think of this as the revenge of the Small Angle Approximation from section 1.5.

We take $a = 0$. Then since $\sin'(x) = \cos(x)$ and so $\sin'(0) = \cos(0) = 1$, we have

$$\sin(x) \approx 1(x - 0) + 0 = x.$$

Thus for small angles, $\sin(x)$ is approximately just x! For instance, our formula says that $\sin(.05) \approx .05$, where the true answer is about .04998. So this is pretty good. In fact, we compute that $\sin''(0) = -\sin(0) = 0$. Since the second derivative is zero, we expect the linear approximation to work well.

That means that in a lot of calculations, if we have a formula with a lot of sines in it, as long as our angles are small we can replace every $\sin(x)$ with an x without losing too much. And that's much easier to think about.

We can do the same thing for cosine. We compute that $\cos'(x) = -\sin(x)$ so $\cos'(0) = 0$. Then

$$\cos(x) \approx 0(x-0) + 1 = 1.$$

This is actually a constant! The line that fits $\cos(x)$ best near 0 is just the horizontal line $y = 1$.

We can calculate, e.g., that $\cos(.05) \approx 1$, where the true answer is about .9986. This is also pretty good, but the approximation isn't quite as good as the one for sine. We compute that $\cos''(0) = -\cos(0) = -1$; while the second derivative isn't huge, it isn't trivial either.

Example 2.33 (Geometric Series)**.** Let's find a formula to linearly approximate $f(x) = \dfrac{1}{1-x}$ near $x = 0$.

We compute that $f'(x) = (1-x)^{-2} = \frac{1}{(1-x)^2}$. Then

$$f(x) \approx 1 + x.$$

This is a special case of what's known as the geometric series formula.

You might ask why we did the slightly funky $\frac{1}{1-x}$ instead of the more normal $\frac{1}{x}$. After thinking about it for a bit, you'll notice that wee can't approximate $\frac{1}{x}$ near zero at all! We see that f is undefined at 0, and equally importantly, $f'(x) = -1/x^2$ is also undefined at zero. So there's no linear approximation.

But if we want to,, we can linearly approximate $f(x) = 1/x$ near 1. We have $f(1) = 1$ and $f'(1) = -(1)^{-2} = -1$ so

$$f(x) \approx 1 - (x-1) = 2 - x.$$

Finally, a bonus fun fact to notice.

Example 2.34. Let's find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.

2.7 Speed and Rates of Change

In this section we'll develop a second way of thinking about the derivative. We'll ask a different question, and see that the derivative is also an answer to that question. We'll talk a little bit about why the two different questions are secretly the same, and thus explain why you might *care* about linear approximation, even if you aren't as much of a nerd for algebra as I am.

2.7.1 The Problem of Speed

An important concept in physics is *speed*, which is defined to be distance covered divided by time spent. That is, $v = \frac{\Delta x}{\Delta t}$. In particular, if your position at time t is given by the function $p(t)$, then your average speed between time t_0 and time t_1 is

$$v = \frac{p(t_1) - p(t_0)}{t_1 - t_0}.$$

This formula should look familiar. It is the slope of a line through the points $(t_0, p(t_0))$ and $(t_1, p(t_1))$. It is *not* the derivative of p, because we didn't take a limit. It is instead a "difference quotient", which is really a fancy way of saying the slope of a line.

Example 2.35. For example, on Earth dropped objects fall about $p(t) = 5t^2$ meters after t seconds. The average speed between time $t = 1$ and time $t = 2$ is

$$v = \frac{p(2) - p(1)}{2 - 1} = \frac{20 - 5}{1} = 15\text{m/s}$$

and the average speed between time $t = 3$ and time $t = 1$ is

$$v = \frac{p(3) - f(1)}{3 - 1} = \frac{45 - 5}{3 - 1} = 20\text{m/s}.$$

It's useful here to look at the units. We know that the result is a speed, so comes out in m/s. But how do we know we get those units? We have to think a bit about what the function p is actually doing.

The function p gives us position as a function of time. Thus the *inputs* to p are given in seconds, and the *outputs* are given in meters. So it's not really fully correct to say that $p(t) = 5t^2$; that would suggest that $p(1\text{s}) = 5(1\text{s})^2 = 5\text{s}^2$. But your position isn't described in square seconds!

Instead, we would write something like $p(t\text{seconds}) = 5t^2\text{m}$. The function takes in seconds as inputs, and gives meters as outputs. Thus our last calculation properly should have been

$$v = \frac{p(3\text{s}) - f(1\text{s})}{3\text{s} - 1\text{s}} = \frac{45\text{m} - 5\text{m}}{3\text{s} - 1\text{s}} = 20\text{m/s}.$$

We see that the numerator—which is made up of the outputs of p—has units of meters, while the denominator, which is made up of the inputs of p, has units of seconds. So the entire fraction has units of m/s, which is what it should be.

We can give a more general formula. What's the average speed between time $t_0 = 1$ and time $t_1 = t$? We have

$$v = \frac{p(t\text{s}) - p(1\text{s})}{t\text{s} - 1\text{s}} = \frac{5t^2\text{m} - 5\text{m}}{t\text{s} - 1\text{s}} = 5(t+1)\frac{t-1}{t-1}\text{m/s}.$$

As long as $t \neq 1$, this gives us a formula for average speed between time t and time 1: the average speed is $5(t+1)$m/s. But what if we want to know the speed "at" the time $t = 1$?

On some level, this question doesn't make any sense. Speed is defined as the change in distance divided by the change in time; if time doesn't change, and distance doesn't change, then this doesn't really mean anything. Maybe what we really mean is, what's a good estimate of our average speed, as long as our time is close to $t = 1$? Our average speed depends on the exact interval we choose; the speed from $t = 1$ to $t = 2$ isn't the same as the speed from $t = 1$ to $t = 1.1$. But can we find one number that gives a good estimate?

This should make you think of the limit idea from section 1.3. We can find a good estimate of the speed from time 1 to time t by taking a limit as t approaches 1. Thus we define your *instantaneous speed* or *speed at time t_0* to be

$$\lim_{t_1 \to t_0} \frac{p(t_1) - p(t_0)}{t_1 - t_0} = \lim_{h \to t_0} \frac{p(t_0 + h) - p(t_0)}{h}.$$

Notice that since the function p has input in seconds and output in meters, the instantaneous speed will be in m/s, as it should be. But also notice that this formula is just the definition of the derivative of p.

Thus from the previous example, we can see that the instantaneous speed at time $t_0 = 1$ is

$$v(1\text{s}) = p'(1\text{s}) = \lim_{t \to 1} 5(t+1)\frac{t-1}{t-1}\text{m/s} = 10\text{m/s}.$$

Alternatively, we know that $p(t) = 5t^2$, so by our derivative rules we know that $p'(t) = 10t$ and thus $p'(1) = 10$. Once we add units, we have $p'(t\text{s}) = 10t$m/s and thus $p'(1\text{s}) = 10$m/s.

The derivative of a function has different units from the original function. Since the derivative is given by a formula with output in the numerator and input in the denominator, the derivative will have the units of the output per units of input.

We can take this one step further and look at the derivative of p'. The function p' takes in a time and outputs a speed; its derivative will be

$$p''(t_0\text{s}) = \lim_{t \to t_0} \frac{p'(t\text{s}) - p'(t_0\text{s})}{t\text{s} - t_0\text{s}}.$$

The units of the denominator are still seconds; but the units of the top are m/s, so the second derivative takes in seconds and outputs meters per second *per second*, or m/s^2. This makes sense: the second derivative is the change in the first derivative, so p'' tells us how quickly the speed is changing. So it tells us how many meters per second your speed changes each second. This is otherwise known as "acceleration".

Once we have the speed of a particle in terms of its derivative, we can apply it to do the sort of things we've already been doing. So for instance, we can ask how far a dropped object will have fallen after 2.2 seconds. We could calculate this exactly, but we can also approximate:

$$p(2.2\text{s}) \approx p(2\text{s}) + p'(2\text{s})(2.2\text{s} - 2\text{s}) = 20\text{m} + 10\text{m/s}(.2\text{s}) = 22\text{m}.$$

How does all this relate to linear approximation? We know that speed is change in distance over time. Another way of saying that is that our final position is our initial position, plus speed times time.

$$p(t) = p(0) + v_{\text{average}}(t - 0).$$

If our speed varies over time, this isn't terribly helpful: we can only compute average speed by knowing our initial and final position. If we only know our speed "at" each moment, this doesn't work—and making it work precisely involves *integrals*, which we will develop in sections 5 and 6.

But if the length of time is small, we can make a pretty good guess by assuming our speed is constant. Thus we compute our instantaneous speed at time 0, and wee have the approximate formula

$$p(t) \approx p(0) + v_0(t - 0).$$

And this is precisely the linear approximation formula we started with in 2.1.

Remark 2.36. This is basically how we reason about speed in real life. If you're driving fifteen miles and your friend calls you and asks how long you'll take, you might say "Well, traffic isn't too bad; I'm going about 30 miles per hour. So I should be there in about half an hour". This doesn't mean you'll get there in exactly half an hour. Traffic might get better or worse, and you might speed up or slow down. But your best guess of your average speed is your speed right now.

Of course, that's not always your best guess. If you're driving into the city you might know that you're about to hit bad traffic. Or if you can see the end of your traffic jam, you might know you're about to speed up. In either case, this is like having information about the second derivative, and you can refine your guess.

The worst-case version of this thought process is the old Windows download boxes, which would give an estimate of how long a file transfer would take. But this estimate was a simple linear approximation of remaining file size divided by your current download speed—and download speeds would vary wildly from second to second. So you'd see an estimate jump from thirty minutes to two hours to five minutes and back up to forty minutes, all within the space of thirty seconds.

2.7.2 Other Rates of Change

We used this to think about physical speed as we move from one location to another. But the same logic applies to basically any time we have a physical process with change over time. If you know how quickly the output is changing "right now", you can use that to build a linear model of what the output will look like over time. And that means that any rate of change is, fundamentally, a derivative.

Another way of thinking about the derivative is the difference between "stocks" and "flows". If your function measures the *level* or something, then the derivative measures the rate at which the level is changing. If the function measures the amount of something you have in stock, then the derivative measures the rate at which new stock is flowing in or out of your warehouse.

Example 2.37 (Debt and Deficit)**.** A lot of discussions of economics and public policy address the deficit and the debt. The "deficit" and the "debt" are easy to confuse but importantly different, in a way that maps cleanly to the idea of a derivative.

A "debt" is the amount of money that is currently owed; it is measured in dollars (or euro or yen or some other currency). The current US national deficit is approximately $22 trillion.

A "deficit" is the rate at which the debt is increasing. So the national deficit is currently about $1 trillion. This means we expect the debt next year to be about $1 trillion bigger than the debt this year.

Mathematically we can define a function $D(t)$ which takes in the year and outputs the number of dollars owed. Then the annual deficit is

$$\frac{D((t+1)\mathrm{y}) - D(t\mathrm{y})}{1\mathrm{y}}.$$

This isn't a derivative, since there's no limit; this is a *difference quotient* that measures a discrete change in debt over a discrete time. It's analogous to average speed, not instantaneous speed.

But we could imagine asking how the deficit is changing from month to month, or from week to week, or from hour to hour. We can take a limit as the time between $t + h$ and t goes to zero, and then the deficit would be the derivative of debt. The function $D'(t)$ will take in years, and output dollars per year.

What about the second derivative? The function D'' will take in years, and output the yearly change in the deficit, measured in dollars per year per year. When people talk about whether the deficit is going up or down, they are looking at the second derivative of the debt.

Example 2.38 (Inflation)**.** We can make a similar point about inflation, and make fun of Richard Nixon at the same time.

Roughly speaking, inflation is the change in the *price level*, which measures how the value of money changes over time. Thus inflation is a rate of change, and thus a derivative. If we oversimplify and measure the price level as the number of liters of gas you can buy with a dollar, then inflation is measured in liters per dollar per year.

In the seventies, inflation was a major political topic, because inflation was both high and rising. What does it mean to say inflation is rising? That's a *second derivative*. Inflation is the rate at which the price level is changing, but that rate is itself increasing.

In Nixon's reelection campaign, he couldn't say inflation was low, because it wasn't. And he couldn't even say it was falling, because it wasn't. So instead he said that "the rate at which the rate of inflation is increasing is decreasing". That's terrible sentence, even before we unpack it into "the rate at which the rate at which the price level is increasing is increasing is decreasing". (I promise that sentence wasn't me losing control of my keyboard.)

I've heard that this is the only known use of the third derivative in political messaging.

Both of these examples have one very important trait in common. The position function $p(t)$ and the debt function $D(t)$ output different types of things with different units, but they both take *time* as an input. But it's easy for a function to take inputs other than time, and these functions are often physically important and meaningful.

One common place they show up is in economics. Economics cares a lot about so called "marginal" effects.

Example 2.39 (Marginal Revenue)**.** If you're deciding how many machines to buy, what really matters isn't the total cost of the machines and the total revenue they'll make you.

Instead, you need to ask how much more you'll have to spend to get *one* more machine, and how much more revenue that one machine will get you. (This is called "marginal thinking", because we care about the effect of getting one more machine on the margin.)

Any of these marginal effects are implicitly asking for a derivative. So suppose we have some revenue curve where $R(m) = 100m - m^2$: your total revenue is \$100 for every machine, minus upkeep costs of the square of the number of machines you have. So with one machine, you make \$99; with two machines, you make \$196; with ten machines you make \$900. The units of the input is "machines" and the units of the output are "dollars".

We compute $R'(m) = 100 - 2m$; each new machine adds roughly \$100 of revenue, minus 2 times the number of machines you already have. Thus the marginal revenue of the first machine is about \$98, and the marginal revenue of the tenth machine is about \$80. We can see that the fiftieth machine has a marginal revenue of \$0; this is our break-even point, where adding another machine neither helps nor hurts. The sixtieth machine has a marginal revenue of about $-$\$20, and we actually lose money by adding it! The units of this derivative are "dollars per machine"; how many more dollars will you get by adding a machine?

But of course the actual revenue of 50 machines is $R(50) = 5000 - 2500 = 2500$ dollars. The actual revenue of 60 machines is $R(60) = 6000 - 3600 = 2400$ dollars, which is less than $R(50)$ but still positive.

Example 2.40 (Marginal Cost)**.** We also often talk about marginal cost. Suppose the cost of buying m machines is $C(m) = 5000 + 10m + .05m^2$. There's some start-up cost to having any machines at all; then each machine costs a bit more than the previous one. The units of the input are "machines" and the units of output are "dollars".

We can see that $C(1) = 5010.05$, and $C(10) = 5105$. Even $C(100) = 6500$ is not that much bigger than $C(1)$.

The marginal cost would be $C'(m) = 10 + .1m$. We have to pay a huge sum to have any machines at all, but each new machine we add costs only 10 plus a tenth of the number of machines we have. So the cost of adding the hundredth machine is about $C'(100) = 20$, which checks out with the numbers we computed earlier. The units of the derivative are, again, dollars per machine.

This shows a really big separation between marginal and average cost. The total cost of all our machines is really high; if this cost is paired with the revenue from the previous example, we'll continually lose money no matter what we do. But once we've already eaten our sunk costs, the marginal cost of adding one more machine is pretty low, so we should go ahead and get a lot of them.

Example 2.41 (Ohm's Law)**.** In physics and electrical engineering, Ohm's Law tells us that current is equal to voltage over resistance, or $I = V/R$. (Here current is generally measured in amperes, voltage in volts, and resistance in, essentially, volts per amp).

The default assumption in most physics problems is that resistance is constant, a property of whatever material you're putting current through. So we have the function $I(V) = \frac{1}{R}V$, which is a linear function and simple to work with.

But this is just an approximation! Most materials will actually have their resistance change as the voltage applied to them changes, so the equation above is just a linear approximation to the actual relationship between current and voltage. This means that the slope $\frac{1}{R}$ is really a derivative.

An incandescent lightbulb works by running a current through a metal wire until it heats up. But as the heat of the wire increases, the resistance goes up. Thus the graph of current as a function of voltage is curving down; the higher the voltage, the less extra current you get from adding another volt. This means that the derivative $\frac{dI}{dV}$ is large when V is small, but small when V is large.

A diode is a material that does the opposite. Resistance is high when the voltage is low, but past some transition point the resistance drops and becomes very low. This means that the derivative is large when V is small, and then small when V is large. The graph of I as a function of V will curve up.

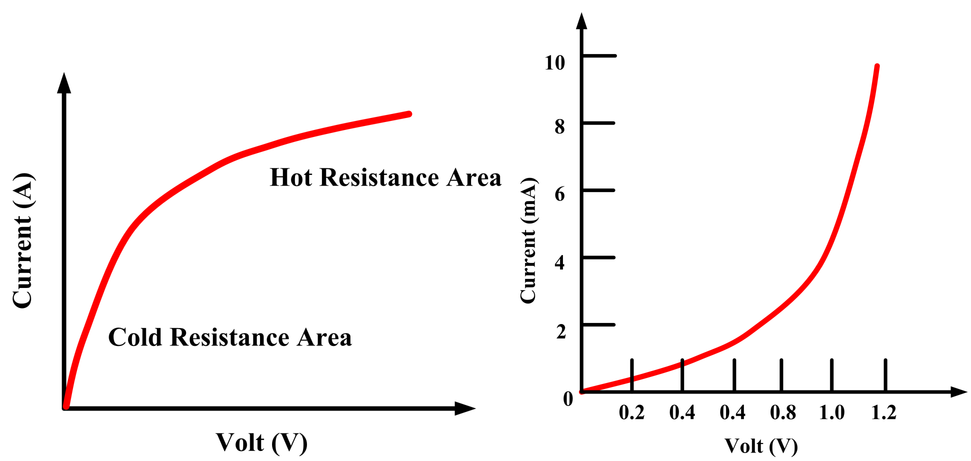


Figure 2.4: Current as a function of resistance for an incandescent bulb filament (left) and a diode (right)
Figures from Nonlinear Resistors — Characteristics Curves of Nonlinear Devices at `https://electricalacademia.com`

In practice. engineers mostly don't want to worry about the whole curve. If they know about what voltage their devices will experience, they don't need to worry what happens in other places. So they take the local linear approximation, call that "the resistance", and use the equation $I = I_0 + \frac{1}{R}(V - V_0)$. And this is just the linear approximation equation we've been using all class.

Example 2.42 (Price Elasticity of Demand)**.** Another common economics question is to see how the demand for a product relates to its price. We can define a function $Q(p)$ that takes in a price in dollars, and outputs the quantity of items that will be bought. So if $Q(p) = 10000 - 10p$, this means that if the price is \$100 then people will buy $Q(100) = 10000 - 1000 = 9000$ widgets.

What's the derivative here? The function $Q'(p)$ takes in a price in dollars and outputs a number of widgets per dollar. It tells you how the quantity demanded changes in response to changes in the price. Thus we see that since $Q'(p) = -10$, we expect to sell ten fewer widgets for each dollar we raise the price.

(Economists call this the Price Elasticity of Demand: "elasticity" is how quickly one thing responds to changes in another thing. So any time the term "elasticity" shows up in economics, there's a derivative inolved somewhere).

What if instead we had the function $Q(p) = 10000 - 5p^2$? Now we see that changing the price doesn't have a huge effect if the price is already small, but it has a dramatic effect if the price is big. We compute that $Q'(p) = -10p$. This means that increasing the price by one dollar will decrease the quantity demanded by ten widgets for every dollar of the price.

Thus if the current price is \$10, we expect raising the price to \$11 to reduce sales by about a hundred widgets. If the current price is \$30 then raising the price will lose us nine hundred widgets in sales.

2.8 Tangent Lines

In this section we'll introduce a third perspective on the derivative. We saw first an *algebraic* perspective, thinking about linear approximation, then a *physical* perspective thinking about rates of change. Now we'll take a *geometric* perspective.

Classically mathematicians were really interested in geometry, which was tied up deeply in questions of philosophy and theology. One obvious-to-them geometric question was to try to find a line *tangent* to the graph of some function.

Definition 2.43. A line that touches a curve at one point without crossing it is *tangent* to the curve at that point, and we call such a line a *tangent line* (from Latin *tangere* "to touch".)

A line crossing a curve in two points is called a *secant* line. (from Latin *secare* "to cut").

Just as the tangent of an angle is the length of a (specific) tangent line segment, the secant of an angle is the length of a (specific) secant line segment.

Suppose we want to find the tangent line to a graph at a point $(a, f(a))$. We need either two points, or a point and a slope. Clearly we have one point. The derivative gives a slope, but why is it the *right* slope?

If we know another point $(b, f(b))$, then we can use the two-points formula to write the equation of a line through those two points:

$$f(x) - f(b) = \frac{f(b) - f(a)}{b - a}(x - a).$$

And this is *almost* the linear approximation formula, since $f'(a) \approx \frac{f(b)-f(a)}{b-a}$. As b gets closer to a, this will get closer and closer to being the linear approximation formula.

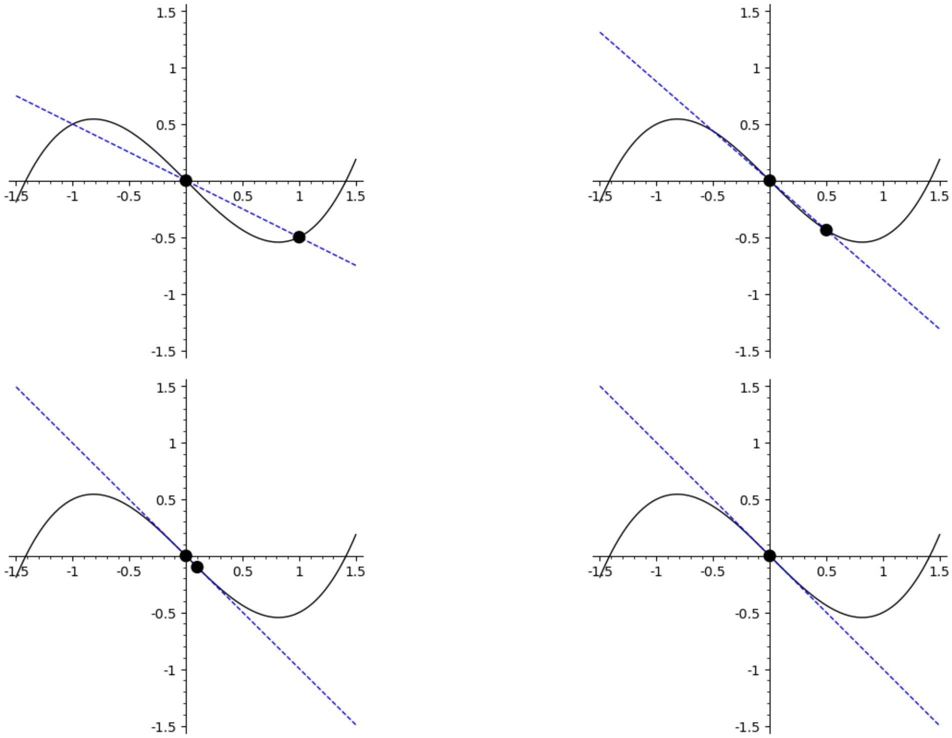
This line through $(a, f(a))$ and $(b, f(b))$ is a secant line. As b gets closer to a, then the two points the secant line goes through get closer together. When we take the limit, our line "goes through the same point twice". Thus it only touches the curve at one point—so it is a tangent line. Thus we see that the linear approximation to a function at a point a is the line tangent at that point a.

Example 2.44. Let $f(x) = \frac{x^3}{2} - x$. We can draw secant lines through the points $(0, f(0))$ and $(b, f(b))$, and see what happens as b gets closer to a. Below, we see the lines for $b = 1, 1/2, 1/10$, and then finally the tangent line given by the linear approximation formula.

We can see that each of the first three lines passes through two points, but as the points get closer and closer together, the secant lines better approximate the tangent line we see in the fourth picture.

We can see that this is, in fact, the same sort of question we asked earlier. The tangent line touches the function graph at one point, and is going in the "same direction" as the graph at that point. Thus it's the line that looks most like the point. So it *should* be the line that best approximates that function. And this is why the geometric tangent line quation is essentially the same as the algebraic linear approximation question.

Example 2.45 (Slope)**.** How can I think of the tangent line as a physical rate of change? If I'm thinking about the graph of a function, then the input to the function is a horizontal position, measured in inches (or some other unit of distance). And the output is a vertical position, also measured in inches. So $f(x)$ takes in inches and outputs inches.



The derivative $f'(x)$ will still take in inches. But if we compute the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, then the denominator is in inches and the numerator is also in inches. This makes the derivative technically unitelss—but in reality, it is measured in inches per inch.

And this has a clear physical interpretation! The slope of a line measures how many units the line goes up for each unit it goes over. Thus, it measures inches of horizontal position per inch of vertical position.

The second derivative $f''(x)$ will take in inches and output 1/inch, which is really inches per inch per inch. It tells us how much the slope, measured in inches per inch, changes if we move one inch horizontally.

Example 2.46.

2.9 Implicit Differentiation

We can push all these ideas about differentiation one step further. This time it makes the most sense to start with the geometric approach, and return to the other two later.

Let's start with a warmup example.

Example 2.47. Consider the curve defined by the equation $x^2 + y = 25$. Can we find a line tangent to this curve at the point $(3, 16)$?

This equation is not written as a function. Recall a function is a rule that takes an input and gives an output. And I haven't described a rule for you. But you can work out a rule that's hidden, or *implicit*, in this equation. A little rearranging gives us

$$\begin{aligned} y &= 25 - x^2 \\ \frac{dy}{dx} &= -2x \end{aligned}$$

and thus the derivative at $x = 3$ is -6. Then the equation for the tangent line is

$$y = 16 - 6(x - 3).$$

Now let's try a hairier example.

Example 2.48. Consider the equation $x^2 + y^2 = 25$, whose graph is a circle of radius 5. Can we find a tangent line to the curve when $x = 3$?

This is trickier, because we can't just reinterpret this equation as a function. We could try, and do something like

$$\begin{aligned} y^2 &= 25 - x^2 \\ y &= \pm\sqrt{25 - x^2}. \end{aligned}$$

But that $\pm$ symbol makes this not a real function. And derivatives are facts about *functions*. So what can we do?

We can't describe the whole circle as a function. But we can describe the top half of it as a function. The formula

$$y = \sqrt{25 - x^2}$$

gives us a perfectly fine function. We can differentiate this to get

$$y' = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}},$$

and thus when $x = 3$ we get $y' = \frac{-3}{4}$. So the equation of our tangent line is

$$y = 4 - \frac{3}{4}(x - 3).$$

But I have two problems with this. The first is simple: why did I take the positive square root and not the negative? It would have been just as valid to look at $y = -\sqrt{25 - x^2}$, and

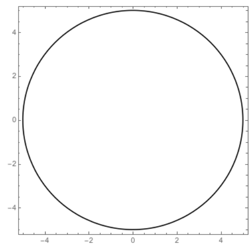


Figure 2.5: The circle $x^2 + y^2 = 25$.

get a derivative of 3/4 and a tangent line of $y = -4 + \frac{3}{4}(x - 3)$. I'd like a method that doesn't force me to make that choice.

The second, bigger problem is that this is too much work, and I'm lazy. The original equation is simple; I don't want to do a ton of work to turn it into something more complicated.

The key *idea* of our argument was that we can find a hidden function that sort of describes our equation. $y = \sqrt{25 - x^2}$ isn't the same as our equation, but as long as we're looking at positive y values, and don't worry too much about what's happening elsewhere, it gives us a good picture. The way I can be lazy now is just to assume that y is *some* function of x. But I won't worry about which function it is, and instead I'll just leave it as a named-but-unspecified function. (This is basically the whole trick of algebra: I don't know what this number is, so let's call it x and move on with our lives.)

If y is a function of x, now we get the equation

$$x^2 + (y(x))^2 = 25.$$

Each side of this equation is a function, and the two functions are the same. And that means that their derivatives are the same. I know the derivative of 25, and the derivative of x^2. I don't really know the derivative of $(y(x))^2$, since I don't even know what $y(x)$ is. But I'll just leave that unspecified again: by the chain rule, we know that

$$\frac{d}{dx}(y(x))^2 = 2y(x) \cdot y'(x).$$

Thus differentiating both sides of our original equation gives

$$2x + 2y(x)y'(x) = 0.$$

This doesn't give us the derivative of y exactly, but it does give us a formula! Rearranging this equation gives

$$2y(x)y'(x) = -2x$$
$$y'(x) = \frac{-2x}{2y(x)} = \frac{-x}{y(x)}.$$

And we get a formula for $y'(x)$ in terms of x and $y(x)$. This might seem like a problem, that I need two numbers to plug in and not just one. But this is actually revealing something deep about the problem. Remember that if $x = 3$, it's possible that $y = 4$ or $y = -4$. If I want to find the slope of the tangent line, I really do need to know which one I'm talking about.

And finally, we can say that if $x = 3$ and $y = 4$, then the derivative is $y'(x) = \frac{-3}{4}$. Which is, of course, what we got earlier.

Remark 2.49. There's one thing to beware of here. What if we look at the point $x = 5, y = 0$? Then our formula would have us dividing by 0, which isn't possible. We can see on the picture that the tangent line would be vertical. But it isn't a function, so the derivative there isn't well-defined.

Basically this is a failure of our idea, that if we zoom in on any point enough, its surroundings will look like a function. No matter how tight our focus, the curve near $(5, 0)$ will never look like the graph of a function, because it will always fail the vertical line test.

Example 2.50 (Folium of Descartes)**.** Let's consider a more complex equation, $x^3 + y^3 = 6xy$. This is known as the Folium of Descartes. We can compute the derivative of both sides:

$$\begin{aligned}
\frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 6\left(y + x\frac{dy}{dx}\right) \\
(3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \\
&= \frac{2y - x^2}{y^2 - 2x}.
\end{aligned}$$

(Notice that I did in fact simplify at the end here. Because I'm about to use this formula to do a bunch of more computations, it's worth it to stop and simplify here to make my life easier.)

Now we can use this formula to find some tangent lines.

At the point $(3,3)$ we compute that

$$\frac{dy}{dx} = \frac{6-9}{9-6} = -1$$

and thus the equation of the tangent line is $y - 3 = -(x-3)$.

At the point $(0,0)$, however, this doesn't actually give us a useful answer; the top and the bottom would both be zero. if you look at the picture in Figure 2.6, you see that there's not a clear tangentline there since the curve crosses itself. You can think of these "self-intersection" points as another way a function can fail to be differentiable, on our earlier list with corners, vertical tangents, and cusps.

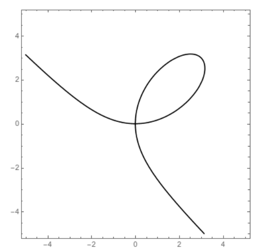


Figure 2.6: The folium of Descartes $x^3 + y^3 = 6xy$

We can also find second derivatives by extending this method. In this problem, we already know that

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

We can differentiate both sides of this. The derivative of the left side is just the derivative of the derivative, which is the second derivative. On the right we can use the quotient rule, so we get

$$\frac{d^2y}{dx^2} = \frac{\left(2\frac{dy}{dx} - 2x\right)(y^2 - 2x) - \left(2y\frac{dy}{dx} - 2\right)(2y - x^2)}{(y^2-2x)^2}.$$

This is okay, but it's a little unsatisfying; I'd like a formula purely in terms of x and y, and this formula also has the $\frac{dy}{dx}$ terms. But I can substitute in my earlier formula for $\frac{dy}{dx}$ and get

$$\frac{d^2y}{dx^2} = \frac{\left(2\frac{2y-x^2}{y^2-2x} - 2x\right)(y^2 - 2x) - \left(2y\frac{2y-x^2}{y^2-2x} - 2\right)(2y - x^2)}{(y^2-2x)^2}.$$

This is a little gross, but it does work. And we can compute now that the second derivative at $(3,3)$ is

$$\frac{d^2y}{dx^2} = \frac{(-2-6)(9-6) - (-6-2)(6-9)}{(9-6)^2} = \frac{-24-24}{9} = \frac{-16}{3}.$$

The exact number here is hard to interpret, but the fact that the second derivative is negative means that the slope of the tangent line decreases as we move to the right, which we can see on the graph.

Example 2.51. If $9x^2 + y^2 = 9$ then we have

$$\begin{aligned} 18x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{9x}{y} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{9x}{y}\right) \\ &= -\frac{9y - 9x\frac{dy}{dx}}{y^2} \\ &= -\frac{9y - 9x(-\frac{9x}{y})}{y^2} \\ &= -\frac{9y + \frac{81x^2}{y}}{y^2} \end{aligned}$$

We see that at the point $(0, 3)$ we have $y' = 0$ and $y'' = -3$. At the point $(\sqrt{5}/3, 2)$, then $y' = -\frac{3\sqrt{5}}{2}$ and $y'' = -\frac{18+\frac{45}{2}}{4}$.

Example 2.52. Find y'' if $x^6 + \sqrt[3]{y} = 1$. Then find the first and second derivatives at the point $(0, 1)$.

$$\begin{aligned} 6x^5 + \frac{1}{3}y^{-2/3}y' &= 0 \\ -18x^5y^{2/3} &= y' \\ -18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}y') &= y'' \\ -18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}(-18x^5y^{2/3})) &= y'' \end{aligned}$$

Thus at $(0, 1)$, we have $y' = 0$ and $y'' = 0$. So the tangent line to the curve is horizontal at the point $(0, 1)$.

So far we've been looking at implicit differentiation as a geometric tool, to find tangent lines. But we can also use it algebraically, on relationships that apply to functions.

Example 2.53. Suppose we have some function f such that $8f(x) + x^2(f(x))^3 = 24$, and we want to find a linear approximation of f near $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Then we have

$$\begin{aligned}\frac{d}{dx}\left(8f(x)+x^2(f(x))^3\right) &= \frac{d}{dx}24\\ 8f'(x)+2x(f(x))^3+3x^2(f(x))^2f'(x) &= 0\\ 8f'(4)+2\cdot 4\cdot 1^3+3\cdot 4^2\cdot 1^2 f'(4) &= 0\\ 8f'(4)+8+48f'(4) &= 0\end{aligned}$$

and thus $f'(4) = -1/7$.

This leaves us with a question, though. We know $f(1)$; can we figure out the value of f at other points?

We have a derivative, so we can again compute a linear approximation. We get

$$f(x) \approx f'(4)(x-4)+f(4) = \frac{-1}{7}(x-4)+1.$$

Thus we compute

$$f(5) \approx \frac{-1}{7}(5-4)+1 = 1+\frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close.

2.10 Related Rates

Finally, let's apply a version of implicit differentiation to physical problems, or word problems.

It's good to take a moment here to talk about why we do work problems, and how to approach them. On a philosophical level, math does not tell us anything about the physical world. It only tells us that if certain properties hold, other things also have to be true. It's our job to take the aspect of the world we care about and translate it into math. Then we can see what the math implies, and hopefully that will still be true when translated back into the world.

Word problems are training for this process. We take verbal (or pictorial etc.) information, and try to turn it into a mathematical description. Then we see the mathematical consequences, and translate those back into a verbal description of physics.

So how do we approach this? **Checklist of steps for solving word problems:**

(a) Draw a picture.

(b) Think about what you expect the answer to look like. What is physically plausible?

(c) Create notation, choose variable names, and label your picture.

  (a) Write down all the information you were given in the problem.

  (b) Write down the question in your notation.

(d) Write down equations that relate the variables you have.

(e) Abstractly: "solve the problem." Concretely differentiate your equation.

(f) Plug in values and read off the answer.

(g) Do a sanity check. Does you answer make sense? Are you running at hundreds of miles an hour, or driving a car twenty gallons per mile to the east?

Example 2.54. A spherical balloon is inflating at 12cm^3/s. How quickly is the radius increasing when the radius is 3cm?

In a perfect world, we want to relate the rate at which the volume is changing to the rate at which the radius is changing. But we don't have any formulas lying around that relate those rates. What we *do* have are formulas that relate the levels: we can relate the "current" volume of the sphere to the "current" radius.

Specifically, we know that a sphere has volume $V = \frac{4}{3}\pi r^3$. Then we can differentiate both sides of that equation: using the chain rule, we compute that

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt}.$$

We know that $\frac{dV}{dr} = 12\text{cm}^3/\text{s}$, and $r = 3\text{cm}$. Substituting those facts in gives us

$$12\text{cm}^3/\text{s} = 4\pi(3\text{cm})^2 r'$$
$$r' = \frac{1}{3\pi}\text{cm/s}$$

So the radius is increasing by $1/3\pi$ cm per second.

Example 2.55. Suppose one car leaves Baltimore at noon, heading due north at 40 mph, and at 1 PM another car leaves Baltimore heading due west at 60 mph. At 2PM, how quickly is the distance between them increasing?

Write a for the distance the first car has traveled, and b for the distance the second car has traveled. We have that $a = 80\text{mi}, b = 60\text{mi}, a' = 40\text{mi/h}, b' = 60\text{mi/h}$. We want a formula that will relate the distances the cars have traveled to the distance between them;

after drawing a picture we see this is the pythagorean theorem $a^2 + b^2 = c^2$, where c is the distance between the two cars.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 2c\frac{dc}{dt} &= 2a\frac{da}{dt} + 2b\frac{db}{dt}. \end{aligned}$$

We can use the pythagorean theorem to tell that $c = 100$, and thus we get

$$\begin{aligned} 2 \cdot (100\text{mi}) \cdot \frac{dc}{dt} &= 2 \cdot (80\text{mi}) \cdot (40\text{mi/h}) + 2 \cdot (60\text{mi}) \cdot (60\text{mi/h}) \\ 200\text{mi}\frac{dc}{dt} &= 6400\text{mi}^2/\text{h} + 7200\text{mi}^2/\text{h} \\ \frac{dc}{dt} &= 68\text{mi/h} \end{aligned}$$

so the distance between the cars is increasing at 68 mph.

The last thing we want to do is ask ourselves if this answer seems basically reasonable. The units are correct; we are expecting the distance to be increasing, so that checks out; and the size of the answer seems basically reasonable, because the cars are traveling at 40 mph and 60 mph and 68 is on the same scale as 40 and 60.

Example 2.56 (Recitation)**.** A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?

Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$\begin{aligned} h^2 + b^2 &= 400 \\ 2hh' + 2bb' &= 0 \\ 2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' &= 0 \\ b' &= \frac{24}{32} = 3/4 \end{aligned}$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Example 2.57 (Recitation)**.** A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

We have $\ell = 5\text{in}, w = 7\text{in}, \frac{d\ell}{dt} = 1\text{in/s}, \frac{dw}{dt} = 2\text{in/s}$. We can relate all our quantities with the formula for the area of a rectangle: $A = \ell w$ relates the area, which we want to know about, to the length and width, which we do know about.

Taking a derivative gives us

$$\begin{aligned}\frac{dA}{dt} &= \ell\frac{dw}{dt} + w\frac{d\ell}{dt}\\ &= 5\text{in} \cdot 2\text{in/s} + 7\text{in} \cdot 1\text{in/s}\\ &= 17\text{in}^2/\text{s}.\end{aligned}$$

The units are right (the rate at which area is changing per second), and the direction is right (the area should be increasing, and this derivative is positive). It's really hard to see if the size is right using our intuition; people in general have bad intuition for the rate at which area changes in response to lengths.

But we can ask what would happen after a full second. One second later, we'd have $\ell = 6\text{in}$ and $w = 9\text{in}$ for a total area of 54in^2. This is an increase of 19in^2 over our starting area of 35in^2, and 17 is a pretty good approximation of 19.

As one final note, this is a problem we've basically seen before, in a different guise. The derivative of the area formula is just the product rule; we saw basically this same picture during the proof of the product rule in section 2.3.

Example 2.58. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the distance between the pole and the tip of his shadow changing when he is forty feet from the pole?

There are actually two ways to set this up. The more obvious is to find an equation that will relate the length of the man's shadow to his distance from the pole, because we know how quickly the man is moving and we want to know how the shadow is changing. We see that we have a similar triangles situation, so if we say that m is the distance between the man and the pole, and s is the length of his shadow, we get the equation

$$\frac{6\text{ft}}{15\text{ft}} = \frac{s}{s+m}.$$

We could differentiate this using the quotient rule, but it's way easier if we collect terms first:

$$\begin{aligned}
6(s+m) &= 15s \\
6m &= 9s \\
6\frac{dm}{dt} &= 9\frac{ds}{dt} \\
\frac{2}{3}\cdot 5\text{ft/s} &= \frac{ds}{dt}.
\end{aligned}$$

So the shadow is growing at a rate of $\frac{10}{3}$ft/s.

However, that is *not* the answer to the question I asked! I don't want to know how fast the shadow is growing; I want to know how fast the tip of the shadow is moving away from the pole. So I need to add $\frac{ds}{dt}$ the rate at which the shadow is growing, to the rate at which the base of the shadow is moving away from the pole, which is $\frac{dm}{dt}$. So my final answer is that the tip of the shadow is moving away from the pole at $(5+10/3)$ft/s $= \frac{25}{3}$ft/s.

But once I realize that $\frac{ds}{dt}$ isn't actually the thing I need to know, maybe I can set the question up differently. Let m be the distance between the man and the pole, and let L be the distance from the pole to the tip of the shadow—which is the thing that we actually care about. We can make the same similar triangles equation, but this time we get

$$\begin{aligned}
\frac{6\text{ft}}{15\text{ft}} &= \frac{L-m}{L} \\
6L &= 15(L-m) \\
15m &= 9L \\
15\frac{dm}{dt} &= 9\frac{dL}{dt} \\
15\cdot 5sift/s &= 9\frac{dL}{dt} \\
\frac{dL}{dt} &= \frac{25}{3}\text{ft/s}
\end{aligned}$$

and thus the distance between the pole and the tip of the shadow is increasing at $\frac{25}{3}$ feet per second.

Example 2.59. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of 2m^3/min. How fast is the water rising when the water level is 3m high in the tank?

We know that we want to relate the height of the water h to the volume of water V. The obvious equation to use here is the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h.$$

But this has a problem; in addition to V and h, this equation includes an r, which we don't know anything about. (The problem gives us a radius, but it's the radius of the *tank*, not the water filling the tank.)

The more naive approach is to plunge boldly ahead. We can take a derivative, and we get

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{3}\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right) \\ 2\text{m}^3/\text{min} &= \frac{\pi}{3}\left(2r\frac{dr}{dt}\cdot 3\text{m} + r^2\frac{dh}{dt}\right), \end{aligned}$$

but we still don't have values for r or $\frac{dr}{dh}$.

We need a new equation, that will relate r to something we already know about. But we know that the water is *in the conical tank*, and should have the same shape. In particular, the sides of our cones are similar triangles! The ratio of the radius of the tank to the height of the tank must be the same as the ratio of the radius of the water to the height of the water. So we get

$$\begin{aligned} \frac{2\text{m}}{4\text{m}} &= \frac{r}{h} \\ r &= \frac{h}{2}\left(3/2\text{m}\right) \\ \frac{dr}{dt} &= \frac{1}{2}\frac{dh}{dt}. \end{aligned}$$

We can substitute this back into our original equation, and we get

$$\begin{aligned} 2\text{m}^3/\text{min} &= \frac{\pi}{3}\left(2\cdot\frac{3}{2}\text{m}\cdot\frac{1}{2}\frac{dh}{dt}\cdot 3\text{m} + \left(\frac{3}{2}\text{m}\right)^2\frac{dh}{dt}\right) \\ &= \frac{\pi}{3}\left(\frac{9}{2}\frac{dh}{dt}\text{m}^2 + \frac{9}{4}\frac{dh}{dt}\text{m}^2\right) \\ &= \frac{9\pi}{4}\text{m}^2\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{8}{9\pi}\text{m/min}. \end{aligned}$$

So we conclude that the water level is rising at $\frac{8}{9\pi}$ meters per minute.

However, while that worked, it was a huge mess algebraically. If we're smart we can do this much more easily. We start with the volume equation

$$V = \frac{1}{3}\pi r^2 h.$$

At this point, we can *notice* that the r will be a problem, so we get rid of it now. We make the similar triangles and see that

$$\frac{2\text{m}}{4\text{m}} = \frac{r}{h}$$
$$r = \frac{h}{2}$$

and substituting that back into the original equation gives

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$
$$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt}$$
$$2\text{m}^3/\text{min} = \frac{\pi}{4}\cdot(3\text{m})^2\frac{dh}{dt}$$
$$\frac{8}{9\pi}\text{m/min} = \frac{dh}{dt}.$$

So we conclude that the water level is rising at $\frac{8}{9\pi}$ meters per minute.

Example 2.60. A lighthouse is located three kilometers away from the nearest point P on shore, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline 1 kilometer from P?

Let's say the angle of the light away from P is θ, and the distance from P is d. Then we have $d = 1$ and $\theta' = 8\pi$ (in radians per minute). We also have the relationship that $\tan\theta = \frac{d}{3}$.

Taking the derivative gives us $\sec^2(\theta)\cdot\theta' = d'/3$. We need to work out $\sec^2(\theta)$, but looking at our triangle we see that the adjacent side is length 3 and the hypotenuse is length $\sqrt{10}$ (by the Pythagorean theorem), so we have $\sec^2(\theta) = (\sqrt{10}/3)^2 = 10/9$.

Thus we have $d' = 3\sec^2(\theta)\cdot 8\pi = \frac{80\pi}{3}$ kilometers per second.

Example 2.61 (Recitation)**.** A kite is flying 100 feet over the ground, moving horizontally at 8 ft/s. At what rate is the angle between the string and the ground decreasing when 200ft of string is let out?

Call the distance between the kite-holder and the kite d and the angle between the string and the ground θ. When the length of string is 200 then $d = \sqrt{200^2 - 100^2} = 100\sqrt{3}$. We have that $d' = 8$ (since the angle is decreasing, the kite must be getting farther away). And finally we have the relationship $\tan\theta = \frac{100}{d}$ by the definition of tan in terms of triangles.

Then we have

$$\begin{aligned}
\tan\theta &= 100d^{-1}\\
\sec^2(\theta)\theta' &= -100d^{-2}d'\\
\theta' &= \frac{-100\cdot 8\cos^2(\theta)}{d^2}.
\end{aligned}$$

We see that $\cos(\theta) = \frac{100\sqrt{3}}{200} = \sqrt{3}2$, so we have

$$\theta' = \frac{-100\cdot 8\cdot 3/4}{(100\sqrt{3})^2} = -\frac{8}{100\cdot 4} = \frac{-1}{50}.$$

So the angle between the string and the ground is decreasing at a rate of $1/50$ per second. (Note: radians are unitless!)

3 Optimization

Let's step back and look at the big picture. In section 2 we defined the derivative, and we also talked about one major application of the derivative: linear approximation, which we saw from many perspectives. We took the algebraic approach of approximating a function, the geometric approach of finding equations for tangent lines, and the physical approach of studying rates of change. We then used implicit differentiation to extend these concepts, and related rates was one more use of the rates-of-change understanding of the derivative.

In this section we're going to look at the *other* major application of the derivative: optimization. And while we'll spend the bulk of this section (until 3.6) working abstractly in the realm of functions and graphs, I want to convince you that we are dealing with a real concrete physical question.

If you're running a factory, you may want to ask how you can make as much money as possible. Or you may want to keep your costs as low as possible. Or, if you're feeling pro-social, you may want to minimize the level of pollution you create.

If you're a biologist studying an ecosystem, you may want to know what the maximum population of wolves you can expect to see is. If you're doing medical research, you may want to know what drug dose will be most effective. If you're a physicist studying the motion of an object, you may want to see where the highest point of its trajectory is, or where it reaches its fastest speed, or what the shortest path it can take is.

All of these questions are problems of *optimization*: we have some function or relationship, and we want to find the maximum (or minimum) value it can take on. And so for the next few sections we'll talk about maximizing or minimizing a function, but we always want to remember that this is potentially a very concrete, practical question.

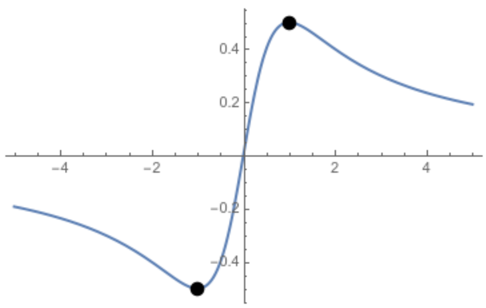
3.1 Extreme Values and Critical Points

So what is it we're looking for? We have a function f, and we want to find the greatest (or least) number it can ever output. If L is the greatest value that f can output, two things need to be true. First, L is actually an output of f; there is some number c such that $f(c) = L$. And second, f never outputs a number that's bigger than that. We can combine those two ideas with the following series of definition.

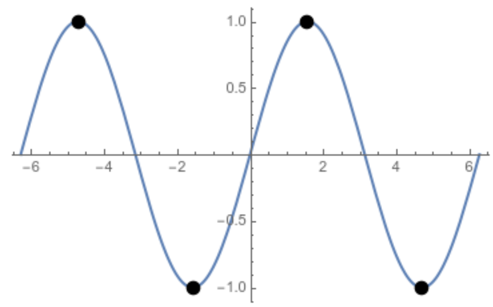
Definition 3.1. If $f(c) \geq f(x)$ for every x in the domain of f, then $f(c)$ is an *absolute maximum* or *global maximum* for f. We say that f has an absolute maximum at c.

Similarly, if $f(c) \leq f(x)$ for every x in the domain of f, then $f(c)$ is an *absolute minimum* or *global minimum* for f, and f has a global minimum at c.

Absolute maxima and absolute minima are somtimes collectively called *extreme values* or *absolute extrema*. ("Extremum" comes from "extreme value," meaning a value that is very big or small or otherwise unusual).

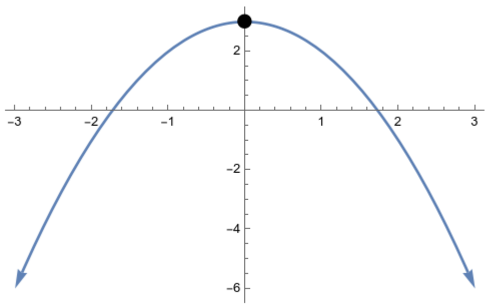


The graph of $\frac{x}{x^2+1}$. This function has a maximum value of $1/2$ which occurs at $x = 1$, and a minimum value of $-1/2$ which occurs at $x = -1$.

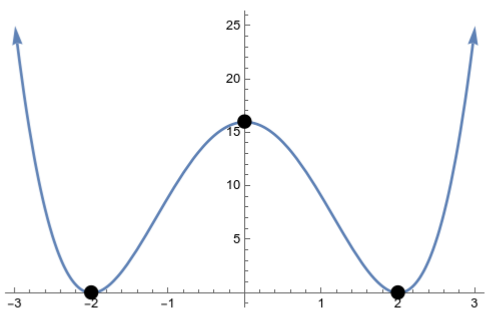


The graph of $\sin(x)$. This function has a maximum value of 1, which occurs at $-3\pi/2, \pi/2, 5\pi/2, \ldots$. Similarly, it has a minimum value of -1, which occurs at $-\pi/2, 3\pi/2, 7\pi/2, \ldots$.

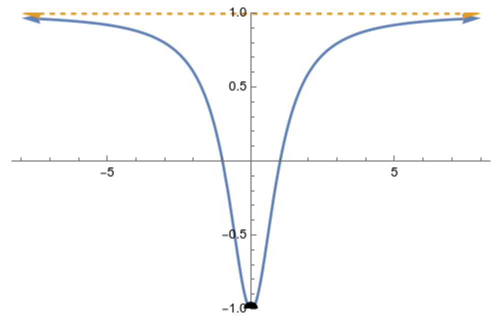
Very important note: the function has *one* maximum value, which occurs in many different places.



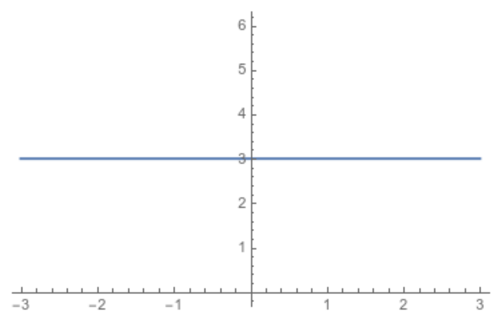
The graph of $3 - x^2$. This function has an absolute maximum of 3, which occurs at zero. It has no absolute minimum: the function goes to $-\infty$



The graph of $x^4 - 8x^2 + 16$. There is a peak in the middle but it's not an absolute maximum, since we get higher values elsewhere. There are relative minima at both -2 and 2, with a value of zero. Again, there is only one absolute minimum: zero. It just happens at two places.

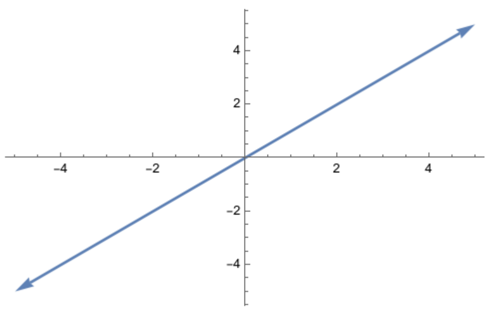


The graph of $\frac{x^2-1}{x^2+1}$. This function has an absolute minimum of -1, which occurs at 0. It has no absolute maximum. We see that the function *almost* reaches a value of 1, but it never actually outputs 1. So 1 can't be the maximum, since it's not an output; but if you pick any number smaller than 1, that can't be the maximum either, because we can always get closer to 1.

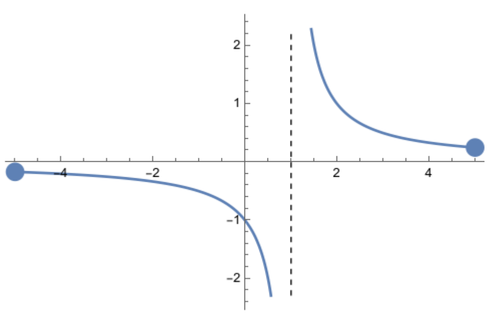


This is a horizontal line, so you might think it doesn't have any maxima or minima. But in fact, the largest value we can get is 3, so the maximum value is three. And the last value we can get is...3. So the minimum value is three. This function has a maximum and a minimum at every single point.

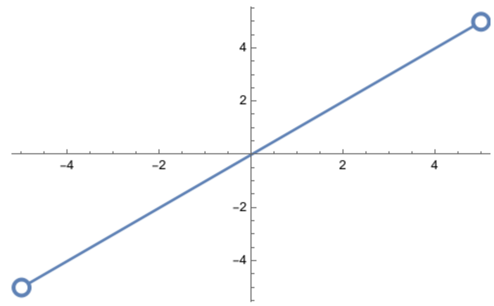
We've noticed that maxima and minima don't necessarily exist. Some of these functions had both a max and a min; some had one but not the other; and some had neither. But if you want to avoid having extrema, there are only a few ways that can happen.



This function $f(x) = x$ has no max or min, because it just keeps going to infinity as x gets bigger or smaller. It's very easy to avoid having extrema if x isn't bounded.

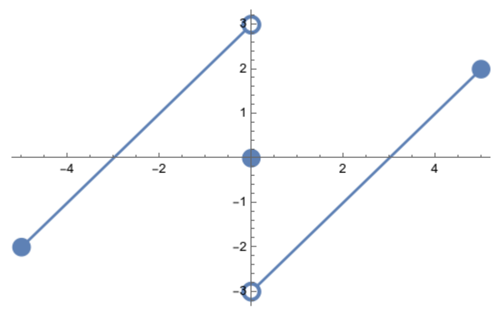


$f(x) = \frac{1}{x-1}$ has endpoints on either side, but there's a discontinuity in the middle where it goes to infinity. This function has no max or min because near 1 the outputs can get arbitrarily large or small.



This is still $f(x) = x$, but it has no max or min for a different reason. We have x bounded by $-5 < x < 5$. And that means our function can get *very close* to 5, but never quite gets there. The number 5 kind of "wants to be" the maximum value here, but there's no way to actually get it as an output.

(If we want to be fancy, we say that 5 is the "supremum" of this function on $(-5, 5)$, and -5 is the "infimum". But you don't need to know those words for this course.)



We get the same problem with just a jump discontinuity. This function has endpoints, and is defined everywhere in $[-5, 5]$. And it stays less than 3 and bigger than -3, but it never actually reaches 3 or -3—it jumps away just before it gets there. So this graph has no maximum or minimum either.

We've seen that we can avoid extrema if our x values are unbounded, or if our function's domain is missing endpoints, or if there's a discontinuity somewhere. But it turns out these are the only ways to avoid having extrema, as stated in the following theorem:

Theorem 3.2 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum $f(c)$ at some point c in the interval $[a, b]$, and an absolute minimum $f(d)$ at some point d in the interval $[a, b]$.*

Note that both the continuity and the closed-ness are important here. I'm not going to try to prove this; it's in a lot of ways the most challenging result to prove in the entire course and depends on advanced ideas like "topological compactness". But looking at the pictures above should convince you there's something here.

Also, this is another "existence theorem", like the Intermediate Value Theorem of section 1.4. It tells us that a global maximum and a global minimum exist, but not anything about where. We can answer this question and find them, but it will require a bit more setup.

If functions are complicated, it's hard to think about the entire function, and find the absolute maximum. So we want to replace this with an easier question. We can look for places where the graph of our function has a peak or a valley, even if it's not the biggest or smallest possible point. This will be much easier to work with, because it allows us to use the tools we've developed already; both limits and derivatives involve focusing on a very small region of the graph, and ignoring everything else.

Definition 3.3. If $f(c) \geq f(x)$ for all x near c, we say that $f(c)$ is a *relative maximum* or a *local maximum* for, and that f has a relative maximum at c.

If $f(c) \leq f(x)$ for all x near c, we say that $f(c)$ is a *relative minimum* or a *local minimum* for f, and that f has a relative minimum at c.

Theorem 3.4 (Fermat's Theorem/Critical Point Theorem)**.** *If f has a local extremum at c, and c is not an endpoint of the domain of f, and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Intuitive idea: If $f'(c) > 0$ then f is increasing, so $f(c + h) > f(c)$ for some small positive h. If $f'(c) < 0$ then f is decreasing, so $f(c + h) > f(c)$ for some small negative h.

To keep things simple, let's suppose f has a local maximum at c, and $f'(c)$ exists. Since $f(c)$ is a local maximum, we know that $f(c) \geq f(c + h)$ for small h, and thus that $f(c + h) - f(c) \leq 0$.

If we take h to be positive, then we can divide both sides by h and we get

$$\frac{f(c+h)-f(c)}{h} \leq 0$$
$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0.$$

But since $f'(c)$ exists, this limit must be $f'(c)$, so $f'(c) \leq 0$.

If we take h to be negative, then dividing both sides of our inequality by h flips the inequality, and we get

$$\frac{f(c+h)-f(c)}{h} \geq 0$$
$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

But since $f'(c)$ exists, this limit must be $f'(c)$, so $f'(c) \geq 0$.

But then $f'(c) \geq 0$ and $f'(c) \leq 0$, so $f'(c) = 0$. □

Remark 3.5.

- The converse of this theorem isn't true: you can have points where $f'(c) = 0$ or $f'(c)$ does not exist that are not local extrema.
- Your textbook uses its words slightly differently, and believes that you cannot have a relative extremum at the endpoint of an interval. I think this is poor word choice, but you should be aware of it when reading the textbook.

Definition 3.6. We say that c is a *critical point* of a function f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Then Fermat's theorem says specifically that if f has a local extremum at c, then c is a critical point. (Again, remember that c can be a critical point without being a local extremum).

Example 3.7.

- Let $f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$; this is defined everywhere, and $f'(x) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$. So the critical points are $\pm\frac{\sqrt{3}}{3}$.
- If $f(x) = x^2$, then $g'(x) = 2x$ and is 0 when $x = 0$. So the only critical point is 0.
- If $h(x) = \sin(x)$ then $h'(x) = \cos(x)$, which is 0 when $x = (n + 1/2)\pi$ for any integer n. Thus the critical points are $\pi/2, 3\pi/2, 5\pi/2, \ldots$.

- If $f(x) = x^3$ then $f'(x) = 3x^2$ which is 0 when x is 0. Thus the only critical point is at 0.

- If $g(x) = |x|$ then
$$g'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ DNE & x = 0 \end{cases}$$
and thus has a critical point at $x = 0$ since the derivative does not exist there.

- If $f(x) = |x^2 - 4|$ then we know that $|x|$ isn't differentiable at 0, so $f(x)$ won't be differentiable at $x^2 - 4 = 0$ and thus at $x = \pm 2$. We see the derivative of the inside is $2x$, so $f'(x) = \pm 2x = 0$ when $x = 0$, and thus the critical points are $0, \pm 2$.

The obvious next question is "how can we determine whether these critical points are a maximum or a minimum or neither?" This is a bit tricky, so we'll hold off for a bit. First we will identify the absolute extrema of a continuous function on a closed interval.

Remember that if f is continuous on $[a, b]$, it must have an absolute maximum and an absolute minimum. By Fermat's theorem, if the absolute extrema are in the interior they must be at critical points. So we can find the absolute extrema by the following method:

(a) List all the critical points.

(b) Evaluate f at each critical point, and at a and b.

(c) The largest value is the maximum and the smallest is the minimum.

Example 3.8.

- If $f(x) = x^3 - x$, we saw the critical points are $\pm\sqrt{3}/3$. If we want the absolute maximum on $[0, 2]$, we compute that $f(0) = 0, f(2) = 6$, and $f(\sqrt{3}/3) = -2\sqrt{3}9$. Thus the absolute maximum is 6 at 2 and the absolute minimum is $-2\sqrt{3}/9$ at $\sqrt{3}/3$.

- Let $h(x) = 2\cos t + \sin(2t)$ on $[0, \pi/2]$. Then $h'(x) = -2\sin(t) + 2\cos(2t) = 0$ when $\sin(t) = \cos(2t)$. On $[0, \pi/2]$ this happens precisely when $x = \pi/6$, so this is the only critical point. We compute $h(0) = 2, h(\pi/2) = 0, h(\pi/6) = 3\sqrt{3}/2$, so the absolute maximum is $3\sqrt{3}/2$ at $\pi/6$ and the absolute minimum is 0 at $\pi/2$.

- Let $f(x) = \frac{x^2+3}{x-1}$ on $[-2, 0]$. Then we see that
$$f'(x) = \frac{2x(x-1) - 1(x^2+3)}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

does not exist at 1. To test when $f'(x) = 0$ we need only consider the numerator, so we have $0 = x^2 - 2x - 3 = (x-3)(x+1)$ and thus $x = 3$ or $x = -1$. So the critical points are $-1, 1, 3$.

f is continuous on $[-2, 0]$ and so must have global extrema. To find them we only need to look at the critical points in $[-2, 0]$, and thus only at -1. So we compute $f(0) = -3, f(-1) = -2, f(-2) = -7/3$. Thus the maximum is -2 (at -1) and the minimum is -3 (at 0).

- What about the global extrema of that same function on $[0, 2]$? We already know the critical points, so we need to check $0, 1, 2$. We have $f(0) = -3$ and $f(2) = 7$, but $f(1)$ is not defined. In fact the function is not defined everywhere on $[0, 2]$ and so not continuous; it has an asymptote at $x = 1$ and thus no minimum or maximum.

We'd still like to determine what each critical point is like, but for that we will need more tools.

3.2 The Mean Value Theorem

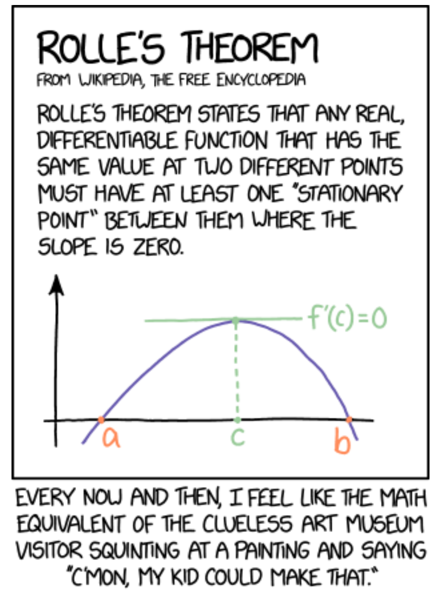
Now we're going to take a brief detour from computation to do something that, in theory, we should have done a while ago: convince ourselves that the derivative actually does what it's supposed to do.

In section 2 we raised a few different questions: how do we approximate a function with a linear function? (2.1) How do we estimate how the output of a function will change when the input changes? (2.7) How can we find an equation for the tangent line to the graph of a function? (2.8) And we argued that all three of these questions should be answered with the same computational tool, the derivative.

But it's important to stop at some point in this process and figure out whether the derivative really does answer those questions. So we're going to take a detour into theory to make sure that the derivative does what we want it to do. Specifically, we're going to pick a few things that should be true if the derivative does what it should, and check that they're actually true of the actual derivative.

And that means that this section will feel a little bit backwards. The things we're going to prove are obviously true. And that's fine! The goal of this section is not to prove those things are true; it's to check that *the derivative*, the one we defined, makes those things true. So we're going to prove that if your top speed is below sixty miles per hour, you can't go more than sixty miles in one hour. That should be obvious; the point is that this works if we interpret your "top speed" as referring to the maximum of the derivative.

Before we can get there, though, we need to start with a somewhat technical result called Rolle's Theorem. Named after the French mathematician Michel Rolle who partially proved it in 1691 without using calculus (which he apparently didn't believe in!), it was fully proven by Cauchy in 1823, and given his name in 1834. This theorem shows that under certain conditions, a function has to have a point where the derivative is zero, and thus the tangent line is horizontal. This isn't especially interesting on its own, although it does have one neat application where we can use it to prove an equation doesn't have too many solutions. But it's easy to prove, and we can leverage it to prove a more powerful and important result.



https://xkcd.com/2042/

Theorem 3.9 (Rolle)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a point c in (a, b) where $f'(c) = 0$.*

Proof. If f is constant everywhere, then the derivative is 0 everywhere.

By the Extreme Value theorem, f has a global maximum on $[a, b]$. If there is some x in (a, b) with $f(x) > f(a)$, then the maximum is in the interior at some point c, and by Fermat's theorem, since $f'(c)$ must exist, we have $f'(c) = 0$.

If f is not constant, and there is no x with $f(x) > f(a)$, then there is some f with $f(x) < f(a)$. Then f has an absolute minimum in the interior at some point c. By Fermat's theorem $f'(c) = 0$. □

Remark 3.10. We need f to be continuous at the endpoints, but it doesn't have to be differentiable there. Rolle's theorem does guarantee a derivative of zero somewhere in the interior–not just at the endpoints.

Example 3.11. If $f(x)$ represents the height of an object, $f'(x)$ represents its speed. If I throw an object up and wait for it to fall back down to the ground, at some point during the process (at the top of its arc) it's instantaneous velocity will be 0.

Example 3.12. We can prove that $f(x) = x^3 + x - 1$ has exactly one real root.

First we use the Intermediate Value Theorem to show that a root exists at all. f is continuous because it's a polynomial. We see that $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by the Intermediate Value Theorem there's some a in $(0, 1)$ with $f(a) = 0$. Thus f has at least one real root.

Now suppose $f(b) = 0$ and $b \neq a$. Then f is continuous and differentiable everywhere, and $f(a) = f(b)$, so by Rolle's theorem there's some c in between a and b with $f'(c) = 0$.

But $f'(c) = 3c^2 + 1$, and since $c^2 \geq 0$, we know that $f'(c) \geq 1$ for every c. Thus there's no c with $f'(c) = 0$, so there's no $b \neq a$ with $f(b) = 0$. Thus f has exactly one real root.

Rolle's theorem can be useful, but it's very limited by the need for $f(a) = f(b)$. The Mean Value Theorem lets us lift that restriction.

Theorem 3.13 (Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), then there's a c in (a, b) with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We prove this using Rolle's theorem, by writing an altered version of f that satisfies the hypotheses of Rolle's theorem. Define

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This is basically just taking $f(x)$ and then subtracting off the line from $(a, f(a))$ to $(b, f(b))$. It's clear that

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 - \frac{f(b) - f(a)}{b - a}0 = 0 \\ h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = (f(b) - f(a)) - (f(b) - f(a)) = 0 \end{aligned}$$

so $h(a) = h(b)$. h is continuous on $[a, b]$ because f is continuous on $[a, b]$, polynomials are continuous, and the sum of two continuous functions is continuous. h is differentiable on (a, b) because f is differentiable on (a, b), polynomials are differentiable, and the sum of two differentiable functions is differentiable.

Thus h satisfies the hypotheses of Rolle's theorem. Then there's some c in (a, b) with $h'(c) = 0$. But

$$\begin{aligned} h'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a}(1 - 0) \\ 0 &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

as we desired. □

Example 3.14. Earlier in the class, we talked about driving to San Diego. That's about 120 miles, so if it takes me two hours to get there, my average speed is 60 mph. That doesn't mean my speed at each point is 60 mph, though; I might go 90 part of the way and then 20 part of the way while I'm stuck in traffic. But the Mean Value Theorem tells me that at some point during that drive the needle on my speedometer pointed at the 60–which makes sense, since it will do that while I'm accelerating up to 90.

Example 3.15. We can also use the mean value theorem to constrain the possible values for a function. For instance, suppose I have a function f, and all I know is that $f(1) = 10$ and $f'(x) \geq 2$ for every x. Then if I want to know about $f(4)$, I can conclude that there is some c in $(1, 4)$, such that:

$$\begin{aligned} f'(c) &= \frac{f(4) - f(1)}{4 - 1} \\ 3f'(c) &= f(4) - 10 \\ f(4) &= 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16. \end{aligned}$$

Thus $f(4) \geq 16$.

Example 3.16. Suppose $|f'(x)| \leq 2$ for all x, and $f(0) = 7$. What do we know about $f(5)$?

We know that for any x, $-2 \leq f(x) \leq 2$. By the mean value theorem, we have

$$\begin{aligned} f'(c) &= \frac{f(5) - f(0)}{5 - 0} \\ -2 \leq &\frac{f(5) - f(0)}{5 - 0} \leq 2 \\ -10 \leq &f(5) - 7 \leq 10 \\ -3 \leq &f(5) \leq 17. \end{aligned}$$

This corresponds to the intuition that if you're travelling less than 2 miles per hour, you won't get more than ten miles in five hours; and if you start at 7, you'll wind up between -3 and 17.

Example 3.17. Show $f(x) = x^5 + x^3 + x$ has exactly one root.

It's pretty clear that f has a root; we could use the intermediate value theorem, but we can also observe that $f(0) = 0$.

Suppose $f(a) = f(b) = 0$. Then by Rolle's Theorem there is some c with $f'(c) = 0$. But $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ and thus $f'(c)$ is never zero; so f has at most one root, and thus exactly one root.

More intuitively, $f(x)$ has at most one root because it's always increasing, and so one it gets above zero it can't come back down and hit zero again. Which leads us to discuss the idea of increasing or decreasing functions.

3.3 Increasing or Decreasing Functions and Finding Relative Extrema

We now want to use the Mean Value Theorem to answer our original question, about which critical points are maxima or minima. We start with a defnition:

Definition 3.18. We say that f is *(strictly) increasing* on an interval (a, b) if, whenever x_1 and x_2 are points in (a, b) and $x_2 > x_1$, then $f(x_2) > f(x_1)$.

We say that f is *(strictly) decreasing* on an interval (a, b) if, whenever x_1 and x_2 are points in (a, b) and $x_2 > x_1$, then $f(x_2) < f(x_1)$.

Notice that these definitions make sense if you assume we're moving to the right; an increasing function is one where $f(x)$ increases as x increases.

Proposition 3.19.

- *If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).*
- *If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).*
- *If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).*

Proof. Let x_1, x_2 be two points in (a, b) with $x_2 > x_1$. Then since f is differentiable (and thus continuous) everywhere in (a, b), it is continuous and differentiable everywhere on $[x_1, x_2]$, and by the mean value theorem there is some c with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$(x_2 - x_1)f'(c) = f(x_2) - f(x_1).$$

- Now, if $f'(x) = 0$ for all x, then $f'(c) = 0$ and thus $f(x_2) - f(x_1) = 0$. This is true for any points x_1 and x_2, and thus f is constant.
- If $f'(x) > 0$ for all x, then $f'(c) > 0$. Since $x_2 - x_1 > 0$, this implies that $f(x_2) - f(x_1) > 0$. This is true for any points $x_1 < x_2$ and thus f is increasing.
- If $f'(x) < 0$ for all x, then $f'(c) < 0$. Since $x_2 - x_1 < 0$, this implies that $f(x_2) - f(x_1) < 0$. This is true for any points $x_1 < x_2$ and thus f is decreasing.

□

Remark 3.20. This theorem doesn't say anything about intervals where f isn't always differentiable. It also doesn't say anything about intervals where f' switches sign in the middle. In practice, we split the domain of our function up into intervals on which exactly one of these things is happening and study each interval separately.

Example 3.21. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Where is f increasing or decreasing?

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ is 0 when $x = 0, -1, 2$. These three points are the critical points. $f'(x)$ has three factors, and it will be positive when one or all three factors are positive. We make a chart:

	$12x$	$x-2$	$x+1$	$f'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Thus $f'(x)$ is positive when $-1 < x < 0$ or $2 < x$, so f is increasing on $(-1, 0)$ and on $(2, +\infty)$. $f'(x)$ is negative when $x < -1$ or $0 < x < 2$, so f is increasing on $(-\infty, -1)$ and $(0, 2)$.

Can we use this information about increasing and decreasing functions to say something about relative maxima and minima? In fact, assuming f is continuous at c, if f is increasing to the left of a point c and decreasing to the right of c, then it must have a maximum at c. Similarly, if f is decreasing to the left and increasing to the right, it must have a minimum. If it increases on both sides or decreases on both sides, then c is neither a maximum nor a minimum. Therefore:

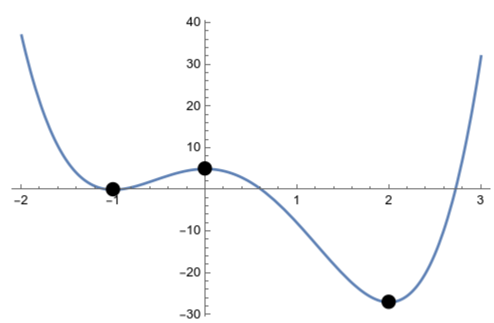


Figure 3.1: a graph of $3x^4 - 4x^3 - 12x^2 + 5$ with critical points marked

Proposition 3.22 (First derivative test for extrema).
If c is a critical point of f and f is continuous at c, then

- *If f' changes from positive to negative at c then f has a relative maximum at c.*
- *If f' changes from negative to positive at c then f has a relative minimum at c.*

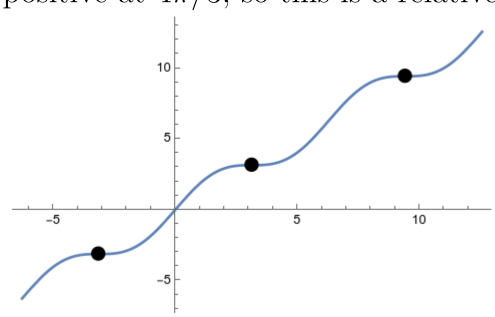
- *If f' "changes" from positive to positive or negative to negative at c then f has neither a relative maximum nor a relative minimum at c.*

Remark 3.23. If f' is continuous, the sign of f' actually only *can* change at a critical point by the intermediate value theorem. So we just have to check the sign of f' at one point in between each critical point.

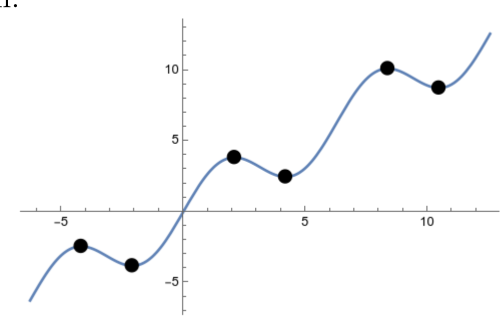
So what does this say about our previous example? We had three critical points, at $-1, 0, 2$. At -1 we saw that f' changed from negative to positive, so f has a relative minimum $f(-1) = 0$ at -1. Similarly, at 0 f' changed from positive to negative and at 2 f' changed from negative to positive, so f has a relative maximum of $f(0) = 5$ at 0 and a relative minimum of $f(2) = -27$ at 2.

Example 3.24. Let $g(x) = x + \sin(x)$. Then $g'(x) = 1 + \cos(x)$ is zero precisely when $x = (2n+1)\pi$ for some integer n. Since we only need to check the sign of g' at one point between each critical point, we check that $g'(2n\pi) = 1 + \cos(2n\pi) = 2$. Thus g' is positive everywhere except at the critical points, so g is increasing everywhere except at the critical points. Thus g has no relative maxima or minima.

Now let $h(x) = x + 2\sin(x)$. We have $h'(x) = 1 + 2\cos(x) = 0$ when $x = 2n\pi + 4\pi/3$ or $x = 2n\pi + 2\pi/3$. We compute that $h'(0) == 3, h'(\pi) = -1$, and $h'(2\pi) = 3$. Thus h' changes from positive to negative at $2\pi/3$, so this is a relative maximum. h' changes from negative to positive at $4\pi/3$, so this is a relative minimum.



Graph of $g(x) = x + \sin(x)$ on $[-2\pi, 4\pi]$ with critical points marked. Notice none of them are relative extrema.



Graph of $h(x) = x + 2\sin(x)$ on $[-2\pi, 4\pi]$ with critical points marked. Every critical point is a relative extremum.

But we'd like to find relative maxima and minima with even less work, which brings us to the subject of concavity.

3.4 Concavity and the Second Derivative Test

Definition 3.25. We say a function f is *concave upward* on an interval (a, b) if every tangent line to a point in (a, b) lies below the graph of f.

We say a function f is *concave downard* on (a, b) if every tangent line to a point in (a, b) lies above the graph of f.

We say a point c is an *inflection point* for a function f if the graph of f changes from concave up to concave down, or concave down to concave up, at c.

Remark 3.26. Functions that are concave upward are curving up, like a bowl. Functions that are concave downward are curving down, like an umbrella.

Example 3.27. Looking at graphs, we can see:

- x^2 is concave upward everywhere. $-x^2$ is concave downward everywhere.
- x^3 is concave downward when $x < 0$ and is concave upward when $x > 0$.
- $\sqrt[3]{x}$ is concave upward when $x < 0$ and concave downward when $x > 0$.
- $\sin(x)$ is concave downward when $0 < x < \pi$ and concave upward when $\pi < x < 2\pi$.

We see that when a function is concave upward, the slopes of its tangent lines are increasing–which means the derivative is increasing. Similarly, a function is concave downward when its derivative is decreasing. But we just showed that we can determine whether a function is increasing or decreasing by looking at its derivative. So we need to study the derivative of the derivative–the second derivative.

Proposition 3.28 (Concavity Test).

- *If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).*
- *If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave downward on (a, b).*

Remark 3.29. It's not necessarily true that f has an inflection point whenever $f''(x) = 0$. But it often is.

Example 3.30.

- $\frac{d}{dx}x^2 = 2x$, so $\frac{d^2}{dx^2}x^2 = 2 > 0$, so x^2 is concave upward everywhere. Similarly, $\frac{d^2}{dx^2} - x^2 = -2, 0$, so $-x^2$ is concave downward everywhere. Neither function has an inflection point.
- $\frac{d^2}{dx^2}x^3 = 6x$ is positive if $x > 0$ and negative if $x < 0$, so the function is concave upward when $x > 0$ and concave downward when $x < 0$. It has an inflection point when $x = 0$.

- $\frac{d^2}{dx^2}\sqrt[3]{x} = \frac{-2}{9\sqrt[3]{x^5}}$ is negative when $x > 0$ and positive when $x < 0$, so the function is concave upward when $x < 0$ and concave downward when $x > 0$. It has an inflection point when $x = 0$.
- $\frac{d^2}{dx^2}\sin(x) = -\sin(x)$, so $\sin(x)$ is concave upwards precisely when it is positive, and concave downwards when it is negative. It has an inflection point at $0, \pi, 2\pi$, and in general at $n\pi$ for any integer n.
- Consider $f(x) = x^4$. $f''(x) = 12x^2$ is positive everywhere except at 0, so the function is concave upwards everywhere except at 0. $f''(0) = 0$, so the second derivative concavity test doesn't tell us anything. But this isn't an inflection point, because the concavity doesn't change on either side–in fact the function is concave at $x = 0$ as well, as you can see from a graph.

Why do we care? Notice that if f is concave upward then the first derivative is increasing; so if $f'(c) = 0$ and f is concave upwards at c, the derivative is changing from negative to positive, and f has a local minimum at c. A similar argument works for local maxima, and thus:

Proposition 3.31 (The Second Derivative Test)**.** *If f'' is continuous near c, then*

- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark 3.32.

- If $f''(c) = 0$ this theorem tells us nothing; almost anything could happen. We can use the increasing/decreasing function test, or we can use the third and fourth derivatives to give us information.
- This rule only works if $f'(c) = 0$; if $f'(c)$ doesn't exist, then $f''(c)$ certainly doesn't exist and this proposition is not helpful.

Example 3.33. We looked at the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in example 3.21. We computed that $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$, so the critical points are $x = -1, 0, 2$.

Then $f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$. We can compute

$$\begin{aligned} f''(-1) &= 12(3 + 2 - 2) = 36 > 0 \\ f(0) &= -24 < 0 \\ f(2) &= 12(12 - 4 - 2) = 72 > 0 \end{aligned}$$

so by the second derivative test, f has a local maximum at 0 and local minima at -1 and 2.

This was a little faster and easier than the way we original classified the maxima and minima of this function. But sometimes the second derivative test just isn't very helpful.

Example 3.34. Let $f(x) = x^{2/3}(6-x)^{1/3}$. Where does f have relative maxima and minima? Where is it increasing or decreasing?

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$
$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}.$$

Then $f'(x) = 0$ when $x = 4$, and $f'(x)$ does not exist when $x = 0$ or $x = 6$, so these are the three critical points.

We can use the second derivative test—or try to. We see that $f''(4) = \frac{-8}{2^{13/3}} = -2^{-4/3} < 0$ so f has a maximum at 4. But at 0 and at 6, the second derivative isn't defined, so the second derivative test isn't useful there.

But we can still use the first derivative test. We get a table:

	$4-x$	$x^{-1/3}$	$(6-x)^{-2/3}$	$f'(x)$
$x < 0$	$+$	$-$	$+$	$-$
$0 < x < 4$	$+$	$+$	$+$	$+$
$4 < x < 6$	$-$	$+$	$+$	$-$
$6 < x$	$-$	$+$	$+$	$-$

This tells us that f has a minimum of $f(0) = 0$ at 0 and a maximum of $f(4) = 2^{5/3}$ at 4. It doesn't have a local maximum or minimum at 6.

But now we can do one more thing. Our table tells us that f is increasing for $0 < x < 4$, and it's decreasing for $x > 0$ or $x > 4$. And further, we can do the same thing for the second derivative. The second derivative is zero, or undefined, at 0 and at 6. So we get

	-8	$x^{-4/3}$	$(6-x)^{-5/3}$	$f'(x)$
$x < 0$	$-$	$+$	$+$	$-$
$0 < x < 6$	$-$	$+$	$+$	$-$
$6 < x$	$-$	$+$	$-$	$+$

So the function is concave down for $x < 6$ and concave up for $x > 6$. We say that $x = 6$ is a *point of inflection* for this function, where the concavity changes. And we can use this information to sketch an effective graph of the function.

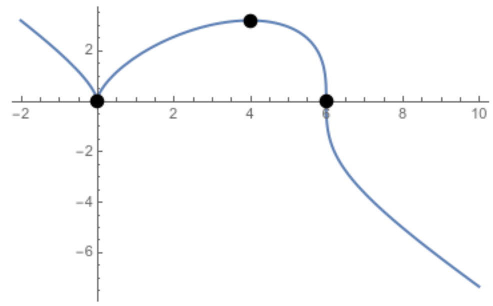


Figure 3.2: The graph of $f(x) = x^{2/3}(6-x)^{1/3}$ with critical points

3.5 Curve sketching

And now we're ready to approach the task of sketching the graph of a function in an organized way. What follows is a good checklist, though not every point is relevant to every function.

(a) Find the domain of the function. If it has holes, what happens near them? Does it go to infinity, or jump, or just skip a point?

(b) Find the roots–where does the function hit the x-axis?

(c) Find the limits as x goes to $\pm\infty$–what happens to the function "far away" from 0?

(d) Compute f' and find the critical points. It can be helpful to evaluate f at the critical points.

(e) Find intervals of increase or decrease. Identify local maxima and minima.

(f) Compute f'' if you haven't already. Determine where the function is concave, and find inflection points.

(g) Use all this information to sketch a graph of the function.

Example 3.35. Let $f(x) = x(x-4)^3 = x^4 - 12x^3 + 48x^2 - 64x$. Then:

(a) The function is a polynomial, so its domain is all real numbers.

(b) The function has roots at 0 and 4.

(c) $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$.

(d) $f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(4x-4) = 4(x-1)(x-4)^2$. So $f'(x) = 0$ when $x = 1$ or $x = 4$. These are the critical points. $f(1) = -27$ and $f(0) = 0$.

(e) Looking at our factorization, it's clear that $f'(x) < 0$ when $x < 1$ and $f'(x) > 0$ when $x > 1$, except $f'(x) = 0$ when $x = 4$. So f is decreasing when $x < 1$ and is increasing when $x > 1$ except at 4. Thus f has a minimum of -27 at 1.

(f) $f''(x) = (x-4)^2 + 2(x-1)(x-4) = (x-4)(3x-6) = 3(x-2)(x-4)$. We see that $f''(x) > 0$ is $x < 2$ or $x > 4$, and $f''(x) < 0$ if $2 < x < 4$. Thus f is concave up on $(-\infty, 2)$ and $(4, +\infty)$, is concave down on $(2, 4)$, and has inflection points at 2 and 4.

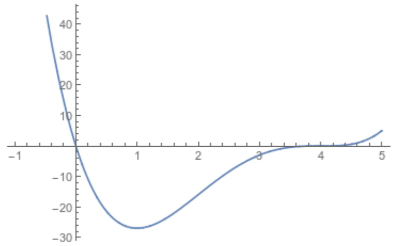


Figure 3.3: The graph of $f(x) = x(x-4)^3$

Example 3.36. Let $g(x) = x\tan(x)$. Then

(a) The domain of g is real numbers except $n\pi + \pi/2$. For simplicity we'll just look at x between $-\pi/2$ and $\pi/2$. $\lim_{x\to-\pi/2^+} g(x) = +\infty$ and $\lim_{x\to\pi/2^-} g(x) = +\infty$.

(b) The function is 0 when $x = 0$ (and when $x = n\pi$ if we look farther out).

(c) This isn't applicable since we're not looking out to $\pm\infty$.

(d) $g'(x) = \tan(x) + x\sec^2(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. It's not hard to see that when $-\pi/2 < x < 0$ then $g'(x) < 0$, and when $0 < x < \pi/2$ then $g'(x) > 0$, and $g'(0) = 0$. So the only critical point is at 0.

(e) And we saw that g is decreasing on $(-\pi/2, 0)$ and increasing on $(0, \pi/2)$. Thus g has a local minimum at 0. $g(0) = 0$.

(f) $g''(x) = \sec^2(x) + \sec^2(x) + 2x\sec(x)\sec(x)\tan(x) = 2\sec^2(x)(1 + x\tan(x))$. $x\tan x \geq 0$ on $(-\pi/2, \pi/2)$, so the function is concave up everywhere.

Example 3.37. Let $h(x) = \frac{x+2}{x-1}$.

(a) The domain of h is all real numbers except 1. We see that $\lim_{x\to1^-} h(x) = -\infty$ and $\lim_{x\to1^+} h(x) = +\infty$.

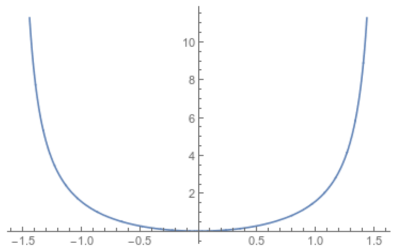


Figure 3.4: The graph of $g(x) = x\tan(x)$

(b) The function has a root at $x = -2$.

(c) We have $\lim_{x \to +\infty} h(x) = \lim_{x \to -\infty} h(x) = 1$. (We can use L'Hôpital's rule or divide the top and bottom by x).

(d) We have $h'(x) = \frac{(x-1)-(x+2)}{(x-1)^2} = -3(x-1)^{-2}$. This has no roots and fails to exist when $x = 1$. Thus there are no "real" critical points.

(e) We make a chart for increase and decrease:

	-3	$(x-1)^{-2}$	$h'(x)$
$x < 1$	$-$	$+$	$-$
$1 < x$	$-$	$+$	$-$

Thus h is decreasing everywhere. It has no local maxima or minima.

(f) $h''(x) = 6(x-1)^{-3}$ is positive when $x > 1$ and negative when $x < 1$, so it is concave down on the left, and concave up on the right.

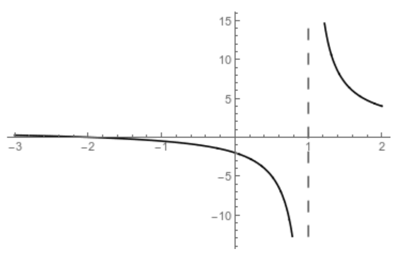


Figure 3.5: The graph of $h(x) = \frac{x+2}{x-1}$

3.6 Physical Optimization Problems

Through most of this section we've been finding the minimum and maximum values of functions purely to understand the functions. But the techniques used to maximize a function are extremely useful in finding optimum inputs to real world processes.

In other words, we're going to do more word problems.

Example 3.38. Suppose we have 2400 feet of fencing and we'd like to build a rectangular fence that encloses the most possible area. How can we do this?

If we have a rectangular fence, then one side will have a length L and another will have a width W. We know that the area $A = W \cdot L$ and that $2W + 2L = 2400$. So we can write $W = 1200 - L$ and see that $A = L(1200 - L)$. We'd like to maximize area.

We observe that our L has to be between 0 and 1200, so we're maximizing the function A on the closed interval $[0, 1200]$. By the extreme value theorem there must be some absolute maximum.

$A' = 1200 - 2L$. We see that the only critical point is $L = 600$. $A(0) = A(1200) = 0$ and $A(600) = 600^2 = 360,000$. $A(600)$ is the largest of these values, and so is the absolute max.

But what if we build the fence against a river, so we only need to build three sides? Then $A = W \cdot L$ but $W + 2L = 2400$, and thus $W = 2400 - 2L$. Then we have $A = L(2400 - 2L)$. A is still a function of L defined on $[0, 1200]$, and we compute $A' = 2400 - 4L$ and the only critical point is $L = 600$, again. $A(0) = A(1200) = 0$, and $A(600) = 600 \cdot 1200 = 720,000$. This last is the largest of the values, and the absolute max.

Example 3.39. Suppose we want to construct a cylindrical can that holds one liter of liquid, and we want to use the least possible metal to construct the can—and thus build the can with the least possible surface area. We have $A = 2\pi r^2 + 2\pi r h$.

To eliminate the h, we note that the can holds one liter or 1000 cm^3, and thus $\pi r^2 h = 1000$ and $h = \frac{1000}{\pi r^2}$. (We also could have written it as one cubic decimeter, but nobody ever works in decimeters). Thus we have $A = 2\pi r^2 + \frac{2000}{r}$.

$A' = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = 0$ when $\pi r^3 = 500$, or when $r = \sqrt[3]{500/\pi}$. So this is the only critical point. Our function A has domain $(0, +\infty)$ so we can't use the extreme value theorem here. But we can see that A' is negative when $r < \sqrt[3]{500/\pi}$ and positive when $r > \sqrt[3]{500/\pi}$, so that must be a global minimum.

(Alternatively: $A'' = 4\pi + \frac{4000}{r^3}$ is always positive, so A is concave upwards everywhere, and has a unique minimum at its critical point).

But now what if the curved material for the sides costs more than the flat material for the ends, and we want to minimize cost? Say the material for the sides costs twice as much as material for the base. Then we have $C = 2\pi r^2 + \frac{4000}{r}$, and $C' = 4\pi r - \frac{4000}{r^2} = 0$ when $\pi r^3 = 1000$, when $r = 10/\sqrt[3]{\pi}$. This is the only critical point, and a similar argument to before shows it must be a global minimum.

We can break down our approach to these problems just as we did for related rates.

(a) Draw a picture of the setup.

(b) Create notation. Give names to all the quantities involved in the problem. Write down any equations that relate them.

(c) Express the quantity you want to maximize or minimize as a function of the other quantities in the problem. Rewrite it so it's a function of a single variable.

(d) Take the derivative and find the critical points.

(e) Determine the absolute maximum or minimum.

(f) Do a sanity check! Does your answer make sense?

Example 3.40. If we have 1200 cm^2 of cardboard to make a box with a square base and an open top, what is the largest possible volume of the box?

Well, we know that the total surface area of the box is $A = 1200$, and we also know that if the height of the box is h and the length of one of the base sides is b, then the area is $A = b^2 + 4bh$. So we can write $h = \frac{1200-b^2}{4b}$. We also know that the volume of the box is $V = b^2h$, so we have

$$\begin{aligned} V &= b^2h = b^2\frac{1200 - b^2}{4b} \\ &= 300b - b^3/4 \\ V' &= 300 - 3b^2/4 \\ 300 &= 3b^2/4 \\ 400 &= b^2 \\ 20 &= b \end{aligned}$$

so the only critical point occurs at 20. We see that $V(20) = 400 \cdot 10 = 4000$, so this is the largest possible volume of the box. (We can see that this is the absolute maximum via the Extreme Value Theorem, and observing that $V(0) = V(\sqrt{1200}) = 0$.

Example 3.41. Suppose a man wishes to cross a 20 m river and reach a house on the other side that is 48m downstream. The man can walk at 5 m/s or swim at 3 m/s. What is the optimal path for him to take to reach the house?

The man will swim for some point on the bank of the river, and then walk the other way. Let b be a number in $[0, 48]$ representing how far he travels towards the house. Then he travels $\sqrt{400+b^2}$ meters in the river, at a speed of 3 m/s, and thus spends $\frac{1}{3}\sqrt{400+b^2}$ seconds in the river. He then spends $(48-b)/5$ seconds walking.

So total time spent is

$$\begin{aligned}
T &= \frac{\sqrt{400+b^2}}{3} + \frac{48-b}{5} \\
T' &= \frac{b}{3\sqrt{400+b^2}} - \frac{1}{5} \\
\frac{1}{5} &= \frac{b}{3\sqrt{400+b^2}} \\
3\sqrt{400+b^2} &= 5b \\
3600 + 9b^2 &= 25b^2 \\
225 &= b^2 \\
15 &= b
\end{aligned}$$

so we have a critical point at $b = 15$. On this path we have $T = 25/3+33/5 = (125+99)/15 = 224/15 \approx 14.9$ seconds.

What about the two other paths? If we head straight to the house, we travel $\sqrt{48^2+20^2} = 52$ meters at a speed of 3 m/s, for a total time of 17.3 seconds. If instead we head straight across the river to begin walking as soon as possible, we travel 20 m at 3 m/2 and then 48 m at 5 m/s, for a total time of $20/3 + 48/5 = (100+144)/15 = 244/15 \approx 16.3$ seconds. So the shortest path has us swim 25 m and deposits us 33 m from the house.

Example 3.42. A piece of wire 10 m long is going to be cut into two pieces. We will fold one piece into a square and the other into an equilateral triangle. What is the largest joint area we can enclose? What is the smallest?

Let L be the length of the wire bent into a triangle (so that $10 - L$ is the length of the wire bent into a square). Then the area of the square is $A_1 = (10-L)^2/16$. The area of the triangle is $bh/2$; the length of the base is $L/3$ and the height of the triangle is $\sin(\pi/3) \cdot L/3 = (1/2) \cdot (\sqrt{3}/2) \cdot L/3 = \sqrt{3}L/12$. So the area of the triangle is $A_2 = (1/2)(L/3)(\sqrt{3}L/6) = L^2\sqrt{3}/36$. Then we have

$$\begin{aligned}
A &= A_1 + A_2 = (100 - 20L + L^2)/16 + L^2\sqrt{3}/36 \\
A' &= -5/4 + L/8 + L\sqrt{3}/18 \\
5/4 &= L/8 + L\sqrt{3}/18 \\
90 &= 9L + 4\sqrt{3}L \\
L &= 90/(9+4\sqrt{3})
\end{aligned}$$

This is the only critical point. At that point,

$$A \approx 1.2 + 1.5 = 2.7.$$

But checking the endpoints, if we use all the wire for the square, we have area $A = 100/16 = 6.25$ and if we use all the wire for the triangle we have $A = 100\sqrt{3}/36 \approx 4.8$. So we get the biggest area when we use all the wire for the square, and the smallest if we use $90/(9+4\sqrt{3})$ m of wire for the triangle.

4 Interlude: Approximation

This section is a bit of an interlude; it'll be a short bridge between section 3 on optimization, and section 5 on integration.

In this section we want to talk a bit more about the idea of approximation. We introduced this in section 4, when we talked about continuous approximation: if $x \approx a$, we can estimate $f(x) \approx f(a)$. We refined this a bit in section 2.1 and 2.6. The derivative allows us to estiamte that $f(x) \approx f(a) + f'(a)(x-a)$. But can we do even better?

4.1 Quadratic Approximation

In this class we've spent a lot of time on *linear approximation*: we can approximate a function with its tangent line, which is the linear function most similar to our starting function. This simplifies a lot of things, but is only an approximation.

$$f(x) \approx f(a) + f'(a)(x-a). \tag{2}$$

How good this approximation is depends on two things. The first is the distance $|x-a|$; the approximation is better when your goal point x is close to your starting point a. There are other techniques (like Fourier series) that don't have this limitation, but we won't discuss them in this course.

The other is the speed at which the derivative changes. If the derivative is constant, your function is just a line and the "approximation" is perfect. But the faster the derivative changes, the faster the function deviates from the line.

Thus we might try to get a better approximation using the second derivative, which tells us how quickly the derivative is changing. So how can we do this?

We're looking for some function $g(x)$ so that

$$f(x) \approx f(a) + f'(a)(x-a) + g(a)(x-a)^2.$$

(We want the linear approximation to be the same as (4), and we want the third derivative to be zero, so the only thing that can change at all is the degree two term). Taking derivatives of both sides gives us

$$\begin{aligned} f'(x) &\approx f'(a) + 2g(a)(x-a) \\ f''(x) &\approx 2g(a). \end{aligned}$$

Thus we set $g(a) = f''(x)/2$, and we get the equation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2. \tag{3}$$

This is the *parabola* that best approximates our function near a.

Example 4.1. Let's again ask our old question: what is $\sqrt{5}$?

We use the function $f(x) = \sqrt{x}$ and we compute $f'(x) = \frac{1}{2\sqrt{x}}$ and $f''(x) = \frac{-1}{4\sqrt{x^3}}$. Then we have

$$\begin{aligned}
f'(4) &= \frac{1}{4} \\
f''(4) &= \frac{-1}{32} \\
f(x) &\approx f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \\
f(5) &\approx 2 + \frac{1}{4} - \frac{1}{64} = 2 + \frac{15}{64} \approx 2.23483.
\end{aligned}$$

We see we've slightly overcorrected: rather than being .014 too big, we're now .0012 too small.

Example 4.2. Compute the quadratic approximations of $\sin(x)$ and $\cos(x)$ centered at zero. Estimate $\sin(.01)$ and $\cos(.01)$? How does this relate to the Small Angle Approximation?

$$\begin{aligned}
\sin'(x) &= \cos(x) \\
\sin'(0) &= 1 \\
\sin''(x) &= -\sin(x) \\
\sin''(0) &= 0 \\
\sin(x) &\approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x \\
\sin(.01) &\approx .01.
\end{aligned}$$

Recall the small angle approximation told us that $\sin(x) \approx x$. Here we see that this is not just a linear approximation, but in fact also the quadratic approximation; the reason the small angle approximation worked so well is that it was correct ot second order.

$$\begin{aligned}
\cos'(x) &= -\sin(x) \\
\cos'(0) &= 0 \\
\cos''(x) &= -\cos(x) \\
\cos''(0) &= -1 \\
\cos(x) &\approx 1 + 0(x-0) - 1(x-0)^2 = 1 - \frac{x^2}{2} \\
\cos(.01) &\approx .99995.
\end{aligned}$$

Example 4.3. Let $g(x) = x^4 - 3x^3 + 4x^2 + 4x - 2$. Compute the quadratic approximations at $a = 0$ and at $a = -2$. Compare them to $g(x)$. Estimate $g(-1.97)$.

$$\begin{aligned}
g(0) &= -2 \\
g'(x) &= 4x^3 - 9x^2 + 8x + 4 \\
g'(0) &= 4 \\
g''(x) &= 12x^2 - 18x + 8 \\
g''(0) &= 8 \\
g(x) &\approx -2 + 4(x-0) + \frac{8}{x}x^2 = 4x^2 + 4x - 2.
\end{aligned}$$

Notice that this is just the lower-degree terms of our original polynomial!

$$\begin{aligned}
g(-2) &= 16 + 24 + 16 - 8 - 2 = 46 \\
g'(x) &= 4x^3 - 9x^2 + 8x + 4 \\
g'(-2) &= -32 - 24 - 16 + 4 = -80 \\
g''(x) &= 12x^2 - 18x + 8 \\
g''(-2) &= 48 + 36 + 8 = 92 \\
g(x) &\approx 46 - 80(x+2) + 46(x+2)^2 \\
f(-1.97) &\approx 46 - 80(.03) + 46(.009) = 43.6414.
\end{aligned}$$

However, if we take $h(x) = 4x^2 + 4x - 2$ and approximate near -2, we get

$$\begin{aligned}
h(-2) &= 6 \\
h'(x) &= 8x + 4 \\
h'(-2) &= -12 \\
h''(x) &= 8 \\
h''(-2) &= 8 \\
h(x) &\approx 6 - 12(x+2) + 4(x+2)^2 = 6 - 12x - 24 + 4x^2 + 16x + 16 \\
&= 4x^2 + 4x - 2 = h(x).
\end{aligned}$$

No matter where we center our approximation, the best quadratic approximation to our parabola is our original parabola.

Example 4.4. Now let's estimate 1.01^{25} using a quadratic approximation. We use the function $f(x) = (1+x)^{25}$, and center our approximation at $x = 0$. (Equivalently we could consider $g(x) = x^{25}$ and center our approximation at $x = 1$; the way I set it up is a bit more common).

We take $f'(x) = 25(1+x)^{24}$ so $f'(0) = 25$, and $f''(x) = 25 \cdot 24(1+x)^{23}$ so $f''(0) = 25 \cdot 24 = 600$. Then we have

$$\begin{aligned}
f(x) &\approx 1 + 25(x-0) + \frac{600}{2}(x-0)^2 = 1 + 25x + 300x^2 \\
1.01^{25} = f(.01) &\approx 1 + 25 \cdot .01 + 300 \cdot .0001 = 1 + .25 + .03 = 1.28.
\end{aligned}$$

Since $1.01^{25} \approx 1.28243$ this is pretty good.

What if we move a bit farther? If we want to estimate 1.04^{25} we get

$$1.04^{25} = f(.04) \approx 1 + 25 \cdot .04 + 300 \cdot .0016 = 1 + 1 + .48 = 2.48$$

while $1.04^{25} \approx 2.66584$. We've lost fidelity because our move away is bigger.

But while .4 is still much smaller than 1, this estimate is much worse than our estimate of $\sqrt{5}$ from earlier. Why is this much worse? Linear are bad for two reasons: either because x and a are far apart, or because the second derivative is large. Here we've taken care of the second derivative, but we haven't taken care of everything. Our quadratic approximations will be bad when the *third* derivative is large.

Finally, let's use this to estimate 2^{25}. We get

$$2^{25} = f(1) \approx 1 + 25 \cdot 1 + 300 \cdot 1^2 = 326.$$

But $2^{25} = 33,554,432$, so this is very far off. We see here even more problems with the largeness of the higher derivatives.

4.1.1 Cubics and Beyond: Taylor Series

We can carry this logic further. We can work out that if we want to match the first *three* derivatives and get a cubic approximation, we get the formula

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3.$$

More generally, we can get a degree-n polynomial approximation, called the *Taylor polynomial of degree n*, with the formula

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If a function is infinitely differentiable, we can take an infinite sum here and get the *Taylor series*:

$$T_f(x,a) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Most functions we're interested in are equal to their own Taylor series. (Not all functions are, though!) In particular, we can work out the following formulas:

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \ldots \\ \cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots. \end{aligned}$$

Taylor series are extremely important in any sort of computational or advanced math, and you will talk about them a lot more if you take Calculus II.

However, in practice, just like we rarely use third or fourth derivatives, we rarely use approximations of degree higher than two. If the quadratic approximation doesn't pick up whatever you need to think about, we will do something else entirely.

4.2 Iterative Approximation: Newton's Method

In section 2.6 we saw that there were two things that make a linear approximation work better or worse. The first was the size of the second derivative; in section 4.1 we leveraged the second derivative to improve our approximations.

To keep things simple, we'll assume that we want to solve $f(x) = 0$. (If not, we can just subtract our number y from both sides of the equation). If we know the value of f and of f' at a point x_0, then recall that by linear approximation we estimate that $f(x_1) =$

$f(x_0) + f'(x_0)(x_1 - x_0)$. Since we want $f(x_1) = 0$, we set $f(x_1) = 0$ and solve this equation for x_1, and get

$$x_1 = x_0 - (f(x_0)/f'(x_0)).$$

In many conditions, we will get the result that x_1 is closer to being a root of f than x_0 is.

We can repeat this process to find x_2, x_3, etc., and ideally each will be a better estimate than the previous estimate was. A good rule of thumb for when to stop: if you want five decimal places of accuracy, you can stop when the nth step and the $n+1$st step agree to five decimal places.

This method does have limitations. First, we have to start with a guess x_1 for our root x. Second, if $f'(x_1)$ is very close to zero, Newton's method will work poorly if it works at all, and we might have to pick a better guess. But it can be very useful for finding approximate solutions to equations.

Example 4.5. Let's approximate the square root of 5, one more time. First, we need to turn this into finding a solution to an equation. We want to solve the equation $x^2 = 5$, which we can rewrite as $f(x) = x^2 - 5 = 0$. We compute $f'(x) = 2x$.

We need to pick a starting estimate, which should probably be $x_0 = 2$. Then we have $f(x_0) = -1$, and $f'(x_0) = 4$. So we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{4} = 9/4 = 2.25.$$

You might notice that this is exactly what we got by doing a simple linear approximation. So what did we get from this new method? Now we can *iterate*.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 9/4 - \frac{81/16 - 5}{9/2} = 161/72 \approx 2.23611 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 161/72 - \frac{1/5184}{161/36} = \frac{51851}{23184} \approx 2.23607 \end{aligned}$$

Checking with a computer tells us that $\sqrt{5} \approx 2.23607$, so we're now correct to five decimal places.

Example 4.6. Let's find a solution to $x^3 - x = 1$. We need to write this as $f(x) = 0$, so let's take $f(x) = x^3 - x - 1$. Then we have $f'(x) = 3x^2 - 1$, and we can guess $x_0 = 1$ as a decent starting point, since $f(1) = -1$ is close to 0. Then we have

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{2} = 3/2$$
$$x_2 = \frac{3}{2} - \frac{f(3/2)}{f'(3/2)} = \frac{3}{2} - \frac{27/8 - 3/2 - 1}{27/4 - 1} = 31/23 \approx 1.34783$$
$$x_3 = \frac{31}{23} - \frac{f(31/23)}{f'(31/23)} = \frac{31}{23} - \frac{1225/12167}{2354}529\frac{71749}{54142} \approx 1.3252.$$

We can notice a couple of things here. the first is that the numerators $f(x_i)$ are getting closer and closer to zero. This is what we should expect: we're trying to get closer and closer to a root of f.

Second, each successive step is smaller. From x_0 to x_1 we change by .5; from x_1 to x_2 we change by about 1.5; from x_2 to x_3 we change by about .02, which means we're probably within .02 of the true answer at x_3.

Example 4.7. Suppose we want to find a solution to $x^5 + x^2 + x - 1 = 0$. If we take $f(x) = x^5 + x^2 + x - 1$, then $f(0) = -1$ and $f(1) = 2$ so there must be at least one solution to this equation. But a result from the field of Galois theory tells us that we cannot express the solution exactly.

However, we can use Newton's method. $f(0) = -1$ so it seems reasonable to start with 0 as a guessed root. We compute $f'(x) = 5x^4 + 2x + 1$, and so if $x_0 = 0$ we have

$$x_1 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$
$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2}{8} = \frac{3}{4}$$
$$x_3 = \frac{3}{4} - \frac{f(3/4)}{f'(3/4)} \approx .75 - \frac{563/1024}{1045/256} = \frac{643}{1045} \approx .615311.$$

If we keep going, we see the true root is about $x = .586544$.

5 Integration

5.1 The Area Problem

For the next month, we will primarily be occupied by the question of *area*.

What is area? This actually gets a little fuzzy. We know how to compute the area of a rectangle: base times height. From that fact, and drawing a quick picture, we know the area of triangle: $\frac{1}{2}bh$, since it's half a rectangle.

We also know the area of a circle. But how? What about an ellipse? Or something funny-looking and squiggly? What does "area" mean, exactly, in these cases?

To measure the area of a shape, we can try filling it up with small squares or rectangles—we know how to measure those. (Similar principle: if you need to measure the length of something curved, run a string along it, straighten it out, measure the string. This idea will reappear in Calculus 2.)

We're going to make our lives easier, and assume our shape has one straight side. (This isn't as strict a condition as it seems; we can always cut our shape in half. We'll talk more about that in section 6.1). In fact, let's look at shapes that are given by graphs of functions.

We want to find the area of the shape "under" the graph. For right now we'll assume the function is always positive, so we get an actual area of an actual shape. (We'll relax that assumption very soon).

When we were trying to get areas earlier, we used a lot of rectangles. We can fill this area with rectangles in a bunch of different ways. But one particular way turns out to work very well, which is to have a bunch of tall skinny rectangles.

So what's the area of these rectangles? If a rectangle goes from a to b, then its width is $b-a$. How tall is it? That depends on where we put the top. There are a few things we can do, but the easiest is to make one of the top corners lie exactly on the graph. If we pick the right corner, then the width is $(b-a)f(b)$.

Example 5.1. Let's find the area under the curve $y = x^2$, between 0 and 1. If we use just one rectangle, with width 1, then we get either 0 or 1. This is true, but not super helpful.

Let's try two rectangles. They each are $\frac{1}{2}$ wide. If we line up the right-hand corners, then the area of thefirst one is $\frac{1}{2} \cdot \frac{1}{2}^2 = \frac{1}{8}$, and the area of the second one is $\frac{1}{2} \cdot 1^2 = \frac{1}{2}$. We get a total area of $\frac{5}{8}$.

What if we used the left-hand corners instead? Then the first rectangle is $\frac{1}{2} \cdot 0^2 = 0$ and the second is $\frac{1}{2} \cdot \frac{1}{2}^2 = \frac{1}{8}$. So the "true" area is somewhere between $\frac{1}{8}$ and $\frac{5}{8}$.

Let's get skinnier. If we use four rectangles, then with the right-hand point, we get

$$A_R \approx \frac{1}{4}\cdot\frac{1}{4}^2 + \frac{1}{4}\cdot\frac{1}{2}^2 + \frac{1}{4}\cdot\frac{3}{4}^2 + \frac{1}{4}\cdot 1^2 = \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} = \frac{30}{64} = \frac{15}{32},$$

and if we line up the left-hand point instead, we get

$$A_L \approx \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\frac{1}{4}^2 + \frac{1}{4}\cdot\frac{1}{2}^2 + \frac{1}{4}\cdot\frac{3}{4}^2 = 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} = \frac{14}{64} = \frac{7}{32}.$$

So the "true" area is between $\frac{7}{32}$ and $\frac{15}{32}$.

Notice that as we draw more rectangles, these numbers are getting closer. If we use 8 rectangles, we see the area is between $\frac{35}{128}$ and $\frac{51}{128}$, and if we use 64 we find that the area is between .326 and .341.

You can probably guess what happens as the number of rectangles gets very big, but let's work it out. If we have n rectangles, then each one has width $1/n$, and if we use the right-hand approximation then each rectangle has height $\left(\frac{i}{n}\right)^2$. So we have

$$\begin{aligned} R_n &= \frac{1}{n}\cdot\frac{1}{n}^2 + \frac{1}{n}\cdot\frac{2}{n}^2 + \cdots + \frac{1}{n}\cdot\frac{n}{n}^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}. \end{aligned}$$

(We had to use a "sum of squares" formula to get to the third line; feel free to check it on your own, but don't worry about it too much.)

What happens to R_n as n gets large? From what we learned about limits in section 1.4, we can compute that this limit is $\frac{1}{3}$.

We can generalize this process to define exactly what we mean by the area under a curve.

Definition 5.2. We define the area under a curve to be the limit of the sums of the areas of these rectangles. We write

$$A = \lim_{n\to+\infty} R_n = \lim_{n\to+\infty}\left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right).$$

Here n is the number of rectangles, and Δx is the width of each rectangle. Thus $\Delta x = \frac{L}{n}$ where L is the length of our shape.

Example 5.3. Estimate the area under the curve of $f(x) = 2x$ between $x = 1$ and $x = 4$, using three rectangles and using six rectangles. Try using both right endpoints and left endpoints. Is it what you expected?

$$\begin{aligned}
R_3 &= \frac{3}{3}(4+6+8) = 18. \\
L_3 &= \frac{3}{3}(2+4+6) = 12. \\
R_6 &= \frac{3}{6}(3+4+5+6+7+8) = 16.5. \\
L_6 &= \frac{3}{6}(2+3+4+5+6+7) = 13.5.
\end{aligned}$$

What if the number of rectangles goes to infinity? We have

$$\begin{aligned}
R_n &= \frac{3}{n} f(1+3/n) + \frac{3}{n} f(1+2\cdot 3/n) + \cdots + \frac{3}{n} f(1+n\cdot 3/n) \\
&= \frac{3}{n}\left(2 + 2\frac{3}{n} + 2 + 4\frac{3}{n} + \cdots + 2 + 2n\frac{3}{n}\right) \\
&= \frac{3}{n}(2+\cdots+2) + \frac{3}{n}\left(2\frac{3}{n} + 4\frac{3}{n} + \cdots + 2n\frac{3}{n}\right) \\
&= 6 + \frac{18}{n^2}(1+2+\cdots+n) \\
&= 6 + \frac{18}{n^2}\frac{n(n+1)}{2} = 6 + 9\frac{n+1}{n}.
\end{aligned}$$

We check that this formula still works for 3 and 6. Then we take the limit:

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} 6 + 9\frac{n+1}{n} = 6 + 9\lim_{n\to+\infty}\frac{1+\frac{1}{n}}{1} = 15.$$

This makes sense, since using the area formula for triangles we get an area of 15. (It's a 4×8 triangle minus a 1×2 triangle).

5.2 Riemann Sums and The Definite Integral

5.2.1 Summation notation

For the next couple weeks we'll be writing a lot of sums, and we'd like to have notation to talk about this.

We write $\sum_{i=1}^{n} a_i$ for $a_1 + a_2 + \cdots + a_n$ to be the sum of a bunch of things. We can index the sums other ways—and in particular, sometimes it's helpful to start from 0 instead of from 1.

You'll learn a lot more about sums in Calculus 2, but for right now, here are a few useful facts, which should all make sense if you just think about what you know about "addition".

$$\begin{aligned}
\sum_{i=1}^{n} c &= c + c + \cdots + c = nc \\
\sum_{1=1}^{n} ca_i &= ca_1 + ca_2 + \cdots + ca_n \\
&= c(a_1 + a_2 + \cdots + a_n) = c\sum_{i=1}^{n} a_i \\
\sum_{i=1}^{n} (a_i \pm b_i) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\
&= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = \left(\sum_{i=1}^{n} a_i\right) \pm \left(\sum_{i=1}^{n} b_i\right)
\end{aligned}$$

There are also three more formulas we're going to want to know, which involve adding up specific numbers.

Let's start by thinking about the sum $\sum_{i=1}^{n} i$. This just means add up the first n numbers $1 + 2 + \cdots + n$, so for instance if $n = 4$ this is $1 + 2 + 3 + 4 = 10$.

But we want what's called a "closed-form formula" for this: that is, instead of telling someone to just add up all the numbers, we want an easy formula we can just plug n into to get the answer. And for this specific case, there's an easy formula that comes with a fun story.

One of the most famous mathematicians ever was Carl Friedrich Guass, who lived from 1777 to 1855. And the (somewhat apocryphal) story goes that when he was a small child, his schoolteacher assigned his class to add up all the numbers from 1 to 100, assuming that would take everyone a fair amount of time and the poor teacher could get a break. But this is just our problem, with $n = 100$; and Gauss immediately gave the answer

What he (apocryphally) realized is that if you add all these numbers twice, but in opposite orders, you get a nice pattern. $1 + 100 = 101$, and then $2 + 99 = 101$, and then $3 + 98 = 101$, and so on; so if you add all the numbers twice, you get 100 copies of 101, or $100 \cdot 101$. But then that's twice the real answer, so we want $\frac{1}{2}(100 \cdot 101 = 50 \cdot 101 = 5050$. Generalizing that formula we get

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

It's much harder to explain where the other two formulas here come from, but we're going to need them. (And we could make a longer list if we wanted; but it's not going to be relevant to our journey.)

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

5.2.2 Riemann Sums

Now we want to use this to put notation on the work we did in subsection 5.1, on finding areas. There's going to be a *ton* of notation, but don't worry too much: the project for the rest of this unit is to get rid of as much of it as possible.

Suppose f is a function defined on a closed interval $[a, b]$. We divide $[a, b]$ into n smaller subintervals by picking points $a = x_0 < x_1 < \cdots < x_n = b$. We get a collection of subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, which we call a *partition* P of $[a, b]$. We will also sometimes use Δx_i to refer to the length $x_i - x_{i-1}$ of the ith subinterval in our partition.

For each subinterval, we can pick a *sample point* x_i^* in the interval. We could use the left endpoints or the right endpoints, as we did last class, or we could pick others; for most of our purposes in this class it doesn't really matter. (In lab next week we'll talk about what to do when it does matter).

Definition 5.4. The *Riemann sum* associated to a partion P and a function f on an interval $[a, b]$ is given by

$$R(P, f) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n.$$

We can think about taking the limit as our partition gets very small—as we use more and more rectangles and the width of each gets close to 0. We define

Definition 5.5. If f is a function defined on $[a, b]$, the *definite integral of f from a to b* is

$$\int_a^b f(x)\,dx = \lim_{P \to 0} R(P, f) = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

if the limit exists. If the limit exists, we say f is *integrable* on $[a, b]$. (otherwise, f is not integrable).

We say a is the *lower limit* of the integral, b is the *upper limit*, and $f(x)$ is the *integrand*.

Remark 5.6. It's important to note that while there are xs inside or "under" the integral sign, after the integral is computed there are no xs left. The x is a "dummy variable" or a "parameter." We'd get the exact same answer if we calculated $\int_a^b f(t)\,dt$ or $\int_a^b f(\text{🦇})\,d\text{🦇}$ or $\int_a^b f(thisisavariable)\,dthisisavariable$.

5.2.3 Signed Area

A very important note: the Riemann sum can give a negative number! That's kind of weird, because area can't be a negative number. So the Riemann sum doesn't literally find the area; when the function is positive it contributes positively to the area, but when the function is negative it counts negatively.

Definition 5.7. The *signed area* under a graph is the area below the graph but above the x-axis, minus the area below the x-axis and above the graph.

You can think of this as the "net area". If a rectangle with a positive height has a positive area, then a rectangle with a negative height has a negative area.

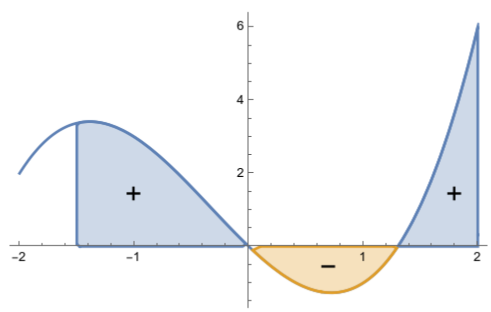


Figure 5.1: If we integrate this function from -1.5 to 2, the blue area counts positively, but the orange area counts negatively.

5.2.4 Simplifying our approach

In our definition, we took the limit over "all" partitions. This is hard to work with in practice, since there are a lot of partitions. (There are infinitely many partitions of $[0, 1]$, for instance, where $x_1 = .99999$. These are in fact partitions but they aren't incredibly helpful).

But if a function is integrable, we can always do our calculations using any collection of partitions that gets small. In particular there's one nice partition we will often use:

Theorem 5.8. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f\left(a + (b-a)\frac{i}{n}\right)\frac{b-a}{n}.$$

In some sense, the dx corresponds to the Δx and the $f(x)$ corresponds to the $f(x_i^*)$. This can be made rigorous, but probably won't be in this course.

Example 5.9.

$$\begin{aligned}
\int_3^5 x^2\,dx &= \lim_{n\to+\infty} \sum_{i=1}^{n} \left(3 + \frac{2i}{n}\right)^2 \frac{2}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^{n} \left(9 + \frac{12i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{18}{n} + \frac{24i}{n^2} + \frac{8i^2}{n^3} \\
&= \lim_{n\to+\infty} \left(\sum_{i=1}^{n} \frac{18}{n} + \sum_{i=1}^{n} \frac{24i}{n^2} + \sum_{i=1}^{n} \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to+\infty} \left(\frac{18}{n}\sum_{i=1}^{n} 1 + \frac{24}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to+\infty} \left(\frac{18}{n}\cdot n + \frac{24}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{(n)(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to+\infty} \left(18 + 12\frac{n(n+1)}{n^2} + \frac{4}{3}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= 18 + 12 + \frac{8}{3} = \frac{98}{3} \approx 32.7.
\end{aligned}$$

5.3 Computing Integrals and The Fundamental Theorem of Calculus Part 1

We want to find a way to compute integrals *without* doing Riemann sums. We can start by taking the same approach with took with derivatives: work out some basic rules that let us manipulate integrals.

Proposition 5.10 (Properties of the Integral)**.** *The following equations are true whenever they make sense, for real numbers a, b, c and functions f, g.*

- $\int_a^b c\,dx = c(b-a)$.
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.
- $\int_a^b cf(x)\,dx = c\int_z^b f(x)\,dx$.
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

Remark 5.11. These properties are derivable from the corresponding properties of sums.

Remark 5.12. Note that while addition and scalar multiplication behave nicely, we didn't make any statements about multiplication or division, because integrals don't actually behave nicely with respect to multiplication. (We call operations like this "linear," and we study them in Math 2184 or 2185).

In Calculus 2, you will return to the idea of "the integral of the product of two functions" when you study integration by parts. But we won't quite get to that in this course.

Example 5.13. Compute $\int_1^0 2 + 3x^2 + 4x^3\,dx$.

We should remember that we've computed $\int_0^1 x^2\,dx = 1/3$ already: that was our very first example 3.21. And in recitation you should compute that $\int_0^1 x^3\,dx = 1/4$.

And with that information and the integral properties we just learned, we can work out this entire integral:

$$\begin{aligned}
\int_1^0 2 + 3x^2 + 4x^3\,dx &= -\int_0^1 2 + 3x^2 + 4x^3\,dx \\
&= -\int_0^1 2 - \int_0^1 3x^2 - \int_0^1 4x^3\,dx \\
&= -\int_0^1 2 - 3\int_0^1 x^2 - 4\int_0^1 x^3\,dx \\
&= -2 - 3(1/3) - 4(1/4) = 4.
\end{aligned}$$

Example 5.14. If $\int_1^5 f(x)\,dx = 3$ and $\int_3^5 f(x)\,dx = 2$, then

$$\int_1^3 f(x)\,dx = 1 = \int_1^5 f(x)\,dx + \int_5^3 f(x)\,dx = \int_1^5 f(x)\,dx - \int_3^5 f(x)\,dx = 3 - 2 = 1.$$

Unfortunately, this doesn't take us very far: we could only do these problems since we'd already done some specific Riemann sums already. (Or, alternately, because I just told you what some of the integrals were.) We need a different approach.

We can start by making some general statements about how integrals relate to each other:

Proposition 5.15 (Comparison Propreties of the Integral)**.** *These properties only work when $a < b$. If we have a case where $a > b$ then we can always rewrite the integral before using them.*

- *If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$.*
- *If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.*
- *If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*

Example 5.16. We've used these implicitly before, when e.g. we said that $0 \leq \int_0^1 x^2 \leq 1$.

Referencing our earlier example, we know that $9 \leq x^2 \leq 25$ on $[3,5]$, so we have $18 \leq \int_3^5 x^2\,dx \leq 50$. Indeed, we calculated that $\int_3^5 x^2\,dx \approx 33$.

Suppose we want to know about $\int_0^\pi \sin(x)\,dx$. We know that $0 \leq \sin(x) \leq 1$ on $[0,\pi]$, so we see that $0 \leq \int_0^\pi \sin(x)\,dx \leq \pi$. (In fact, the integral is equal to 2, but we don't yet have the tools to calculate that).

For derivatives, this basic approach took us pretty far. But integrals are more complex and this isn't going to get us all the way there. Instead, we're going to do something very mathematician-y, that sounds dumb at first. We're going to make things more abstract, and harder now, in order to make things easier later.

From the perspective of section 5.2, the definite integral $\int_a^b f(t)\,dt$ is always a number, as long as f is integrable.(Technically the integral is a function from the set of integrable functions to the set of real numbers, but we don't need to worry about that in this class). In fact the integral is just "the area of a shape I just described," so it should always be a number. If I asked you for the area of a shape you shouldn't ever tell me $y = x^2$, for instance.

But we can use the integral to define a function (in the same way that we can have the function "input a number x and return the area of a square with side length x"—that is, $f(x) = x^2$). In particular, we want to consider functions of the form

$$F(x) = \int_a^x f(t)\,dt \tag{4}$$

where a is some fixed constant, and x is a variable. So our function is "put in a number x, and output the number $\int_a^x f(t)\,dt$, which is the area of some shape, determined by x."

Now that we have a function, there are a bunch of questions we can ask about it. What is its domain? Is it continuous? Is it differentiable?

The domain of $F(x) = \int_a^x f(t)\,dt$ is all x so that f is integrable on $[a,x]$; this answer isn't terribly satisfying, since it boils down to "The domain of F is the domain of F." It's not possible to do better without knowing something about f. But if we impose a fairly mild condition, we can say a bit more:

Theorem 5.17. *If f is continuous on $[a,b]$, or if it is continuous except for finitely many jump discontinuities, then f is integrable on $[a,b]$.*

Sketch of proof. If f has finitely many jump discontinuities, we can pick our partition to chop it up into a finite collection of continuous functions. So we just have to worry about continuous functions.

For any partition, you can always pick a "biggest" sample point in each interval, and a "smallest." The first will give you an upper bound to the integral, and the second will give you a lower bound. If the function is continuous, we can show that those two sums will always get closer together, and every other possible sum will be between the two; so all possible sums converge to the same integral. □

Example 5.18. $f(x) = x^n$ is integrable, as is $|x|$ and $\sqrt[n]{x}$ on any interval on which it is defined. The Heaviside (step) function is integrable. $1/x$ is not integrable on $[0,1]$. The characteristic function of the rationals is not integrable (At least, not until grad school, when they change the definitions on you).

We can see a bit more. It's not too hard to show that F is continuous on its domain. Geometrically, changing x a little bit will change $F(x)$ by about the height of the function times the change in input; if the change in input is small, the change in output will also be small. Algebraically:

$$\begin{aligned}
\lim_{x\to b} F(x) - F(b) &= \lim_{x\to b} \int_a^x f(t)\,dt - \int_a^b f(t)\,dt \\
&= \lim_{x\to b} \int_a^x f(t)\,dt + \int_b^a f(t)\,dt \\
&= \lim_{x\to b} \int_b^x f(t)\,dt.
\end{aligned}$$

If x and b are close enough we can always find m, M such that $m \leq f(t) \leq m$ on $[x,b]$, so we get

$$\begin{aligned}
\lim_{x\to b} m(x-b) \leq \lim_{x\to b} \int_b^x f(t)\,dt &\leq \lim_{x\to b} M(x-b) \\
0 \leq \lim_{x\to b} \int_b^x f(t)\,dt &\leq 0 \\
0 = \lim_{x\to b} \int_b^x f(t)\,dt&.
\end{aligned}$$

The question of differentiability is a little trickier, but significantly more important. Intuitively and geometrically, we can simply look at pictures and ask how much the area under a curve changes if we widen our x-values a bit. After drawing some pictures we conclude that the area should change by "about" the height of the curve on one end.

We can in fact prove this fact. It's important enough for us to give it a silly name:

Theorem 5.19 (The Fundamental Theorem of Calculus, Part 1)**.** *Suppose f is continuous on $[a, b]$, and set*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $\frac{d}{dx}F(x) = f(x)$ for $a < x < b$.

Remark 5.20. As we'll discuss shortly, this theorem is the key to calculuating integrals. Note that it only applies to continuous functions. But if we have a function that's continuous in pieces, we can just split it up into separate integrals, and we see it has the correct derivative on each piece.

Proof. We want to capture our geometric intuitions. Recall that by definition, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

(This calculuation should look similar to the one above for continuity.) Let's assume for now that $h > 0$. By the extreme value theorem, f has an absolute minimum m and an absolute maximum M on $[x, x+h]$, and further we can write $f(u) = m$ and $f(v) = M$ for u, v in $[x, x+h]$. Then

$$\begin{aligned} f(u)h \le \int_x^{x+h} f(t)\,dt \le f(v)h \\ f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v). \end{aligned}$$

As $h \to 0$, the numbers u and v must get closer together, and in fact closer to x, and so by continuity $\lim_{h\to 0} f(u) = \lim_{h\to 0} f(v) = f(x)$. So we have $F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$ as desired. □

Example 5.21.

- If $F(x) = \int_a^x \sqrt{x^3+1}\,dt$ then $F'(x) = \sqrt{x^3+1}$.
- If $G(x) = \int_a^x \sin(\pi t)\cos(\pi t)\,dt$ then $G'(x) = \sin(\pi x)\cos(\pi x)$.
- If $H(x) = \int_a^{x^3} \sqrt{1+t}\,dt$ then we have to be careful. We can write $H(x) = H_1(x^3)$ where $H_1(x) = \int_a^x \sqrt{1+t}\,dt$. So by the chain rule, we have $H'(x) = \sqrt{1+x^3} \cdot 3x^2$.

5.4 Computing Integrals and the FTC 2

We still haven't quite figured out how to compute integrals without going back to the Riemann sum formulation. But we're almost there!

The Fundamental Theorem of Calculus tells us that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. But it isn't the only function with this property. We can give this a name:

Definition 5.22. If $F'(x) = f(x)$, we call F an *antiderivative* of f.

Example 5.23. $\frac{1}{3}x^3$ is an antiderivative of x^2.

$\sin(x)$ is an antiderivative of $\cos(x)$.

7 is an antiderivative of 0.

So $\int_a^x f(t)\,dt$ is an antiderivative of f. Further, we know a lot about what antiderivatives look like:

Proposition 5.24. *If $F'(x) = G'(x)$ for all x, then $F(x) = G(x) + C$ for some constant C.*

Proof. Differentiation is additive, so $(F - G)'(x) = F'(x) - G'(x) = 0$. But since the derivative is the rate of change, any function with zero derivative is constant. (We proved this in proposition 3.19 in section 3.3, using the Mean Value Theorem.) Thus $(F-G)(x) = C$ for some constant C, and so $F(x) = G(x) + C$. □

This proposition incredibly useful, because it means *any* function whose derivative is $f(x)$ is "almost" the same as $\int_a^x f(t)\,dt$. We have some sort of constant hanging around, which we need to get rid of; it turns out that this constant is essentially related to the a, the lower limit of integration.

Theorem 5.25 (Fundamental Theorem of Calculus, Part 2)**.** *Suppose f is continuous on $[a, b]$, and F is any antiderivative of f. Then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Proof. Since $F(x)$ and $\int_a^x f(t)\,dt$ are both antiderivatives of $f(x)$, we know that $F(x) = \int_a^x f(t)\,dt + C$ for some constant C. Then

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - \left(\int_a^a f(t)\,dt + C\right) = \int_a^b f(t)\,dt + C - 0 - C = \int_a^b f(t)\,dt.$$

□

Example 5.26. What is $\int_1^3 3x^2\,dx$?

We can see that $F(x) = x^3$ is an antiderivative of $3x^2$. (It's not the only one, but that's okay.) So $\int_1^3 3x^2\,dx = F(3) - F(1) = 27 - 1 = 26$.

What if we'd picked, say, $G(x) = x^3 + 5$? Then we'd have $\int_1^3 3x^2\,dx = G(3) - G(1) = 32 - 6 = 26$ again.

Example 5.27. What is $\int_{\pi/4}^{3\pi/4} \cos(x)\,dx$?

We see that $\sin(x)$ is an antiderivative for $\cos(x)$. So we have

$$\int_{\pi/4}^{3\pi/4} \cos(x)\,dx = \sin(3\pi/4) - \sin(\pi/4) = \sqrt{2}/2 - \sqrt{2}/2 = 0.$$

5.4.1 Indefinite Integrals

Because antiderivatives are so important, we want a notation for them that is less awkward than having to write the word "antiderivative" over and over. Because they are so closely tied to integrals, we use notation specifically designed to confuse you about what the integral sign means.

Definition 5.28. The *indefinite integral* of a function f, written $\int f(t)\,dt$, is any antiderivative of f. That is, $\int f(t)\,dt$ refers to any function $F(x)$ such that $F'(x) = f(x)$.

The *general form of the indefinite integral* is $\int f(x)\,dx = F(x) + C$. The constant represents the fact that there are many possible antiderivatives of f.

Very Important Note: Remember the difference between the definite and indefinite integrals. The definite integral $\int_a^b f(x)\,dx$ is a number. It is the area of some region under a graph. The indefinite integral $\int f(x)\,dx$ is a collection of functions, which are all antiderivatives of f and are all the same up to a constant. They are related by

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b = F(b) - F(a).$$

In general the notation $|_a^b$ means "the value at b minus the value at a. We will use it a lot while doing integrals.

Example 5.29. We can write $\int x^5\,dx = \frac{1}{6}x^6 + C$, and $\int \sec^2(x)\,dx = \tan(x) + C$.

5.4.2 Antiderivatives, Net Change, and Linear Approximation

We can look at all of what we've done from another perspective, and connect it back to the work we did earlier on linear approximation.

Suppose we have a function F that we want to know about, but we only know about the derivative $F'(x)$. For instance, we may want to know the position of an object but only have measured the speed, or want to know the speed after measuring the acceleration. Or we want to figure out how much money we owe from a record of our annual deficits; we've seen a lot of examples of derivatives.

The example of deficit and debt makes this maybe easy to think of. Suppose you have a deficit of \$3000 one year, \$5000 the second year, and \$2000 the third year. At the end of three years, the debt has increased by \$10,000, which we get by adding the three deficits up.

This works exactly because we have a discrete set of payments, but if we don't have that we can still approximate it. Suppose that $F(t)$ gives the position of a particle at time t, and we know the velocity $F'(t)$. If we also know the starting position $F(0)$, we could estimate $F(4) \approx F(0) + F'(0)(4 - 0)$, but that might not be very good.

One way we could make this better is to do something like a quadratic approximation, or a Taylor series, but that gets messy. Another option is to do *multiple approximations*. Since the approximation gets worse the further x gets from a, we can try to bring it closer, and approximate in multiple steps.

Thus maybe we have

$$\begin{aligned} F(2) &\approx F(0) + F'(0)(2-0) \\ F(4) &\approx F(2) + F'(2)(4-2) \approx F(0) + F'(0)(2-0) + F'(2)(4-2). \end{aligned}$$

So if we take, say, $F'(t) = 10t$ and $F(0) = 0$, this would give us

$$\begin{aligned} F(2) &\approx 0 + 0(2-0) \\ F(4) &\approx 0 + 20(2) = 40 \end{aligned}$$

which is close-ish but not super close to the true answer of 80 (as we'll see soon).

What if we take more steps? We get

$$\begin{aligned} F(1) &\approx F(0) + F'(0)(1-0) && \approx 0 + 0(1-0) \\ F(2) &\approx F(1) + F'(1)(2-1) && \approx 0 + 10(2-1) = 10 \\ F(3) &\approx F(2) + F'(2)(3-2) && \approx 10 + 20(3-2) = 30 \\ F(4) &\approx F(3) + F'(3)(4-3) && \approx 30 + 30(4-3) = 60. \end{aligned}$$

But what is this last formula, really? It's

$$F(4) \approx F(0) + F'(0)(1-0) + F'(1)(2-1) + F'(2)(3-2) + F'(3)(4-3).$$

If we rearrange this a bit, we just get

$$F(4) - F(0) \approx F'(0)(1-0) + F'(1)(2-1) + F'(2)(3-2) + F'(3)(4-3)$$

and the right-hand side is a sum of terms that look like $F'(x_i)\Delta x_i$. So we have

$$F(4) - F(0) \approx \sum_{i=1}^{n} F'(x_i)\frac{4}{n}.$$

This is just a Riemann sum! And as we take the limit, we get an integral

$$F(4) - F(0) = \lim_{n\to\infty} \sum_{i=1}^{n} F'(x_i)\frac{4}{n} = \int_0^4 F'(x)\,dx.$$

Early on in the class, we saw that if you know the value of F and the derivative of F at 0, then you can use a linear approximation to estimate the value at any point. What we see now is that if you know the derivative of F everywhere, and the value at one point, you can find the value exactly, by taking an infinite collection of very small linear approximations.

Specifically, if you know the derivative, you can figure out the net change of F between any two values; so if you have one value, you can find any value.

Corollary 5.30 (Net Change Theorem)**.** *The integral of a rate of change is the total (net) change.*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Remark 5.31. Note that to find the value of $F(b)$ this way, we need to start by knowing $F(a)$ for some a. If we think of F as just being an antiderivative of F', the starting value is nailing down exactly the constant C.

Remark 5.32. This process of taking a large number of linear approximations is used in the real world a lot. If you have an integral that you *can't* find an exact formula for, this is very useful. It generalizes even more to solving differential equations, which are equations that specify F using a *formula* for $F'(x)$. They are more complicated than simple integrals, and you will see a little of them in calculus 2. But they are also the fundamental underpinning of most mathematical models, in the physical sciences and the social scienes.

5.4.3 Computing Integrals for the Practical Person

We've learned that computing integrals is reducible to finding antiderivatives. Now we're finally ready to practice actually computing integrals. In order to do this, we start by recalling a number of antiderivatives.

I'll list a few in these notes. There is an extensive card listing many of htese rules on page 6 of the reference in the back of Stewart, and a shorter table on page 331 in section 4.4.

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- $\int cf(x)\,dx = c\int f(x)\,dx$.
- $\int x^n\,dx = \frac{x^n}{n+1} + C$ if $n \neq -1$.
- $\int \sin(x)\,dx = -\cos(x) + C$.
- $\int \cos(x)\,dx = \sin(x) + C$.
- $\int \sec^2(x)\,dx = \tan(x) + C$.
- $\int \csc^2(x)\,dx = -\cot(x) + C$.
- $\int \sec(x)\tan(x)\,dx = \sec(x) + C$.
- $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$.

Example 5.33.

- What is $\int_1^4 x^2\,dx$? We know that $\int x^2\,dx = \frac{1}{3}x^3 + C$, so $\int_1^4 x^2\,dx = \frac{1}{3}x^3|_1^4 = \frac{1}{3}(64-1) = 21$. Note the Cs cancel each other out so it doesn't matter what they are.
- What is $\int_2^3 x + x^3\,dx$? We can work out that $\int x + x^3 = \frac{x^2}{2} + \frac{x^4}{4}$, so

$$\int_2^3 x + x^3\,dx = \frac{x^2}{2} + \frac{x^4}{4}\Big|_2^3 = \frac{9}{2} + \frac{81}{4} - \frac{4}{2} - \frac{16}{4} = \frac{99}{4} - 6 = \frac{75}{4}.$$

- Calculuate $\int_{-1}^2 |x|\,dx$. We don't really have an antiderivative of $|x|$, so the easiest way to approach this is probably to break it up into two distinct integrals.

  If $x \geq 0$ then $|x| = x$, so we have $\int_0^2 |x|\,dx = \int_0^2 x\,dx = \frac{x^2}{2}|_0^2 = 2 - 0 = 2$.

  If $x \leq 0$ then $|x| = -x$ and we have $\int_{-1}^0 |x|\,dx = \int_{-1}^0 -x\,dx = \frac{-x^2}{2}|_{-1}^0 = 0 - \frac{-1}{2} = \frac{1}{2}$.

  Thus $\int_{-1}^2 |x|\,dx = \int_{-1}^0 |x|\,dx + \int_0^2 |x|\,dx = \frac{1}{2} + 2 = \frac{5}{2}$.

- Calculate $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$. At first blush this looks hard, until you remember that $\sec'(x) = \sec(x)\tan(x)$. So we have

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \sec(x)|_0^{\pi/4} = \sec(\pi/4) - \sec(0) = \sqrt{2} - 1.$$

- What if we want $\int_0^\pi \sec(x)\tan(x)$? This is a much bigger problem, because $\sec(x)\tan(x)$ is not continuous on $[0, \pi]$. We actually won't be able to do that one without new ideas that we won't develop in this course.

Leading question: can you do $\int 3x^2\sqrt{9+x^3}\,dx$?

5.5 Integration by Substitution

The Fundamental Theorem of Calculus is a powerful tool for computing integrals. And with functions that are obviously the derivatives of some other function, like x^2 or $\cos(x)$, it's very easy to apply. With more complicated functions it takes a bit more work.

Example 5.34. What is $\int 3x^2\sqrt{9+x^3}\,dx$?

There are two ways to approach this problem. The first is to notice that you almost have an antiderivative to $\sqrt{9+x^3}$, because $(9+x^3)^{3/2}$ has $\frac{3}{2}(9+x^3)^{1/2}\cdot 3x^2$ as its derivative. The extra $3x^2$ from the chain rule precisely matches up with the extra $3x^2$ from the problem, so we just have to correct for the constant, and we have that $\int 3x^2\sqrt{9+x^3} = \frac{2}{3}(9+x^3)^{3/2}+C$.

If that made sense, great. Whenever you can "just see" the antiderivative, you can go for it; the fact that you can check your work by taking a derivative means that you are safe. But for the cases where you can't just see the answer, we'd like to be a little more systematic in our approach.

We know how to take the antiderivative of $\sqrt{x}$. So let's try using a new variable, which we traditionally call u. We write $u = 9+x^3$ so the thing under the radical is a u. We also notice that $\frac{du}{dx} = 3x^2$; by "abuse of notation" (by which I mean we won't justify it, but just assume it works) we write $du = 3x^2dx$. Since our original integral was $\int \sqrt{9+x^3}\cdot 3x^2dx$, we can rewrite this as $\int \sqrt{u}\,du$, or just $\int u^{1/2}\,du$.

From our integral table, we know that $\int u^{1/2}\,du = \frac{2}{3}u^{3/2}+C$. Now we can replace the u with $9+x^3$ to get $\int 3x^2\sqrt{9+x^3}\,dx = \frac{2}{3}(9+x^3)^{3/2}+C$.

We can formalize this into a rule:

Proposition 5.35 (The Substitution Rule for Indefinite Integrals)**.** *If $u = g(x)$ is differentiable, and $f(x)$ is continuous on the range of g, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)du.$$

Proof. This follows from the chain rule. Let F be an antiderivative of f; then $(F(g(x)))' = F'(g(x))\cdot g'(x) = f(g(x)g'(x)$. Thus $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

I'd like to give you geometric intuition here, but it's a bit hard to communicate. In essence we're changing to a new coordinate system where the integral is easy, but it's hard to make that observation *useful* until you get to multivariable calculus. For right now, yo ushould probably think of this as a way of keeping track of algebraic manipulations. □

How do we use this? Basically, when we see a complicated integral, there are a couple things we can look for. The first is to check whether one part is a derivative of another part, in a way that could reflect a chain rule. The other is to find the most complicated chunk of the expression and replace it with a u, and see how much of our problem that solves.

Choosing the right variable to substitute is a bit of an art; I can't possibly give you a complete set of rules, but I can give you a lot of examples to model off of.

Example 5.36.

- Consider $\int x^2 \sin(x^3+3)\,dx$. We can take $u = x^3+3$, and then $du = 3x^2\,dx$ so $dx = \frac{du}{3x^2}$. So this becomes $\int \sin(u)/3\,du = -\cos(u)/3 + C = -\cos(x^3+3)/3 + C$.
- Consider $\int \sqrt{5x+2}\,dx$. It makes sense to take $u = 5x+2$, so $du = 5dx$. Then $\int \sqrt{u}/5\,du = \frac{2}{15}u^{3/2} + C = \frac{2}{15}(5x+2)^{3/2} + C$.

  Alternatively, we could take $u = \sqrt{5x+2}$. Then $du = \frac{5}{2\sqrt{5x+2}}dx$ and we get $dx = \frac{2}{5}\sqrt{5x+2} = \frac{2}{5}u$. So we have $\int \frac{2}{5}u^2\,du = \frac{2}{15}u^3 + C = \frac{2}{15}(5x+2)^{3/2} + C$.
- For a more complex example, we can look at $\int \sqrt{1+x^2}x^5\,dx$. This doesn't look like it will happen automatically, and indeed it doesn't. But we can still get rid of the complicated bit by taking $u = 1+x^2$, so $du = 2x\,dx$ or $dx = du/2x$.

  This gives us $\int \sqrt{u}x^4\frac{1}{2}du$, but what do we do with the other x^4 term? Well, if $u = 1+x^2$ that means that $x^2 = u-1$, so our integral is

$$\begin{aligned}\int \frac{1}{2}\sqrt{u}(u-1)^2\,du &= \int \frac{1}{2}\left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du \\ &= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C.\end{aligned}$$

5.5.1 Substitution and Definite Integrals

The above talked about indefinite integrals. When we have a definite integral, we can be more specific. We can use substitution in two ways: one is to do what we did above, where we substitute in a u, then integrate, then switch the us back to xs. But we can avoid switching back at all by changing the limits of integration.

Proposition 5.37 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a,b]$, and f is continuous on the range of $g(x)$, then*

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. If F is an antiderivative of f, then the left side is clearly $F(g(b)) - F(g(a))$. But the antiderivative of $f(g(x))g'(x)$ is $F(g(x)$, so the left side is also $F(g(b)) - F(g(a))$. □

Example 5.38.

- Find $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$. We take $u = g(x) = 1+2x^2$ so that $du = 4dx$, so $dx = du/4$, and $g(0) = 1, g(2) = 9$. We have
$$\frac{1}{4}\int_1^9 u^{-1/2}\,du = \frac{1}{4}2u^{1/2}|_1^9 = \frac{1}{2}(3-1) = 1.$$
- Find $\int_1^3 \frac{dx}{(1-2x)^2}$. Set $u = g(x) = 1-2x$, then $du = -2dx$ and $g(1) = -1, g(3) = -5$. So
$$\int_1^3 \frac{dx}{(1-2x)^2} = \int_{-1}^{-5} \frac{-du}{2u^2} = \left.\frac{1}{2u}\right|_{-1}^{-5} = \frac{1}{-10} - \frac{1}{-2} = \frac{2}{5}.$$

A nice bonus application of this is to look at symmetric functions. Since even and odd functions have nice geometric symmetries, integrals, which are about the area under the curve, should also have nice properties.

Corollary 5.39 (Integrals of Symmetric Functions)**.** *Suppose f is a continuous function on $[-a,a]$. Then*

- *If f is even, then $\int_{-a}^a f(t)\,dt = 2\int_0^a f(t)\,dt$.*
- *If f is odd, then $\int_{-a}^a f(t)\,dt = 0$.*

Proof. Intuitively this should be plausible; even functions look the same on either side of the y-axis, and so you should get the same area on both sides, while odd functions are the same but upside down, so you should get the opposite area. (Try sketching a picture of sin and cos to see this).

For either integral, notice that $\int_{-a}^a f(t)\,dt = \int_{-a}^0 f(t)\,dt + \int_0^a f(t)\,dt$. Consider the first integral, and use the substitution $u = g(t) = -t$, and thus $-du = -dt$. Then $\int_{-a}^0 f(t)\,dt = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)\,dt$.

If f is even then $f(-t) = f(t)$, so $\int_{-a}^0 f(t)\,dt = \int_0^a f(t)\,dt$. If f is odd then $f(-t) = -f(t)$ and thus $\int_{-a}^0 f(t)\,dt = -\int_0^a f(t)\,dt$. □

Example 5.40.

- $\int_{-3}^3 x^5 - x^3\,dx = 0$.
- $\int_{-2}^2 x^6 + 1\,dx = 2\int_0^2 x^6 + 1\,dx = 2(x^7/7 + x)|_0^2 = 2(128/7 + 2) = \frac{284}{7}$.

5.6 A Brief Note on How to Cheat

We've now learned how to compute basic integrals. There are a lot of integrals we haven't yet learned to compute; a prominent example is $\int \frac{1}{x}\,dx$, but there are many. In calculus 2 you will develop many other techniques of integration which allow us to integrate more difficult functions. However, as good mathematicians we're also fundamentally lazy and would prefer to avoid work when we can manage it. There are two common solutions here.

First, the back of your textbook has an extensive integral table, and even more extensive tables can be found online. It often requires minor massaging to get your integral into the form of the table, but for complex integrals the table will be much easier than figuring things out from scratch. (For instance, the table incorporates the results of trig subsitution without making you work through it explicitly).

Second, computers are very good at doing integrals. Wolfram Alpha can often integrate a function for you, as can Mathematica and other computer tools. It's dangerous to become overly reliant on these tools—it's easy to make a mistake if you don't understand what's going on, and sometimes the computer will return the answer in a less useful form. They are very good for automated computations and checking your work, however.

A final cautionary note: there are some functions that don't have a nice closed-form antiderivative. Famously, there's no way to write $\int e^{x^2}\,dx$ in terms of "elementary functions." That doesn't mean there is no antiderivative; the obvious one is $\int_0^x e^{t^2}\,dt$. But while correct, that answer isn't terribly enlightening.

We can't easily compute these definite integrals exactly, but we can approximate them using various approximation techniques (among other things, just computing a finite Riemann sum). We can also use the concept of "infinite series" to handle this sort of situation; those techniques occur towards the end of Calculus 2.

6 Applications of Integrals

Now that we know how to compute integrals, we want to finish the course by talking about why integrals are so useful. We'll start with the idea of computing area, which was the original motivation for integration; but we'll see they also answer many other questions.

6.1 Finding Areas

Recall that we originally constructe the integral to find the area of some shape, in particular of shapes that lie under the graph of some function. We can use the same tools to find the area of a region that is not, properly speaking, the graph of one function.

The simplest (well, second-simplest) case is the case where we want the area of a region that lies in between the graph of two functions. We can approximate area by drawing, as before, a great many skinny rectangles which are approximately the right height to cover our region. If our region lies in between two functions f and g, the combined area of our rectangles is

$$\sum_{i=1}^{n}(x_i - x_{i-1})(f(x_i^*) - g(x_i^*))$$

and as the number of rectangles increases this approximation gets increasingly good. We say the area of the region is

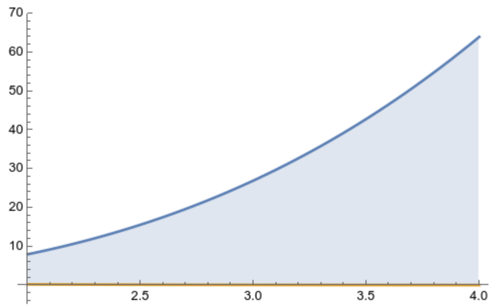
$$A = \lim_{n \to +\infty} \sum_{i=1}^{n}(x_i - x_{i-1})(f(x_i^*) - g(x_i^*)).$$

You may recognize this formula as the integral of the function $f - g$; indeed, if we have a region with x coordinates varying from a to b and y coordinates varying from $g(x)$ to $f(x)$, then its area is $\int_a^b (f(x) - g(x))\, x$.

Remark 6.1. Remember that actual areas are always positive! The integral by itself computes the "signed area"; if you want an actual area you must be careful to make sure you're integrating the correct function.

Example 6.2. Let's start with a trivial example: what's the area of a rectangle with base 3 and height 4? We know that the area of a rectangle is length times width, so the answer should be 12. But we can also compute the area as $\int_0^4 3\, dx = 3x|_0^4 = 12$, which matches what we expect.

Example 6.3. What is the area of the region between $y = x^3$ and $y = 1/x^2$ between $x = 2$ and $x = 4$?

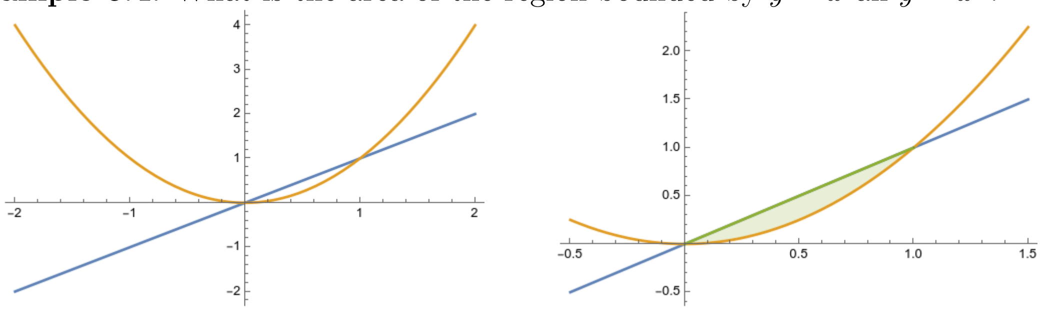


We have

$$\int_2^4 x^3 - (1/x^2)\,dx = \left(\frac{x^4}{4} + \frac{1}{x}\right)\Big|_2^4 = (64 + 1/4) - (4 + 1/2) = 60 - 1/4 = 239/4.$$

Sometimes (usually!) we need to have a visual idea of what our region looks like before we can set up an appropriate integral.

Example 6.4. What is the area of the region bounded by $y = x$ an $y = x^2$?



After we draw a picture, we see that these two graphs enclose a region between $x = 0$ and $x = 1$, and that in that region, $x \geq x^2$. So we compute the integral

$$\int_0^1 x - x^2\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

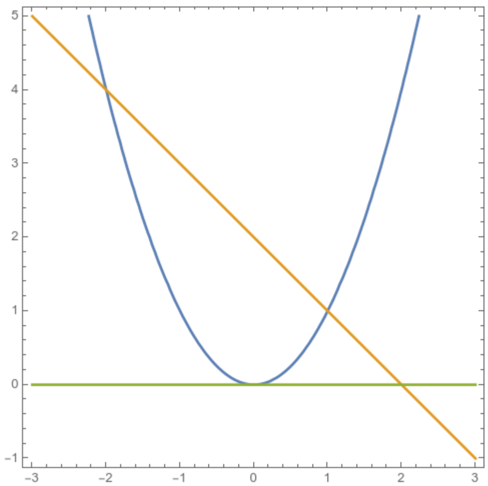
Example 6.5 (recitation)**.** Compute the total area of the "valley" between two peaks of the sine function.

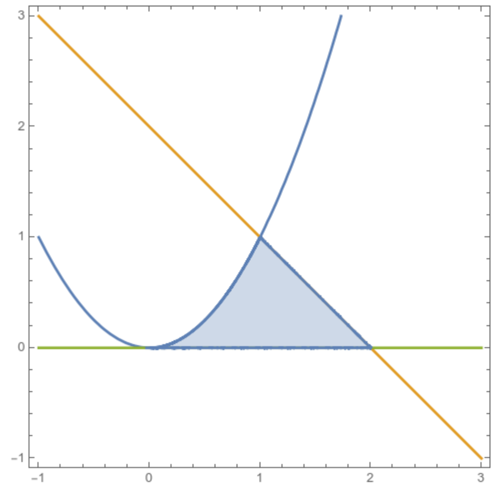
We see that this area is the area of the region between $y = 1$ and $y = \sin x$ between $\pi/2$ and $5\pi/2$. (There are other ways to set this up, but this way works). So we compute

$$\int_{\pi/2}^{5\pi/2} 1 - \sin x\,dx = x + \cos(x)|_{\pi/2}^{5\pi/2} = (5\pi/2 + 0) - (\pi/2 + 0) = 2\pi.$$

Sometimes you have to break your region up into separate pieces/integrals

Example 6.6. What is the area of the region bounded by $y = x^2, y = 2 - x$, and $y = 0$?





We sketch the region and see that we get a sort of collapsed triangle. We compute

$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$$
$$= \frac{1}{3} - 0 + (4-2) - (2 - 1/2) = \frac{5}{6}.$$

We can also do the same problem another way. Notice that we might as well write $x = \sqrt{y}, x = 2 - y$. So we can just as well integrate with respect to y—that is, draw our rectangles stretching horizontally instead of vertically. We have

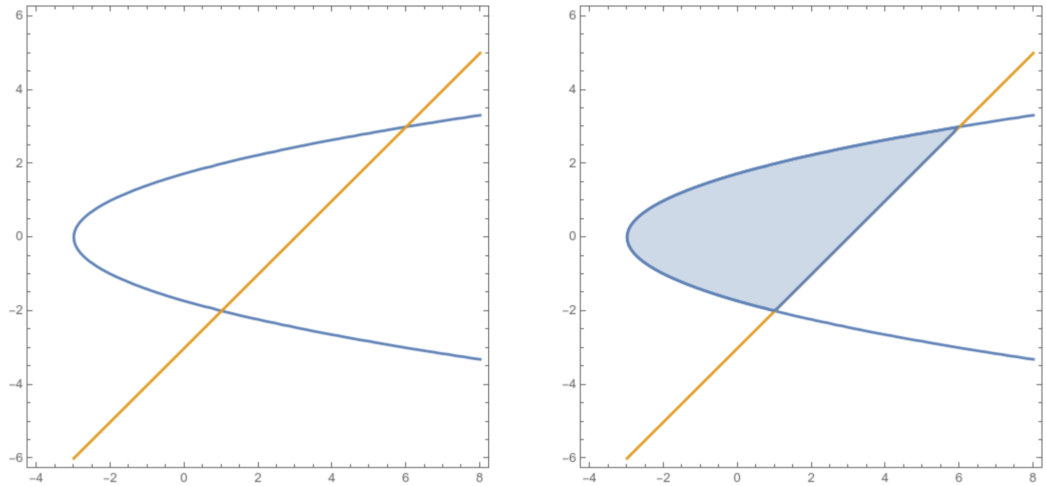
$$A = \int_0^1 (2-y) - \sqrt{y}\,dy = \left(2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right)\Big|_0^1 = \left(2 - \frac{1}{2} - \frac{2}{3}\right) - 0 = \frac{5}{6}.$$

As expected, we get the same answer.

We see here that we can set up these area integrals in two different ways: we can integrate "with respect to x" or "with respect to y". When we integrate with respect to x, we chop our region up into tall, skinny rectangles, as we have been since the beginning of section 5. This works well when we have a shape with straight-line boundaries to the left and right, and can write the top and bottom curves as functions of x.

When we integrate with respect to y, we chop our region up into short wide rectangles. This is the same basic process, but we're looking at the picture from a different angle. It works well when our shape has straight-line boundaries on the top and bottom, and we can describe the left and right boundaries as functions of y.

Example 6.7. What is the area of the region between $y^2 = x + 3$ and $y = x - 3$?

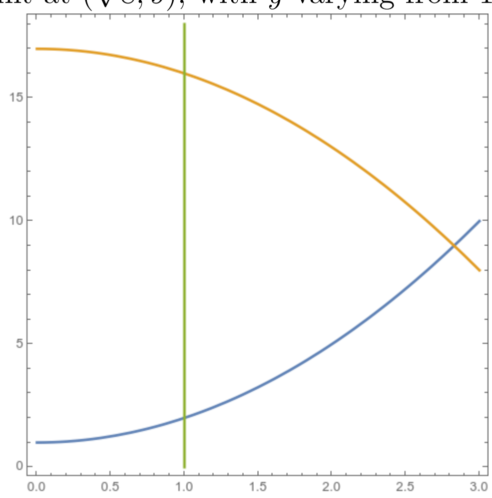


These curves intercept when $y^2 = y + 6$, which happens when $y = 3$ or $y = -2$, and thus at $(6, 3)$ and $(1, -2)$. It's more natural to integrate with respect to y, so we write

$$A = \int_{-2}^{3} (y+3) - (y^2 - 3)\, dy = \int_{-2}^{3} 6 + y - y^2\, dy$$
$$= \left(6y + \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-2}^{3} = \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right) = \frac{27}{2} + 10 - \frac{8}{3} = \frac{125}{6}$$

Example 6.8. What is the area of the region bounded by $y = x^2 + 1, y = 17 - x^2$, and $x = 1$?

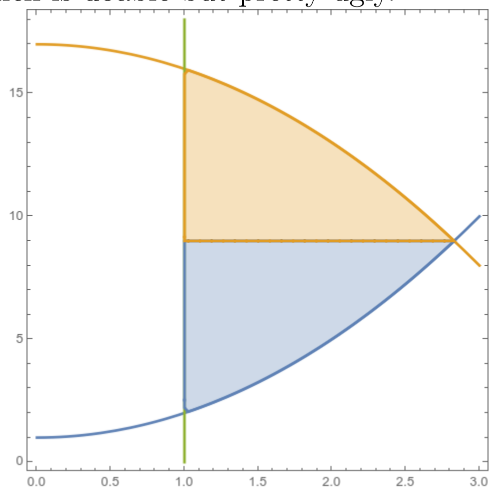
We first draw the region, and see a sort of sideways triangle with a base at $x = 1$ and a point at $(\sqrt{8}, 9)$, with y varying from 1 to 17.



We have two options: integrate with respect to x, or with respect to y by writing $x = \sqrt{y-1}$ and $x = \sqrt{17 - y}$. The second involves breaking our region into two integrals, and gives us

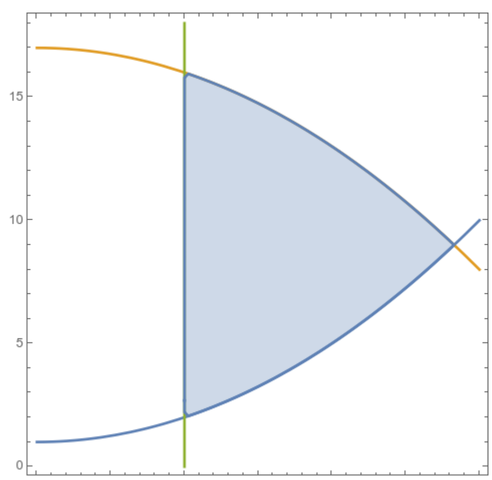
$$A = \int_2^9 \sqrt{17-y} - 1\,dy + \int_9^{16} \sqrt{17-y} - 1\,dy,$$

which is doable but pretty ugly.



Instead, if we integrate with respect to x, we get

$$A = \int_1^{\sqrt{8}} (17-x^2) - (x^2-1)\,dx = \int_1^{\sqrt{8}} 18 - 2x^2\,dx$$
$$= 18x - \frac{2}{3}x^3\Big|_1^{\sqrt{8}} = 36\sqrt{2} - 32\sqrt{2}/3 - 18 + 2/3 = \frac{76\sqrt{2}-52}{3}.$$



6.2 Physical and Economic Applications

At this point we've fully developed the theory of the integral, and answered the original question about area we posed in section 5.1. But much like with the derivative, this tool is really flexible, and can answer many other questions than the original one we asked ourselves.

We can essentially divide integral applications into two categories. The most obvious thing the integral does is that it *adds stuff up.* Remember we have

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

So if we have something we want to compute that looks like adding up a bunch of small numbers, we can usually turn that into an integral.

Less obvious is that the integral represents a sort of *multiplication.* Here, we're going to emphasize that the things we're summing up look like $f(x_k)\Delta x$, and thus represent some sort of product. In particular, we do an integral when we want to multiply two things together but one of them is changing.

We saw this with area: in general area is "width times height" but for the irregular shapes we've been studying, the height isn't the same everywhere. So we chop it up into a bunch of small pieces, and then on each one we pretend the height is constant so we can multiply width by height, and then we add all the small pieces up at the end.

We'll see examples of both of these ideas.

6.2.1 Economic Application: Consumer and Producer Surplus

Let's start with an application that's often closely tied to area. Economists like to talk about the idea of consumer and producer surpluses, the amount of benefit the buyers and sellers get from transacting. If I would be willing to buy a toy for \$18 and you'd be willing to sell it for \$5, then that trade creates \$13 of value. But who gets the value depends on the price—if you sell for ten dollars, then you get \$5 of value and I get \$8, whereas if you sell for \$15 then you get \$10 and I get \$2. Either way we're both better off. My increase in value is "consumer surplus", yours is "producer surplus", and the two together are total surplus.

But we really want to think about the *total* surplus in the market. If 100 people buy toys, then there is some consumer surplus for each one, and some producer surplus for each one. If we want to know the total consumer surplus, we have to add up all the individual surpluses.

If everyone has the same surplus, this is easy. A hundred people value a toy at \$18, and buy it for \$10. Each person gets \$8 of surplus, so the total consumer surplus is \$800. But in reality, different people value the toy different amounts: maybe one person values it at \$25 and another person only values it at \$12. Then we have a hundred different surpluses to add up—which means we should use an integral.

To think about theses sorts of markets, economists will often draw supply and demand curves to describe a market. Conceptually, if you want to sell more goods you have to accept a lower price—if you charge \$15, the guy who values the toy at \$25 will still buy it, but the guy who values it at \$12 won't. We can write a *demand function* $d(q)$, which tells you the price at which people *demand* (are willing to buy) q toys.

Similarly, if you want to buy more goods you have to pay a higher price for them, so more people are willing to sell; this gives a *supply function* $s(q)$ that tells the price at which people will supply q toys. The market clears at a price where these two quantities of goods are equal, that is, when $s(q) = d(q)$. (It might make more sense to write q as a function of p, but in practice we usually write p as a function of q.)

We can draw a graph of this situation, which is the famous "supply curve" and "demand curve". But once we draw supply and demand curves, the total surplus is the area between those curves. So we can use our approach of finding the area between two curves to compute total surplus.

Example 6.9. Suppose the demand for a product is given by $p = d(q) = 20 - .05q$ and the supply for the same product is given by $p = s(q) = 2 + .0002q^2$. For both functions, q is the quantity and p is the price, in dollars.

Then demand when $d(q) = s(q)$, so $20 - .05q = 2 + .0002q^2$ and thus we have $.0002q^2 + .05q - 18 = 0$. Using the quadratic formula, we get the solutions -450 and 200, so the equilibrium happens when 200 units are produced, and plugging this back into either d or q tells us the equilibrium price was \$10.

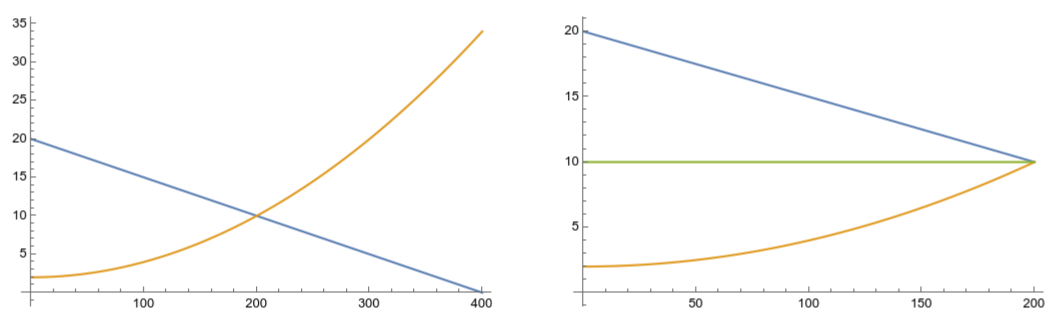


Figure 6.1: Left: the supply and demand curves. Right: the equilibrium price is \$10.

What then is the surplus? The total surplus is

$$\begin{aligned}
\int_0^{200} d(q) - s(q)\,dq &= \int_0^{200} 18 - .05q - .0002q^2\,dq \\
&= 18q - .025q^2 - \frac{.0002}{3}q^3\Big|0^{200} \\
&= 3600 - 1000 - 533.33 = 2066.67.
\end{aligned}$$

Thus the total surplus value created is a bit more than \$2000.

But how is this surplus distributed? We can compute that the equilibrium price is \$10, so we have a consumer surplus of

$$\begin{aligned}
\int_0^{200} d(q) - p\,dq &= \int_0^{200} 10 - .05q\,dq \\
&= 10q - .025q^2\Big|_0^{200} \\
&= 2000 - 1000 = 1000.
\end{aligned}$$

Conversely we can compute the producer surplus of

$$\begin{aligned}
\int_0^{200} p - s(q)\,dq &= \int_0^{200} 8 - .0002q^2\,dq \\
&= 8q - \frac{.0002}{3}q^3\Big|_0^{200} \\
&= 1600 - 533.33 = 1066.67
\end{aligned}$$

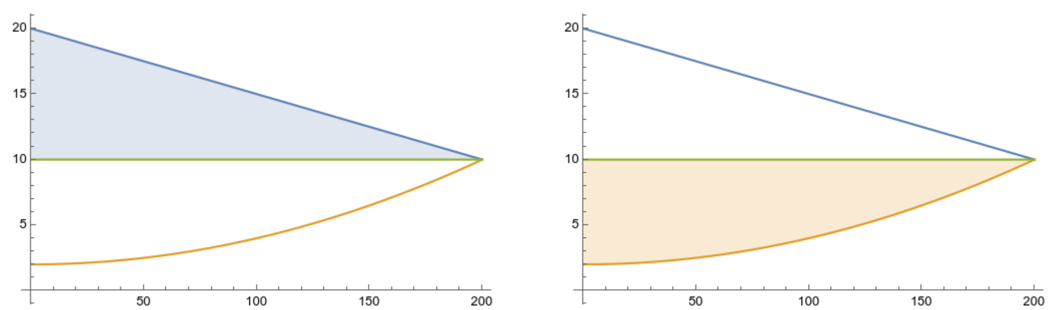


Figure 6.2: Left: the consumer surplus. Right: the producer surplus.

6.2.2 The Average Value of a Function

Another type of problem that involves adding up a bunch of numbers is computing averages. If we have some finite collection of numbers, the average is what we get when we add them up, and divide by the number of numbers:

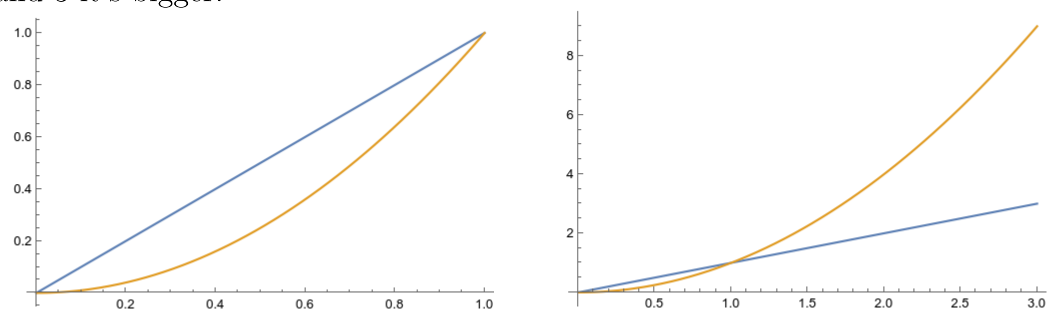
$$\frac{1}{n}\sum_{k=1}^{n} a_k.$$

This gives a straightforward high school problem.

Example 6.10. What is the average value of the set $\{1, 2, 4, 5, 5, 7\}$?

$$\overline{x} = \frac{1+2+4+5+5+7}{6} = \frac{24}{6} = 4.$$

However, sometimes we want to find the average value of a function. This seems like something we should be able to compute; and the question makes some intuitive sense. For instance, between 0 and 1, it seems clear that x^2 is "on average" smaller than x, but between 0 and 3 it's bigger.



A function gives us infinitely many numbers; but integration is in some sense a sensible way to add infinitely many numbers up, and so hopefully to average them.

In particular, if we sample the function at n evenly spaced points, our average is

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*)$$

which you should recognize as a Riemann sum (times $\frac{1}{b-a}$). If we take the limit—which represents taking the average value after "infinitely many" sample points—we get the following definition:

Definition 6.11. The *average value of a function* f over an interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)dt.$$

Example 6.12. What is the average value of $f(x) = x^2$ on $[0, 1]$? We have

$$f_{ave} = \frac{1}{1}\int_0^1 x^2\,dx = \frac{1}{3}.$$

The biggest value is 1, the smallest is 0, and the one in the middle is $\frac{1}{4}$, but the "average" value is $\frac{1}{3}$.

If I have a finite set of numbers and take the average, my average might not be anywhere in the set; for instance, if I roll a six-sided die, the average output will be 3.5, which isn't on the die at all. When I average continuous quantities, however, this can't happen.

Theorem 6.13 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that*

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

In other words,

$$\int_a^b f(t)\,dt = f(c)(b-a).$$

Proof. This statement, as well as its name, might look familiar. In fact this is just the mean value theorem from differential calculus repackaged. Let $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a,b]$ and differentiable on (a,b), and so by the Mean Value Theorem there is some c such that $F(b) - F(a) = F'(c)(b-a)$.

But by the Fundamental Theorem of Calculus, $F'(c) = f(c)$. And it's easy to see that $F(b) = \int_a^b f(t)\,dt$, and $F(a) = \int_a^a f(t)\,dt = 0$. So we have

$$\int_a^b f(t)\,dt - 0 = f(c)(b-a).$$

□

Remark 6.14. Geometrically, this essentially tells us that there is some rectangle with the same area as the region under the graph of f. In particular, we can take a rectangle with width $b-a$, whose top edge intersects the graph of our function *somewhere*, and whose area is the same as the area of the region under the curve.

Now we should discuss some physical and other practical processes that are well-described by integration—which is just a fancy way of saying that integrals let us solve these problems.

6.2.3 Multiplication, Units, and Distance

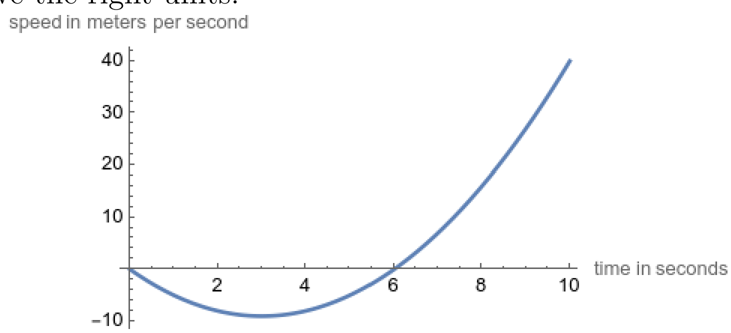
The second way we can use integrals for applications involves *multiplication*. Recall that the integral computes

$$\int_a^b f(t)\,dt = \lim_{n\to+\infty}\sum_{k=1}^n f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x.$$

This is the sum of a bunch of products $f(x_k)\Delta x$.

There are a lot of physical quanties that we can compute as the product of two other things, like distance as the product of speed and time. As long as speed is constant, we can get distance by multiplying speed and time. If you travel 5 meters per second for ten seconds, you will go $5 \cdot 10 = 50$ meters. But if your speed is changing over time, that doesn't quite work.

So imagine we know that our speed is $v(t) = t^2 - 6t$ meters per second. If we travel for ten seconds, we can't really say our total distance is $10(t^2 - 6t)$ meters; that doesn't even have the right units!



Instead, we can split our big time interval up into a bunch of little time intervals, and pretend that our speed is constant on each interval. So for a small time interval, we approximate $d_k \approx v(t_k)\Delta t_k$ for the distance traveled in that interval. The units here make sense, because $v(t_k)$ is in meters per second, and t_k and thus $\Delta(t_k)$ is in seconds; when we multiply them together we get meters. Thus we have total distance

$$D \approx \sum_{k=1}^{n} d_k \approx \sum_{k=1}^{n} v(t_k)\Delta t_k.$$

So for instance we could approximate our total distance traveled as

$$\begin{aligned} D &\approx 2 \cdot v(2) + 2 \cdot v(4) + 2 \cdot v(6) + 2 \cdot v(8) + 2 \cdot v(10) \\ &- 16 - 16 + 0 + 32 + 80 = 80 \end{aligned}$$

so we travel *roughly* 80 meters in ten seconds. (And specifically we travel 16 meters backwards in the first two seconds, then another 16 backwards in the next two seconds, then stay roughly still for the third two seconds, then travel forwards 32 meters in the fourth two seconds, and travel 80 meters forward in the last two seconds.)

But this is just a Riemann sum! So we can get the exact distance traveled by computing

$$D = \int_a^b v(t)\, dt.$$

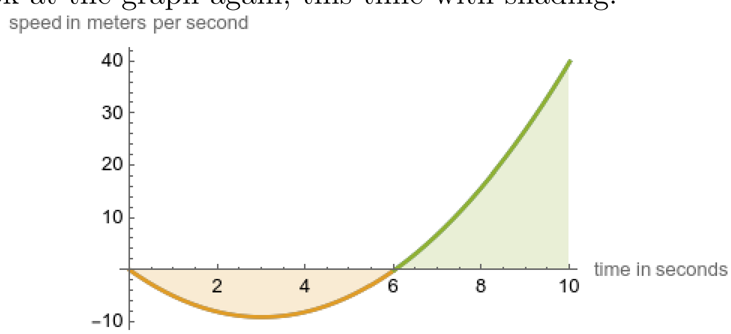
Thus we see that total distance traveled is the integral of velocity. (This shouldn't be a surprise, since velocity is the derivative of distance.)

In our example, we would get

$$\begin{aligned} D &= \int_0^{10} t^2 - 6t\,dt \\ &= \frac{t^3}{3} - 3t^2 \Big|_0^{10} \\ &= \frac{1000}{3} - 300 = \frac{100}{3} = 33.\overline{33}. \end{aligned}$$

This is quite a bit smaller than our estimate; if you look at the graph, you can see that our right-endpoint approximation is wildly overestimating the importance of our high speed at the very end of the process. (If we'd taken a left-endpoint approximation, we'd have estimated that we finish exactly where we started; the truth is roughly in the middle.)

But notice this explains the "net signed area" thing we discussed in section 5.2. Let's look at the graph again, this time with shading:



The integral computes the (green) area above the x axis, minus the (yellow) area below the x axis. In this problem, the green area represents distance moved to the right, and the yellow area represents distance moved to the left; the total distance moved is the green area minus the yellow area.

And one final thing to take away from this is our discussion of units. Any time we compute an integral $\int_a^b f(x)\,dx$, then dx has the units of the variable x, and $f(x)$ of course has whatever units f outputs. So just like the derivative gave units of output-over-input (see section 2.7.2), the integral gives units of output-times-input. If v gives velocity as a function of time, then the integral gives velocity-times-time—which is speed.

6.2.4 Work

Another calculation that works similarly is the physical quantity of *work*.

In physics, *force* is the product of mass and acceleration; intuitively, force is what causes a mass to accelerate, and the more acceleration/the more massive the object, the more force is required. This is often written $F = m \cdot a$, but in our context it is better to say that the position of an object is given by the function $s(t)$, and then $F = m \cdot \frac{d^2s}{dt^2}$, since acceleration is the second derivative of position.

Remark 6.15. In the SI system, mass is measured in kilograms, and force is measured in newtons, where $N = kg \cdot m/s^2$. In the Imperial system most Americans use, the pound is a unit of force; the unit of mass is the *slug*, and one pound is one slug-foot per second squared. I bring this up primarily because the name "slug" is funny.

Intuitively, moving things aroudn takes work, and moving them faster takes more work. Formally, we say that *work* is force times distance: the amount of force applied to an object, times the distance the object is moved. The SI unit for work is the Newton-meter or *joule*, which is $J = kg \cdot m^2/s^2$. The imperial unit for work is the foot-pound, which is about 1.36 joules.

If you lift a 2 kg object a meter, then you have to exert $2 \cdot 9.8$ newtons of force (since acceleration due to gravity is $9.8m/s^2$, and thus do 19.6 joules of work. If a 20 pound weight is lifted five feet, than 100 foot-pounds of work are done.

When force is constant, work is easy to calculuate–just multiply the force by the distance. Things become more interesting when the force varies. As usual, we can approximate by chopping the movement up into lots of little pieces, assuming the force is constant on each small piece, and adding them up. That is, if the force at position x is $F(x)$, then when an object moves from a to b the work done is approximately

$$W \approx \sum_{i=1}^{n} F(x_i) \frac{b-a}{n}.$$

This is a Riemann sum, so taking the limit gives an integral: the total work done is

$$\int_a^b F(x)\, dx.$$

Remark 6.16. Unlike most of the geometric integrals we've been doing for the past few weeks, work can be a negative number; this just indicates that the force is in the opposite direction of the motion.

Example 6.17. A particle is controlled by a force field such that the force on it is $x^3 + x$ pounds when it is x feet away from the origin. How much work does it take to move the particle from $x = 2$ to $x = 4$?

$$W = \int_2^4 x^3 + x\, dx = \frac{x^4}{4} + \frac{x^2}{2}\Big|_2^4 = 64 + 8 - 4 - 2 = 66.$$

Example 6.18. A physical law called Hooke's Law says that the force exerted by a string stretched x units beyond its natural length is kx, where k is the "spring constant" and depends on the particular spring.

Suppose a spring is naturally 20 cm and it takes 50 N to stretch it to 30 cm. How much work is needed to stretch the spring from 30cm to 35cm?

We have $50 = k \cdot .1$ and so $k = 500$. Thus the force when the spring is stretched x meters beyond its normal length is kx, and the work done is

$$W = \int_{.1}^{.15} 500x\,dx = 250x^2|_{.1}^{.15} = 3.125J.$$

Example 6.19. A 50 meter cable has a mass of 50kg and hangs from the top of a cliff. How much work does it take to raise the cable up the cliff?

The thing that makes this difficult is that the mass of the remaining rope depends on how much mass we've lifted already. Conceptually, you can think about having to lift the first meter of rope one meter, and the second meter of rope two meters, etc. Each meter of rope masses 1 kg, so this would give us a Riemann sum

$$W \approx \sum_{i=1}^{50} 1 \cdot 9.8 \cdot i$$

Or more generally

$$W \approx \sum_{i=1}^{n} \Delta x \cdot 9.8 \cdot x_i.$$

Taking the limit gives the integral

$$W = \int_0^{50} 9.8x\,dx = 4.9x^2|_0^{50} = 2500 \cdot 4.9 = 12250J.$$

Example 6.20. A tank of water is shaped like an upside-down pyramid. (No, I don't know why people keep building tanks shaped like upside-down pyramids). The pyramid has a base side length of 4m and a height of 12m, and it is filled with water to a depth of 8m. How much work will it take to pump the water out of the top of the tank? (water has a density of 1000kg / m^3).

Again, to figure out our integral we may want to set up the Riemann sum, or at least fake set it up. Let 0 be the point of the pyramid and 12 be the base (at the top). The volume of a small cross-sectional volume is $A(h)\Delta h$, thus the mass is $1000A(h)\delta h$ and the force is $1000A(h)\Delta h \cdot 9.8$. The distance we have to pump the water is $12 - h$, so the total work on each cross-section is $(12 - h)9800A\Delta h$ Newtons.

Now we just have to work out area in terms of height. Using a similar triangles argument, we see that $\frac{s(h)}{h} = \frac{4}{12}$ and thus $s(h) = h/3$, and $A(h) = h^2/9$. We integrate from 0 to 8 becasue we're integrating over the height that contains water. Then we have

$$\int_0^8 (12-h)9800 \cdot h^2/9 \cdot \, dy = \frac{9800}{9}\left(4h^3 - \frac{h^4}{4}\right)\Big|_0^8 = \frac{9800}{9}(2048 - 1024 - 0) = \frac{10,035,200}{9} J.$$

6.2.5 Bonus: Hydrostatic Pressure

Another problem we can handle easily with these tools is the idea of water (or fluid) pressure. If you imagine a flat surface submerged in some fluid with density ρ to a depth of d meters, then the weight of the fluid over it is $A\rho dg$ where A is the area of the surface (and thus $Ad\rho$ is the mass of the fluid) and $g = 9.8$ is acceleration due to gravity. We define the pressure to the be the force divided by the area, and thus $P = \frac{F}{A} = \rho dg$.

(In SI units we measure this in Newtons per square meter, otherwise known as Pascals. In Imperial units there are a number of different units used, including "inches of mercury.")

Fact 6.21. *If an object is submerged in a fluid to a given depth, the pressure exerted by the fluid is the same in all directions.*

This means that fluid pressure is effectively a function of height/depth and nothing else. If the pressure is varying and we want to find the total force acting on a surface, we can effectively add up the pressure on each little patch of a surface to find the total force acting on it.

Example 6.22. A 3 by 3 meter square is submerged in water until it is just covered, edge-first. What is the total force the water exerts on the square?

We want to chop the square into strips that are all at roughly the same depth. If we slice the square into three horizontal strips, then the ith strip is roughly at depth i meters and has width 3, and thus has roughly the force $3 \cdot 1 \cdot i \cdot \rho \cdot g$. Adding up the force on all thirty strips gives

$$F \approx \sum_{i=1}^{3} 3 \cdot 1 \cdot i \cdot \rho \cdot g = \sum_{i=1}^{3} 3 \cdot 1000 \cdot 9.8 \cdot h\Delta h$$

In the limit, we get the following integral:

$$\int_0^3 3 \cdot \rho \cdot g \cdot h \, dh = \int_0^3 3000 \cdot 9.8 \cdot h \, dh = 29400(h^2/2)\big|_0^3 = 29400 \cdot \frac{9}{2} = 132,300.$$

Example 6.23. A cylindrical drum is lying on its side underwater. The drum has radius of 5 feet and is submerged in 20 feet of water. What is the force exerted on one circular face of the drum?

Let's set 0 to be the center of the circle, so that the equation for the circle is $x^2+y^2=25$. Then the width of the object at height y is $2\sqrt{25-y^2}$. The depth at height y is $15-y$ (which ranges from 10 to 20), and the pressure due to water is 62.5 * depth. So we get the integral

$$F=\int_{-5}^{5} 62.5(15-y)2\sqrt{25-y^2}\,dy = 125\int_{-5}^{5} 15\sqrt{25-y^2}\,dy - 125\int_{-5}^{5} y\sqrt{25-y^2}\,dy.$$

The second integral is 0 because $y\sqrt{25-y^2}$ is an odd function. The first integral can be done by setting $y=5\sin\theta$, but we can also observe that it is the integral of a semicircle of radius 5 and thus is equal to 12.5π. So we have

$$F = 125\cdot 15\cdot 12.5\pi = 23437.5 lb.$$

6.3 Finding Volumes by Cross-Sections

Area is fundamentally length times width, and we computed areas by integrating the length against the width–by which I mean, we wrote the length at a point as a function of the width at that point, and took the integral across the whole width.

Volume is area times height. (Or area times length, depending on your perspective). We will compute volume by finding the area of a cross-section and integrating along the entire length of our shape. Geometrically, the Riemann sum corresponds to slicing our shape into many thin cylinders and adding their areas up.

Remark 6.24. In our terminology, a "cylinder" is any solid that has a flat base and an identical flat top, connected by straight sides at right angles. A traditional circular cylinder qualifies, but so does a rectangular box, and so do stranger shapes.

Definition 6.25. If S is a solid, we say the *cross-sectional area* at a point x is the area of the intersection of our solid with the plane which passes through x and is perpendicular to the x-axis (and thus parallel to the yz plane).

If S is a solid lying between $x=a$ and $x=b$, and $A(x)$ is a function giving the cross-sectional area at x, then we say the *volume* V of S is

$$V = \lim_{\max \Delta x_i\to 0}\sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx.$$

Example 6.26. What is the volume of a cone with height 2 and base radius 4?

We draw a picture. By a similar triangles argument, we see that when we are x distance from the point, the radius is $2x$ and thus the area of the cross-section is $4\pi x^2$. Thus the volume is

$$\int_0^2 4\pi x^2 \, dx = \frac{4\pi x^3}{3}|_0^2 = \frac{32}{3}\pi.$$

This matches the formula for the volume of a cone, which is $\frac{1}{3}\pi r^2 h$.

In fact, we can also rederive that formula. If a cone has height h and base radius b, then the radius at x distance from the height is $x\frac{b}{h}$ and the area is $\pi x^2 b^2/h^2$. So the volume of the cylinder is

$$\int_0^h \pi x^2 b^2/h^2 \, dx = \pi b^2/h^2 \frac{x^3}{3}|_0^h = \frac{b^2 h\pi}{3}.$$

Example 6.27. What is the volume of a solid with a circular base of radius one, where each cross-section is an equilateral triangle?

Make the circle $x^2 + y^2 = 1$. Then the width of the base of the cross-section at x is $2\sqrt{1-x^2}$. Since $\sin 60° = \sqrt{3}2$, we know the height of each triangle is $\sqrt{3}b/2$, and thus the area of the triangle is $\sqrt{3}(1-x^2)$. Thus the volume is

$$\int_{-1}^1 \sqrt{3}(1-x^2) \, dx = \sqrt{3}x - \frac{\sqrt{3}x^3}{3}|_{-1}^1 = \left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) - \left(-\sqrt{3} - \frac{-\sqrt{3}}{3}\right) = \frac{4\sqrt{3}}{3}.$$

These problems are sometimes known as volumes of "solids of rotation," because this technique is particularly good at solving problems like the following:

Example 6.28. What is the volume of the solid obtained by rotating the region bounded by $y = x^2, x = 5, y = 0$ about the x-axis?

We draw a picture, and see that the region has height x^2 at a point x, and thus the solid has a cross-section which is a circle of radius x^2, and thus an area of $\pi(x^2)^2$. It's clear that x varies from 0 to 5. So

$$V = \int_0^5 \pi x^4 \, dx = \frac{\pi x^5}{5}|_0^5 = 5^4\pi - 0 = 625\pi.$$

Example 6.29. What is the volume of the solid obtained by rotatin the region bounded by $y = x^2, y = 25$ with $x \geq 0$ around the y-axis?

As before, we draw a picture. Our region has width $\sqrt{y}$ at a point y, and thus has cross-sectional area πy. Then y varies from 0 to 25, and the volume is

$$V = \int_0^{25} \pi y \, dy = \frac{\pi y^2}{2}|_0^2 5 = \frac{625\pi}{2}.$$

Note that in these problems it's easy to see which way to take our "slices": we want to get the circular cross-sections from the rotation, so we slice accordingly, and integrate along the axis we rotate around.

If our region touches the axis we rotate it around, these problems are straightforward: the cross-sectional area is the height (or width!) of the region squared times π. The problem is trickier if we have a hollow inside. We can still compute the cross-sectional area; it is the area of a *washer*, a circle with a smaller circle cut out of the center.

Remark 6.30. If a washer has outer radius R and inner radius r, then the area is $\pi R^2 - \pi r^2$, the area of the outer circle minus the radius of the inner.

Example 6.31. What is the volume of the solid given by rotating the region bounded by $y = x^2$ and $y = x$ around the x-axis.

At a point x, the cross-section of this solid is a washer. The outer circle has radius x and the inner circle has radius x^2, and thus the area of the cross-section is $\pi x^2 - \pi x^4$. So the volume is

$$V = \int_0^1 (\pi x^2 - \pi x^4)\, dx = \frac{\pi x^3}{3} - \frac{\pi x^5}{5} = \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}.$$

We often find ourselves rotating these regions around lines other than the x- or y-axes. In this case we have to use our geometric intuition a bit more to sort out our cross-sectional areas.

Example 6.32. Rotate the same region about $y = 2$. We draw a picture; we see that we will get a solid whose cross-sections are washers centered at $y = 2$. The outer radius will be $2 - x^2$ and the inner radius will be $2 - x$, so the volume is

$$\begin{aligned} V &= \int_0^1 \pi(2 - x^2)^2 - \pi(2 - x)^2\, dx \\ &= \pi \int_0^1 4 - 4x^2 + x^4 - 4 + 4x - x^2\, dx \\ &= \pi \int_0^1 x^4 - 5x^2 + 4x\, dx \\ &= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2 \right) \Big|_0^1 = \pi(1/5 - 5/3 + 2) = \frac{4\pi}{15}. \end{aligned}$$

Example 6.33. Find the volume of the solid generated by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

We will integrate with respect to x since we rotate about a line parallel to the x-axis. We see that the curves intersect at $x = y = 0$ and $x = y = 1$. Our cross-sections are washers, and we see the outer radius is $1-x$ and the inner radius is $1-\sqrt{x}$. So the volume is

$$V = \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2 \, dx = \pi \int_0^1 x^2 - 3x + 2\sqrt{x} \, dx$$
$$= \pi \left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4}{3} x^{3/2} \right) \Big|_0^1 = \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} \right) = \frac{\pi}{6}.$$

6.4 Bonus material: Finding Volumes with Cylindrical Shells

Recall we want to find the volume of the solid obtained by rotating the region bounded by $x = 1, y = 2, y = \ln x$ about the x-axis. Slicing it into washers as before generates a difficult integral, so we will try to slice it a different way, by slicing it into *cylindrical shells*.

A cylindrical shell is what we get when we take a cylinder and remove a slightly smaller cylinder from the inside. If the outer radius is r_2 and the inner radius is r_1, it's not hard to see that the volume of the shell is $\pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2)$. Less obviously, we factor $r_2^2 - r_1^2 = (r_2 + r_1)(r_2 - r_1)$ and write that the volume is $2\pi \frac{r_1+r_2}{2} h(r_2 - r_1) \approx 2\pi r h \Delta r$.

In many solids of rotation, we can slice the solid into a collection of cylindrical shells to approximate the volume, where the height of each cylinder is $f(x)$ for some x. We get the formula

$$V \approx \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x.$$

As before, our approximation gets better as we use more and thinner cylinders, and when we take the limit, we get

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 1\pi x f(x) \, dx,$$

where a is the inner radius of our entire solid, and b is the outer radius of the entire solid. (Note that this formula is essentially the surface area of the cylinder; this isn't an accident).

So for our earlier example, we can slice into cylinders whose height is in the x-direction. We see that

$$V = \int_0^2 2\pi y(e^y - 1) \, dy = 2\pi \left(ye^y - e^y - \frac{y^2}{2} \right) \Big|_0^2 = 2\pi(e^2 - 1).$$

Remark 6.34. Unlike in the method of washers, this time we will typically integrate with respect to x when we rotate around the y-axis, and vice versa.

Example 6.35. Find the volume of the solid obtained by rotating the region bounded by $y = 0$ and $y = x - x^2$ around the line $x = 2$.

Inverting the function $y = x - x^2$ would be a huge pain; so we'd like to integrate with respect to x, and thus use the cylinder method. Note that in this case the radius r is not x, but is $2 - x$. So the volume is

$$\begin{aligned} V &= \int_0^1 2\pi(2-x)(x-x^2)\,dx \\ &= 2\pi \int_0^1 2x - 3x^2 + x^3\,dx \\ &= 2\pi \left(\frac{x^4}{4} - x^3 + x^2 \right)\Big|_0^1 \\ &= 2\pi(1/4 - 1 + 1) = \frac{\pi}{2}. \end{aligned}$$

Example 6.36. What is the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 0, x = 1$ around the line $x = 1$?

$$V = \int_0^1 2\pi(1-x)x^3\,dx = 2\pi \left(\frac{x^4}{4} - \frac{x^5}{5} \right)\Big|_0^1 = 2\pi \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{\pi}{10}.$$

Example 6.37. What is the volume of the solid obtained by rotating the same region around the line $x = 4$?

$$V = \int_0^1 2\pi(4-x)x^3\,dx = 2\pi \left(x^4 - \frac{x^5}{5} \right)\Big|_0^1 = 2\pi \left(1 - \frac{1}{5} \right) = \frac{8\pi}{5}.$$

Example 6.38 (recitation)**.** What is the volume of the solid obtained by rotating the region bounded by $xy = 1, x = 0, y = 1, y = 3$ about the x-axis?

We draw a picture, and conclude that to use the method of washers we'd have to break the region up into two pieces. Instead we integrate with respect to y and use cylindrical shells. We have y varying from 1 to 3, and the "height" of each cylinder is $1/y - 0$. So the volume is

$$V = \int_1^3 2\pi y(1/y)\,dy = \int_1^3 2\pi\,dy = 2\pi y|_1^3 = 4\pi.$$

Example 6.39. A word has to be said at this point about finding the volume of a sphere. We can view the sphere as a solid of rotation and find its volume using cross-sections:

$$\begin{aligned} V &= \int_{-r}^r \pi(\sqrt{r^2 - x^2})^2\,dx = \pi \int_{-r}^r r^2 - x^2\,dx = \pi \left(r^2 x - \frac{x^3}{3} \right)\Big|_{-r}^r \\ &= \pi \left((r^3 - r^3/3) - (-r^3 + r^3/3) \right) = 4\pi r^3/3. \end{aligned}$$

But we can actually use another approach, similar in spirit to the method of cylindrical shells. We can look at the sphere as being made up of a collection of spherical shells. Taking inspiration from the cylindrical shells method, we see that the volume of each spherical shell will be "about" the surface area of the sphere times thickness; so we integrate the surface area of a sphere of radius x, as x varies from 0 to r. We get

$$V = \int_0^r 4\pi x^2 \, dx = \frac{4\pi x^3}{3}\Big|_0^r = \frac{4\pi r^3}{3}.$$

We haven't entirely justified our argument, but with more care we certainly could.