Math 1231: Single-Variable Calculus I

The George Washington University   Fall 2025

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Contents

3 Optimization

Let's step back and look at the big picture. In section 2 we defined the derivative, and we also talked about one major application of the derivative: linear approximation, which we saw from many perspectives. We took the algebraic approach of approximating a function, the geometric approach of finding equations for tangent lines, and the physical approach of studying rates of change. We then used implicit differentiation to extend these concepts, and related rates was one more use of the rates-of-change understanding of the derivative.

In this section we're going to look at the *other* major application of the derivative: optimization. And while we'll spend the bulk of this section (until 3.6) working abstractly in the realm of functions and graphs, I want to convince you that we are dealing with a real concrete physical question.

If you're running a factory, you may want to ask how you can make as much money as possible. Or you may want to keep your costs as low as possible. Or, if you're feeling pro-social, you may want to minimize the level of pollution you create.

If you're a biologist studying an ecosystem, you may want to know what the maximum population of wolves you can expect to see is. If you're doing medical research, you may want to know what drug dose will be most effective. If you're a physicist studying the motion of an object, you may want to see where the highest point of its trajectory is, or where it reaches its fastest speed, or what the shortest path it can take is.

All of these questions are problems of *optimization*: we have some function or relationship, and we want to find the maximum (or minimum) value it can take on. And so for the next few sections we'll talk about maximizing or minimizing a function, but we always want to remember that this is potentially a very concrete, practical question.

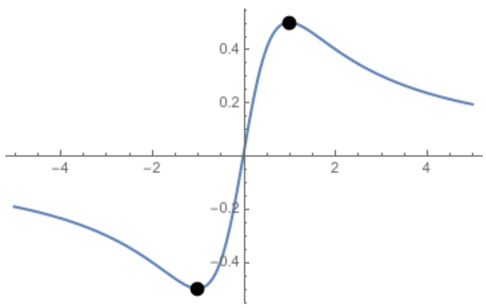
3.1 Extreme Values and Critical Points

So what is it we're looking for? We have a function f, and we want to find the greatest (or least) number it can ever output. If L is the greatest value that f can output, two things need to be true. First, L is actually an output of f; there is some number c such that $f(c) = L$. And second, f never outputs a number that's bigger than that. We can combine those two ideas with the following series of definition.

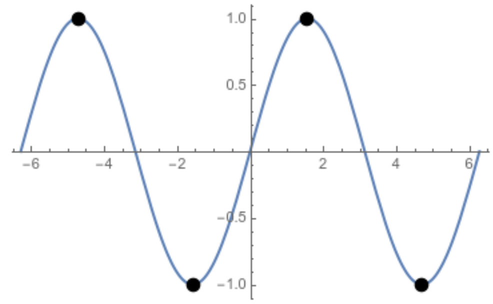
Definition 3.1. If $f(c) \geq f(x)$ for every x in the domain of f, then $f(c)$ is an *absolute maximum* or *global maximum* for f. We say that f has an absolute maximum at c.

Similarly, if $f(c) \leq f(x)$ for every x in the domain of f, then $f(c)$ is an *absolute minimum* or *global minimum* for f, and f has a global minimum at c.

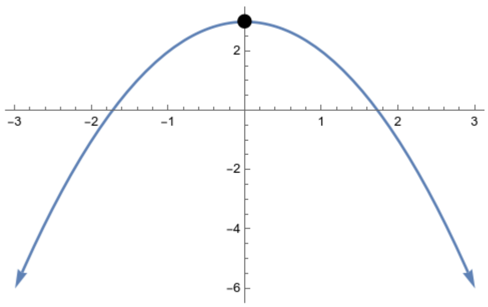
Absolute maxima and absolute minima are somtimes collectively called *extreme values* or *absolute extrema*. ("Extremum" comes from "extreme value," meaning a value that is very big or small or otherwise unusual).



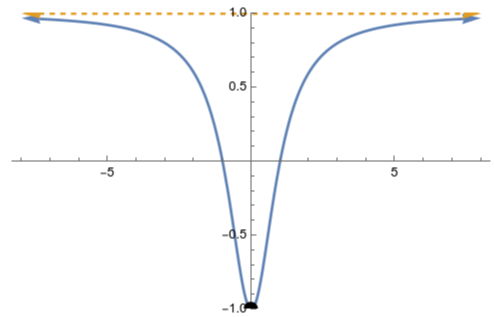
The graph of $\frac{x}{x^2+1}$. This function has a maximum value of 1/2 which occurs at $x = 1$, and a minimum value of $-1/2$ which occurs at $x = -1$.



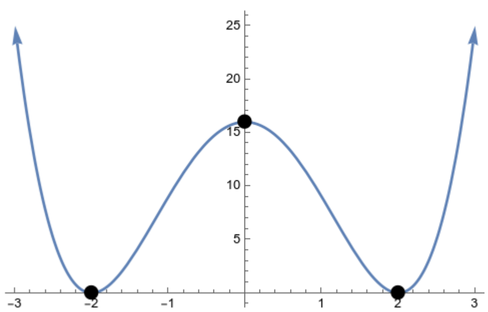
The graph of $\sin(x)$. This function has a maximum value of 1, which occurs at $-3\pi/2, \pi/2, 5\pi/2, \dots$. Similarly, it has a minimum value of -1, which occurs at $-\pi/2, 3\pi/2, 7\pi/2, \dots$.
Very important note: the function has *one* maximum value, which occurs in many different places.



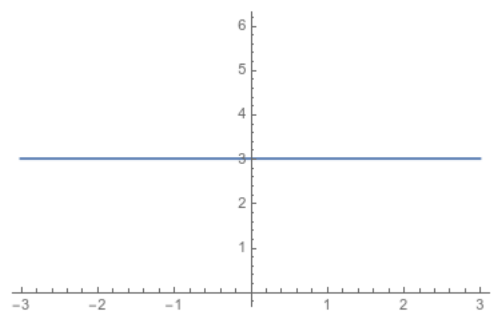
The graph of $3 - x^2$. This function has an absolute maximum of 3, which occurs at zero. It has no absolute minimum: the function goes to $-\infty$



The graph of $\frac{x^2-1}{x^2+1}$. This function has an absolute minimum of -1, which occurs at 0. It has no absolute maximum. We see that the function *almost* reaches a value of 1, but it never actually outputs 1. So 1 can't be the maximum, since it's not an output; but if you pick any number smaller than 1, that can't be the maximum either, because we can always get closer to 1.

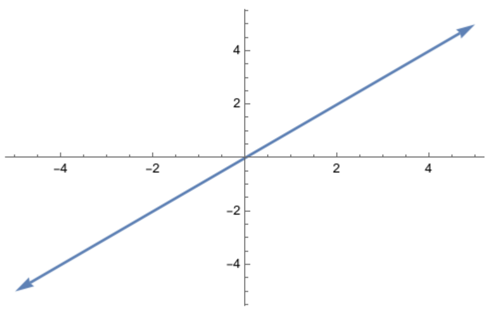


The graph of $x^4 - 8x^2 + 16$. There is a peak in the middle but it's not an absolute maximum, since we get higher values elsewhere. There are relative minima at both -2 and 2, with a value of zero. Again, there is only one absolute minimum: zero. It just happens at two places.

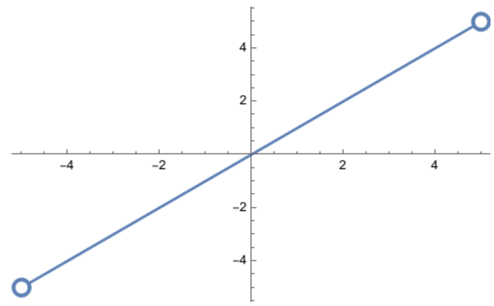


This is a horizontal line, so you might think it doesn't have any maxima or minima. But in fact, the largest value we can get is 3, so the maximum value is three. And the last value we can get is...3. So the minimum value is three. This function has a maximum and a minimum at every single point.

We've noticed that maxima and minima don't necessarily exist. Some of these functions had both a max and a min; some had one but not the other; and some had neither. But if you want to avoid having extrema, there are only a few ways that can happen.

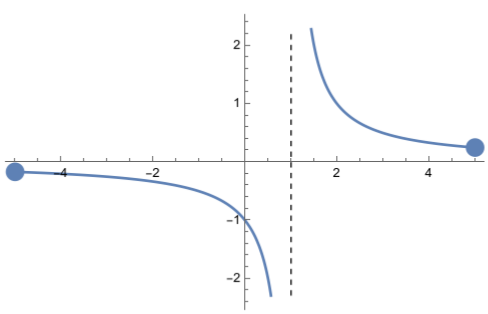


This function $f(x) = x$ has no max or min, because it just keeps going to infinity as x gets bigger or smaller. It's very easy to avoid having extrema if x isn't bounded.

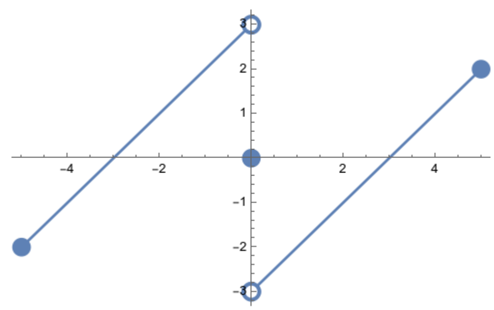


This is still $f(x) = x$, but it has no max or min for a different reason. We have x bounded by $-5 < x < 5$. And that means our function can get *very close* to 5, but never quite gets there. The number 5 kind of "wants to be" the maximum value here, but there's no way to actually get it as an output.

(If we want to be fancy, we say that 5 is the "supremum" of this function on $(-5, 5)$, and -5 is the "infimum". But you don't need to know those words for this course.)



$f(x) = \frac{1}{x-1}$ has endpoints on either side, but there's a discontinuity in the middle where it goes to infinity. This function has no max or min because near 1 the outputs can get arbitrarily large or small.



We get the same problem with just a jump discontinuity. This function has endpoints, and is defined everywhere in $[-5, 5]$. And it stays less than 3 and bigger than -3, but it never actually reaches 3 or -3—it jumps away just before it gets there. So this graph has no maximum or minimum either.

We've seen that we can avoid extrema if our x values are unbounded, or if our function's domain is missing endpoints, or if there's a discontinuity somewhere. But it turns out these are the only ways to avoid having extrema, as stated in the following theorem:

Theorem 3.2 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum $f(c)$ at some point c in the interval $[a, b]$, and an absolute minimum $f(d)$ at some point d in the interval $[a, b]$.*

Note that both the continuity and the closed-ness are important here. I'm not going to try to prove this; it's in a lot of ways the most challenging result to prove in the entire course and depends on advanced ideas like "topological compactness". But looking at the pictures above should convince you there's something here.

Also, this is another "existence theorem", like the Intermediate Value Theorem of section 1.4. It tells us that a global maximum and a global minimum exist, but not anything about where. We can answer this question and find them, but it will require a bit more setup.

If functions are complicated, it's hard to think about the entire function, and find the absolute maximum. So we want to replace this with an easier question. We can look for places where the graph of our function has a peak or a valley, even if it's not the biggest or smallest possible point. This will be much easier to work with, because it allows us to use the tools we've developed already; both limits and derivatives involve focusing on a very small region of the graph, and ignoring everything else.

Definition 3.3. If $f(c) \geq f(x)$ for all x near c, we say that $f(c)$ is a *relative maximum* or a *local maximum* for, and that f has a relative maximum at c.

If $f(c) \leq f(x)$ for all x near c, we say that $f(c)$ is a *relative minimum* or a *local minimum* for f, and that f has a relative minimum at c.

Theorem 3.4 (Fermat's Theorem/Critical Point Theorem)**.** *If f has a local extremum at c, and c is not an endpoint of the domain of f, and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Intuitive idea: If $f'(c) > 0$ then f is increasing, so $f(c + h) > f(c)$ for some small positive h. If $f'(c) < 0$ then f is decreasing, so $f(c + h) > f(c)$ for some small negative h.

To keep things simple, let's suppose f has a local maximum at c, and $f'(c)$ exists. Since $f(c)$ is a local maximum, we know that $f(c) \geq f(c + h)$ for small h, and thus that $f(c + h) - f(c) \leq 0$.

If we take h to be positive, then we can divide both sides by h and we get

$$\frac{f(c+h)-f(c)}{h} \leq 0$$
$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0.$$

But since $f'(c)$ exists, this limit must be $f'(c)$, so $f'(c) \leq 0$.

If we take h to be negative, then dividing both sides of our inequality by h flips the inequality, and we get

$$\frac{f(c+h)-f(c)}{h} \geq 0$$
$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

But since $f'(c)$ exists, this limit must be $f'(c)$, so $f'(c) \geq 0$.

But then $f'(c) \geq 0$ and $f'(c) \leq 0$, so $f'(c) = 0$. □

Remark 3.5.

- The converse of this theorem isn't true: you can have points where $f'(c) = 0$ or $f'(c)$ does not exist that are not local extrema.
- Your textbook uses its words slightly differently, and believes that you cannot have a relative extremum at the endpoint of an interval. I think this is poor word choice, but you should be aware of it when reading the textbook.

Definition 3.6. We say that c is a *critical point* of a function f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Then Fermat's theorem says specifically that if f has a local extremum at c, then c is a critical point. (Again, remember that c can be a critical point without being a local extremum).

Example 3.7.

- Let $f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$; this is defined everywhere, and $f'(x) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$. So the critical points are $\pm\frac{\sqrt{3}}{3}$.
- If $f(x) = x^2$, then $g'(x) = 2x$ and is 0 when $x = 0$. So the only critical point is 0.
- If $h(x) = \sin(x)$ then $h'(x) = \cos(x)$, which is 0 when $x = (n + 1/2)\pi$ for any integer n. Thus the critical points are $\pi/2, 3\pi/2, 5\pi/2, \ldots$.

- If $f(x) = x^3$ then $f'(x) = 3x^2$ which is 0 when x is 0. Thus the only critical point is at 0.

- If $g(x) = |x|$ then
$$g'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ DNE & x = 0 \end{cases}$$
and thus has a critical point at $x = 0$ since the derivative does not exist there.

- If $f(x) = |x^2 - 4|$ then we know that $|x|$ isn't differentiable at 0, so $f(x)$ won't be differentiable at $x^2 - 4 = 0$ and thus at $x = \pm 2$. We see the derivative of the inside is $2x$, so $f'(x) = \pm 2x = 0$ when $x = 0$, and thus the critical points are $0, \pm 2$.

The obvious next question is "how can we determine whether these critical points are a maximum or a minimum or neither?" This is a bit tricky, so we'll hold off for a bit. First we will identify the absolute extrema of a continuous function on a closed interval.

Remember that if f is continuous on $[a, b]$, it must have an absolute maximum and an absolute minimum. By Fermat's theorem, if the absolute extrema are in the interior they must be at critical points. So we can find the absolute extrema by the following method:

(a) List all the critical points.

(b) Evaluate f at each critical point, and at a and b.

(c) The largest value is the maximum and the smallest is the minimum.

Example 3.8.
- If $f(x) = x^3 - x$, we saw the critical points are $\pm\sqrt{3}/3$. If we want the absolute maximum on $[0, 2]$, we compute that $f(0) = 0, f(2) = 6$, and $f(\sqrt{3}/3) = -2\sqrt{3}9$. Thus the absolute maximum is 6 at 2 and the absolute minimum is $-2\sqrt{3}/9$ at $\sqrt{3}/3$.

- Let $h(x) = 2\cos t + \sin(2t)$ on $[0, \pi/2]$. Then $h'(x) = -2\sin(t) + 2\cos(2t) = 0$ when $\sin(t) = \cos(2t)$. On $[0, \pi/2]$ this happens precisely when $x = \pi/6$, so this is the only critical point. We compute $h(0) = 2, h(\pi/2) = 0, h(\pi/6) = 3\sqrt{3}/2$, so the absolute maximum is $3\sqrt{3}/2$ at $\pi/6$ and the absolute minimum is 0 at $\pi/2$.

- Let $f(x) = \frac{x^2+3}{x-1}$ on $[-2, 0]$. Then we see that
$$f'(x) = \frac{2x(x-1) - 1(x^2+3)}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

does not exist at 1. To test when $f'(x) = 0$ we need only consider the numerator, so we have $0 = x^2 - 2x - 3 = (x-3)(x+1)$ and thus $x = 3$ or $x = -1$. So the critical points are $-1, 1, 3$.

f is continuous on $[-2, 0]$ and so must have global extrema. To find them we only need to look at the critical points in $[-2, 0]$, and thus only at -1. So we compute $f(0) = -3, f(-1) = -2, f(-2) = -7/3$. Thus the maximum is -2 (at -1) and the minimum is -3 (at 0).

- What about the global extrema of that same function on $[0, 2]$? We already know the critical points, so we need to check $0, 1, 2$. We have $f(0) = -3$ and $f(2) = 7$, but $f(1)$ is not defined. In fact the function is not defined everywhere on $[0, 2]$ and so not continuous; it has an asymptote at $x = 1$ and thus no minimum or maximum.

We'd still like to determine what each critical point is like, but for that we will need more tools.

3.2 The Mean Value Theorem

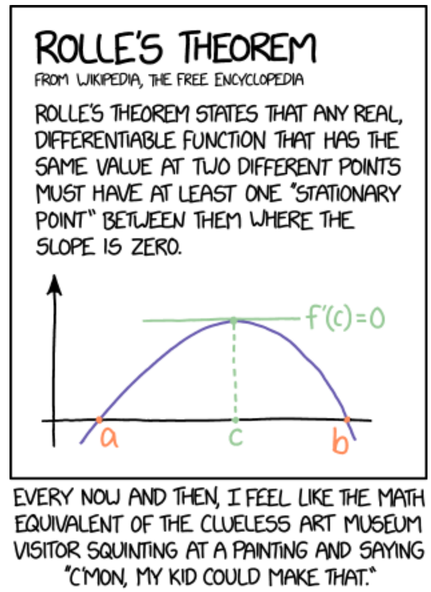
Now we're going to take a brief detour from computation to do something that, in theory, we should have done a while ago: convince ourselves that the derivative actually does what it's supposed to do.

In section 2 we raised a few different questions: how do we approximate a function with a linear function? (2.1) How do we estimate how the output of a function will change when the input changes? (2.7) How can we find an equation for the tangent line to the graph of a function? (2.8) And we argued that all three of these questions should be answered with the same computational tool, the derivative.

But it's important to stop at some point in this process and figure out whether the derivative really does answer those questions. So we're going to take a detour into theory to make sure that the derivative does what we want it to do. Specifically, we're going to pick a few things that should be true if the derivative does what it should, and check that they're actually true of the actual derivative.

And that means that this section will feel a little bit backwards. The things we're going to prove are obviously true. And that's fine! The goal of this section is not to prove those things are true; it's to check that *the derivative*, the one we defined, makes those things true. So we're going to prove that if your top speed is below sixty miles per hour, you can't go more than sixty miles in one hour. That should be obvious; the point is that this works if we interpret your "top speed" as referring to the maximum of the derivative.

Before we can get there, though, we need to start with a somewhat technical result called Rolle's Theorem. Named after the French mathematician Michel Rolle who partially proved it in 1691 without using calculus (which he apparently didn't believe in!), it was fully proven by Cauchy in 1823, and given his name in 1834. This theorem shows that under certain conditions, a function has to have a point where the derivative is zero, and thus the tangent line is horizontal. This isn't especially interesting on its own, although it does have one neat application where we can use it to prove an equation doesn't have too many solutions. But it's easy to prove, and we can leverage it to prove a more powerful and important result.



https://xkcd.com/2042/

Theorem 3.9 (Rolle)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a point c in (a, b) where $f'(c) = 0$.*

Proof. If f is constant everywhere, then the derivative is 0 everywhere.

By the Extreme Value theorem, f has a global maximum on $[a, b]$. If there is some x in (a, b) with $f(x) > f(a)$, then the maximum is in the interior at some point c, and by Fermat's theorem, since $f'(c)$ must exist, we have $f'(c) = 0$.

If f is not constant, and there is no x with $f(x) > f(a)$, then there is some f with $f(x) < f(a)$. Then f has an absolute minimum in the interior at some point c. By Fermat's theorem $f'(c) = 0$. □

Remark 3.10. We need f to be continuous at the endpoints, but it doesn't have to be differentiable there. Rolle's theorem does guarantee a derivative of zero somewhere in the interior–not just at the endpoints.

Example 3.11. If $f(x)$ represents the height of an object, $f'(x)$ represents its speed. If I throw an object up and wait for it to fall back down to the ground, at some point during the process (at the top of its arc) it's instantaneous velocity will be 0.

Example 3.12. We can prove that $f(x) = x^3 + x - 1$ has exactly one real root.

First we use the Intermediate Value Theorem to show that a root exists at all. f is continuous because it's a polynomial. We see that $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by the Intermediate Value Theorem there's some a in $(0, 1)$ with $f(a) = 0$. Thus f has at least one real root.

Now suppose $f(b) = 0$ and $b \neq a$. Then f is continuous and differentiable everywhere, and $f(a) = f(b)$, so by Rolle's theorem there's some c in between a and b with $f'(c) = 0$.

But $f'(c) = 3c^2 + 1$, and since $c^2 \geq 0$, we know that $f'(c) \geq 1$ for every c. Thus there's no c with $f'(c) = 0$, so there's no $b \neq a$ with $f(b) = 0$. Thus f has exactly one real root.

Rolle's theorem can be useful, but it's very limited by the need for $f(a) = f(b)$. The Mean Value Theorem lets us lift that restriction.

Theorem 3.13 (Mean Value Theorem). *If f is continuous on $[a, b]$ and differentiable on (a, b), then there's a c in (a, b) with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We prove this using Rolle's theorem, by writing an altered version of f that satisfies the hypotheses of Rolle's theorem. Define

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This is basically just taking $f(x)$ and then subtracting off the line from $(a, f(a))$ to $(b, f(b))$. It's clear that

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 - \frac{f(b) - f(a)}{b - a}0 = 0 \\ h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = (f(b) - f(a)) - (f(b) - f(a)) = 0 \end{aligned}$$

so $h(a) = h(b)$. h is continuous on $[a, b]$ because f is continuous on $[a, b]$, polynomials are continuous, and the sum of two continuous functions is continuous. h is differentiable on (a, b) because f is differentiable on (a, b), polynomials are differentiable, and the sum of two differentiable functions is differentiable.

Thus h satisfies the hypotheses of Rolle's theorem. Then there's some c in (a, b) with $h'(c) = 0$. But

$$\begin{aligned} h'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a}(1 - 0) \\ 0 &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

as we desired. □

Example 3.14. Earlier in the class, we talked about driving to San Diego. That's about 120 miles, so if it takes me two hours to get there, my average speed is 60 mph. That doesn't mean my speed at each point is 60 mph, though; I might go 90 part of the way and then 20 part of the way while I'm stuck in traffic. But the Mean Value Theorem tells me that at some point during that drive the needle on my speedometer pointed at the 60–which makes sense, since it will do that while I'm accelerating up to 90.

Example 3.15. We can also use the mean value theorem to constrain the possible values for a function. For instance, suppose I have a function f, and all I know is that $f(1) = 10$ and $f'(x) \geq 2$ for every x. Then if I want to know about $f(4)$, I can conclude that there is some c in $(1, 4)$, such that:

$$\begin{aligned} f'(c) &= \frac{f(4) - f(1)}{4 - 1} \\ 3f'(c) &= f(4) - 10 \\ f(4) &= 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16. \end{aligned}$$

Thus $f(4) \geq 16$.

Example 3.16. Suppose $|f'(x)| \leq 2$ for all x, and $f(0) = 7$. What do we know about $f(5)$?

We know that for any x, $-2 \leq f(x) \leq 2$. By the mean value theorem, we have

$$\begin{aligned} f'(c) &= \frac{f(5) - f(0)}{5 - 0} \\ -2 &\leq \frac{f(5) - f(0)}{5 - 0} \leq 2 \\ -10 &\leq f(5) - 7 \leq 10 \\ -3 &\leq f(5) \leq 17. \end{aligned}$$

This corresponds to the intuition that if you're travelling less than 2 miles per hour, you won't get more than ten miles in five hours; and if you start at 7, you'll wind up between -3 and 17.

Example 3.17. Show $f(x) = x^5 + x^3 + x$ has exactly one root.

It's pretty clear that f has a root; we could use the intermediate value theorem, but we can also observe that $f(0) = 0$.

Suppose $f(a) = f(b) = 0$. Then by Rolle's Theorem there is some c with $f'(c) = 0$. But $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ and thus $f'(c)$ is never zero; so f has at most one root, and thus exactly one root.

More intuitively, $f(x)$ has at most one root because it's always increasing, and so one it gets above zero it can't come back down and hit zero again. Which leads us to discuss the idea of increasing or decreasing functions.

3.3 Increasing or Decreasing Functions and Finding Relative Extrema

We now want to use the Mean Value Theorem to answer our original question, about which critical points are maxima or minima. We start with a defnition:

Definition 3.18. We say that f is *(strictly) increasing* on an interval (a, b) if, whenever x_1 and x_2 are points in (a, b) and $x_2 > x_1$, then $f(x_2) > f(x_1)$.

We say that f is *(strictly) decreasing* on an interval (a, b) if, whenever x_1 and x_2 are points in (a, b) and $x_2 > x_1$, then $f(x_2) < f(x_1)$.

Notice that these definitions make sense if you assume we're moving to the right; an increasing function is one where $f(x)$ increases as x increases.

Proposition 3.19.

- *If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).*
- *If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).*
- *If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).*

Proof. Let x_1, x_2 be two points in (a, b) with $x_2 > x_1$. Then since f is differentiable (and thus continuous) everywhere in (a, b), it is continuous and differentiable everywhere on $[x_1, x_2]$, and by the mean value theorem there is some c with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$(x_2 - x_1)f'(c) = f(x_2) - f(x_1).$$

- Now, if $f'(x) = 0$ for all x, then $f'(c) = 0$ and thus $f(x_2) - f(x_1) = 0$. This is true for any points x_1 and x_2, and thus f is constant.
- If $f'(x) > 0$ for all x, then $f'(c) > 0$. Since $x_2 - x_1 > 0$, this implies that $f(x_2) - f(x_1) > 0$. This is true for any points $x_1 < x_2$ and thus f is increasing.
- If $f'(x) < 0$ for all x, then $f'(c) < 0$. Since $x_2 - x_1 < 0$, this implies that $f(x_2) - f(x_1) < 0$. This is true for any points $x_1 < x_2$ and thus f is decreasing.

□

Remark 3.20. This theorem doesn't say anything about intervals where f isn't always differentiable. It also doesn't say anything about intervals where f' switches sign in the middle. In practice, we split the domain of our function up into intervals on which exactly one of these things is happening and study each interval separately.

Example 3.21. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Where is f increasing or decreasing?

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ is 0 when $x = 0, -1, 2$. These three points are the critical points. $f'(x)$ has three factors, and it will be positive when one or all three factors are positive. We make a chart:

	$12x$	$x-2$	$x+1$	$f'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Thus $f'(x)$ is positive when $-1 < x < 0$ or $2 < x$, so f is increasing on $(-1, 0)$ and on $(2, +\infty)$. $f'(x)$ is negative when $x < -1$ or $0 < x < 2$, so f is increasing on $(-\infty, -1)$ and $(0, 2)$.

Can we use this information about increasing and decreasing functions to say something about relative maxima and minima? In fact, assuming f is continuous at c, if f is increasing to the left of a point c and decreasing to the right of c, then it must have a maximum at c. Similarly, if f is decreasing to the left and increasing to the right, it must have a minimum. If it increases on both sides or decreases on both sides, then c is neither a maximum nor a minimum. Therefore:

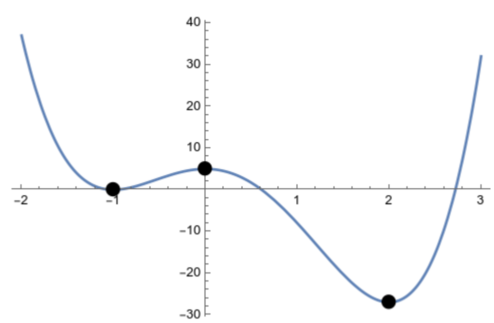


Figure 3.1: a graph of $3x^4 - 4x^3 - 12x^2 + 5$ with critical points marked

Proposition 3.22 (First derivative test for extrema).
If c is a critical point of f and f is continuous at c, then

- *If f' changes from positive to negative at c then f has a relative maximum at c.*
- *If f' changes from negative to positive at c then f has a relative minimum at c.*

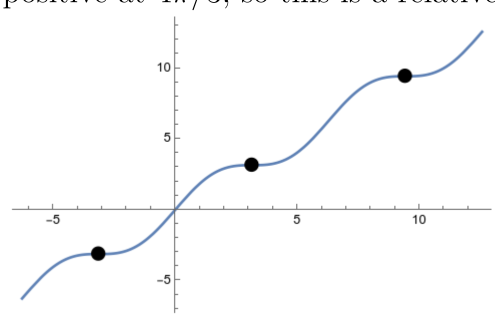
- *If f' "changes" from positive to positive or negative to negative at c then f has neither a relative maximum nor a relative minimum at c.*

Remark 3.23. If f' is continuous, the sign of f' actually only *can* change at a critical point by the intermediate value theorem. So we just have to check the sign of f' at one point in between each critical point.

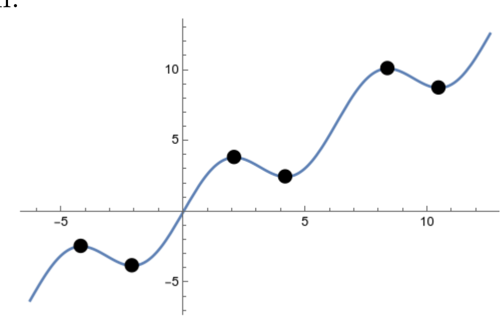
So what does this say about our previous example? We had three critical points, at $-1, 0, 2$. At -1 we saw that f' changed from negative to positive, so f has a relative minimum $f(-1) = 0$ at -1. Similarly, at 0 f' changed from positive to negative and at 2 f' changed from negative to positive, so f has a relative maximum of $f(0) = 5$ at 0 and a relative minimum of $f(2) = -27$ at 2.

Example 3.24. Let $g(x) = x + \sin(x)$. Then $g'(x) = 1 + \cos(x)$ is zero precisely when $x = (2n+1)\pi$ for some integer n. Since we only need to check the sign of g' at one point between each critical point, we check that $g'(2n\pi) = 1 + \cos(2n\pi) = 2$. Thus g' is positive everywhere except at the critical points, so g is increasing everywhere except at the critical points. Thus g has no relative maxima or minima.

Now let $h(x) = x + 2\sin(x)$. We have $h'(x) = 1 + 2\cos(x) = 0$ when $x = 2n\pi + 4\pi/3$ or $x = 2n\pi + 2\pi/3$. We compute that $h'(0) == 3, h'(\pi) = -1$, and $h'(2\pi) = 3$. Thus h' changes from positive to negative at $2\pi/3$, so this is a relative maximum. h' changes from negative to positive at $4\pi/3$, so this is a relative minimum.



Graph of $g(x) = x + \sin(x)$ on $[-2\pi, 4\pi]$ with critical points marked. Notice none of them are relative extrema.



Graph of $h(x) = x + 2\sin(x)$ on $[-2\pi, 4\pi]$ with critical points marked. Every critical point is a relative extremum.

But we'd like to find relative maxima and minima with even less work, which brings us to the subject of concavity.

3.4 Concavity and the Second Derivative Test

Definition 3.25. We say a function f is *concave upward* on an interval (a, b) if every tangent line to a point in (a, b) lies below the graph of f.

We say a function f is *concave downard* on (a, b) if every tangent line to a point in (a, b) lies above the graph of f.

We say a point c is an *inflection point* for a function f if the graph of f changes from concave up to concave down, or concave down to concave up, at c.

Remark 3.26. Functions that are concave upward are curving up, like a bowl. Functions that are concave downward are curving down, like an umbrella.

Example 3.27. Looking at graphs, we can see:

- x^2 is concave upward everywhere. $-x^2$ is concave downward everywhere.
- x^3 is concave downward when $x < 0$ and is concave upward when $x > 0$.
- $\sqrt[3]{x}$ is concave upward when $x < 0$ and concave downward when $x > 0$.
- $\sin(x)$ is concave downward when $0 < x < \pi$ and concave upward when $\pi < x < 2\pi$.

We see that when a function is concave upward, the slopes of its tangent lines are increasing–which means the derivative is increasing. Similarly, a function is concave downward when its derivative is decreasing. But we just showed that we can determine whether a function is increasing or decreasing by looking at its derivative. So we need to study the derivative of the derivative–the second derivative.

Proposition 3.28 (Concavity Test).

- *If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).*
- *If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave downward on (a, b).*

Remark 3.29. It's not necessarily true that f has an inflection point whenever $f''(x) = 0$. But it often is.

Example 3.30.

- $\frac{d}{dx}x^2 = 2x$, so $\frac{d^2}{dx^2}x^2 = 2 > 0$, so x^2 is concave upward everywhere. Similarly, $\frac{d^2}{dx^2} - x^2 = -2, 0$, so $-x^2$ is concave downward everywhere. Neither function has an inflection point.
- $\frac{d^2}{dx^2}x^3 = 6x$ is positive if $x > 0$ and negative if $x < 0$, so the function is concave upward when $x > 0$ and concave downward when $x < 0$. It has an inflection point when $x = 0$.

- $\frac{d^2}{dx^2}\sqrt[3]{x} = \frac{-2}{9\sqrt[3]{x^5}}$ is negative when $x > 0$ and positive when $x < 0$, so the function is concave upward when $x < 0$ and concave downward when $x > 0$. It has an inflection point when $x = 0$.
- $\frac{d^2}{dx^2}\sin(x) = -\sin(x)$, so $\sin(x)$ is concave upwards precisely when it is positive, and concave downwards when it is negative. It has an inflection point at $0, \pi, 2\pi$, and in general at $n\pi$ for any integer n.
- Consider $f(x) = x^4$. $f''(x) = 12x^2$ is positive everywhere except at 0, so the function is concave upwards everywhere except at 0. $f''(0) = 0$, so the second derivative concavity test doesn't tell us anything. But this isn't an inflection point, because the concavity doesn't change on either side–in fact the function is concave at $x = 0$ as well, as you can see from a graph.

Why do we care? Notice that if f is concave upward then the first derivative is increasing; so if $f'(c) = 0$ and f is concave upwards at c, the derivative is changing from negative to positive, and f has a local minimum at c. A similar argument works for local maxima, and thus:

Proposition 3.31 (The Second Derivative Test). *If f'' is continuous near c, then*

- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark 3.32.

- If $f''(c) = 0$ this theorem tells us nothing; almost anything could happen. We can use the increasing/decreasing function test, or we can use the third and fourth derivatives to give us information.
- This rule only works if $f'(c) = 0$; if $f'(c)$ doesn't exist, then $f''(c)$ certainly doesn't exist and this proposition is not helpful.

Example 3.33. We looked at the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in example 3.21. We computed that $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$, so the critical points are $x = -1, 0, 2$.

Then $f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$. We can compute

$$f''(-1) = 12(3 + 2 - 2) = 36 > 0$$
$$f(0) = -24 < 0$$
$$f(2) = 12(12 - 4 - 2) = 72 > 0$$

so by the second derivative test, f has a local maximum at 0 and local minima at -1 and 2.

This was a little faster and easier than the way we original classified the maxima and minima of this function. But sometimes the second derivative test just isn't very helpful.

Example 3.34. Let $f(x) = x^{2/3}(6-x)^{1/3}$. Where does f have relative maxima and minima? Where is it increasing or decreasing?

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$
$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}.$$

Then $f'(x) = 0$ when $x = 4$, and $f'(x)$ does not exist when $x = 0$ or $x = 6$, so these are the three critical points.

We can use the second derivative test—or try to. We see that $f''(4) = \frac{-8}{2^{13/3}} = -2^{-4/3} < 0$ so f has a maximum at 4. But at 0 and at 6, the second derivative isn't defined, so the second derivative test isn't useful there.

But we can still use the first derivative test. We get a table:

	$4-x$	$x^{-1/3}$	$(6-x)^{-2/3}$	$f'(x)$
$x < 0$	$+$	$-$	$+$	$-$
$0 < x < 4$	$+$	$+$	$+$	$+$
$4 < x < 6$	$-$	$+$	$+$	$-$
$6 < x$	$-$	$+$	$+$	$-$

This tells us that f has a minimum of $f(0) = 0$ at 0 and a maximum of $f(4) = 2^{5/3}$ at 4. It doesn't have a local maximum or minimum at 6.

But now we can do one more thing. Our table tells us that f is increasing for $0 < x < 4$, and it's decreasing for $x > 0$ or $x > 4$. And further, we can do the same thing for the second derivative. The second derivative is zero, or undefined, at 0 and at 6. So we get

	-8	$x^{-4/3}$	$(6-x)^{-5/3}$	$f'(x)$
$x < 0$	$-$	$+$	$+$	$-$
$0 < x < 6$	$-$	$+$	$+$	$-$
$6 < x$	$-$	$+$	$-$	$+$

So the function is concave down for $x < 6$ and concave up for $x > 6$. We say that $x = 6$ is a *point of inflection* for this function, where the concavity changes. And we can use this information to sketch an effective graph of the function.

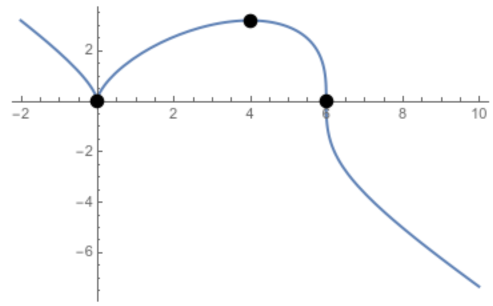


Figure 3.2: The graph of $f(x) = x^{2/3}(6-x)^{1/3}$ with critical points

3.5 Curve sketching

And now we're ready to approach the task of sketching the graph of a function in an organized way. What follows is a good checklist, though not every point is relevant to every function.

(a) Find the domain of the function. If it has holes, what happens near them? Does it go to infinity, or jump, or just skip a point?

(b) Find the roots–where does the function hit the x-axis?

(c) Find the limits as x goes to $\pm\infty$–what happens to the function "far away" from 0?

(d) Compute f' and find the critical points. It can be helpful to evaluate f at the critical points.

(e) Find intervals of increase or decrease. Identify local maxima and minima.

(f) Compute f'' if you haven't already. Determine where the function is concave, and find inflection points.

(g) Use all this information to sketch a graph of the function.

Example 3.35. Let $f(x) = x(x-4)^3 = x^4 - 12x^3 + 48x^2 - 64x$. Then:

(a) The function is a polynomial, so its domain is all real numbers.

(b) The function has roots at 0 and 4.

(c) $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$.

(d) $f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(4x-4) = 4(x-1)(x-4)^2$. So $f'(x) = 0$ when $x = 1$ or $x = 4$. These are the critical points. $f(1) = -27$ and $f(0) = 0$.

(e) Looking at our factorization, it's clear that $f'(x) < 0$ when $x < 1$ and $f'(x) > 0$ when $x > 1$, except $f'(x) = 0$ when $x = 4$. So f is decreasing when $x < 1$ and is increasing when $x > 1$ except at 4. Thus f has a minimum of -27 at 1.

(f) $f''(x) = (x-4)^2 + 2(x-1)(x-4) = (x-4)(3x-6) = 3(x-2)(x-4)$. We see that $f''(x) > 0$ is $x < 2$ or $x > 4$, and $f''(x) < 0$ if $2 < x < 4$. Thus f is concave up on $(-\infty, 2)$ and $(4, +\infty)$, is concave down on $(2, 4)$, and has inflection points at 2 and 4.

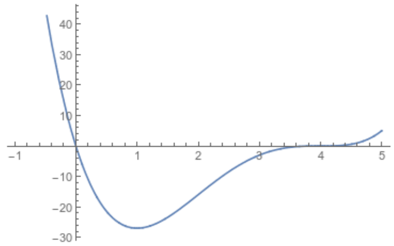


Figure 3.3: The graph of $f(x) = x(x-4)^3$

Example 3.36. Let $g(x) = x\tan(x)$. Then

(a) The domain of g is real numbers except $n\pi + \pi/2$. For simplicity we'll just look at x between $-\pi/2$ and $\pi/2$. $\lim_{x\to-\pi/2^+} g(x) = +\infty$ and $\lim_{x\to\pi/2^-} g(x) = +\infty$.

(b) The function is 0 when $x = 0$ (and when $x = n\pi$ if we look farther out).

(c) This isn't applicable since we're not looking out to $\pm\infty$.

(d) $g'(x) = \tan(x) + x\sec^2(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. It's not hard to see that when $-\pi/2 < x < 0$ then $g'(x) < 0$, and when $0 < x < \pi/2$ then $g'(x) > 0$, and $g'(0) = 0$. So the only critical point is at 0.

(e) And we saw that g is decreasing on $(-\pi/2, 0)$ and increasing on $(0, \pi/2)$. Thus g has a local minimum at 0. $g(0) = 0$.

(f) $g''(x) = \sec^2(x) + \sec^2(x) + 2x\sec(x)\sec(x)\tan(x) = 2\sec^2(x)(1 + x\tan(x))$. $x\tan x \geq 0$ on $(-\pi/2, \pi/2)$, so the function is concave up everywhere.

Example 3.37. Let $h(x) = \frac{x+2}{x-1}$.

(a) The domain of h is all real numbers except 1. We see that $\lim_{x\to1^-} h(x) = -\infty$ and $\lim_{x\to1^+} h(x) = +\infty$.

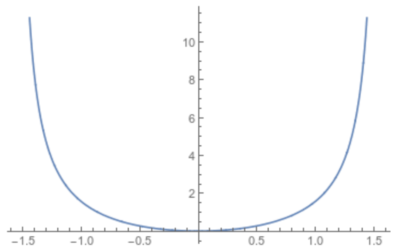


Figure 3.4: The graph of $g(x) = x\tan(x)$

(b) The function has a root at $x = -2$.

(c) We have $\lim_{x\to+\infty} h(x) = \lim_{x\to-\infty} h(x) = 1$. (We can use L'Hôpital's rule or divide the top and bottom by x).

(d) We have $h'(x) = \frac{(x-1)-(x+2)}{(x-1)^2} = -3(x-1)^{-2}$. This has no roots and fails to exist when $x = 1$. Thus there are no "real" critical points.

(e) We make a chart for increase and decrease:

	-3	$(x-1)^{-2}$	$h'(x)$
$x < 1$	$-$	$+$	$-$
$1 < x$	$-$	$+$	$-$

Thus h is decreasing everywhere. It has no local maxima or minima.

(f) $h''(x) = 6(x-1)^{-3}$ is positive when $x > 1$ and negative when $x < 1$, so it is concave down on the left, and concave up on the right.

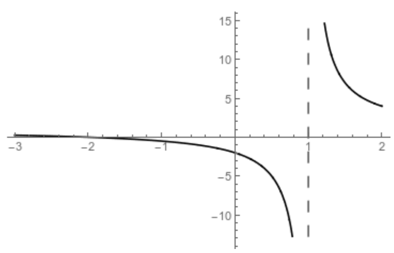


Figure 3.5: The graph of $h(x) = \frac{x+2}{x-1}$

3.6 Physical Optimization Problems

Through most of this section we've been finding the minimum and maximum values of functions purely to understand the functions. But the techniques used to maximize a function are extremely useful in finding optimum inputs to real world processes.

In other words, we're going to do more word problems.

Example 3.38. Suppose we have 2400 feet of fencing and we'd like to build a rectangular fence that encloses the most possible area. How can we do this?

If we have a rectangular fence, then one side will have a length L and another will have a width W. We know that the area $A = W \cdot L$ and that $2W + 2L = 2400$. So we can write $W = 1200 - L$ and see that $A = L(1200 - L)$. We'd like to maximize area.

We observe that our L has to be between 0 and 1200, so we're maximizing the function A on the closed interval $[0, 1200]$. By the extreme value theorem there must be some absolute maximum.

$A' = 1200 - 2L$. We see that the only critical point is $L = 600$. $A(0) = A(1200) = 0$ and $A(600) = 600^2 = 360,000$. $A(600)$ is the largest of these values, and so is the absolute max.

But what if we build the fence against a river, so we only need to build three sides? Then $A = W \cdot L$ but $W + 2L = 2400$, and thus $W = 2400 - 2L$. Then we have $A = L(2400 - 2L)$. A is still a function of L defined on $[0, 1200]$, and we compute $A' = 2400 - 4L$ and the only critical point is $L = 600$, again. $A(0) = A(1200) = 0$, and $A(600) = 600 \cdot 1200 = 720,000$. This last is the largest of the values, and the absolute max.

Example 3.39. Suppose we want to construct a cylindrical can that holds one liter of liquid, and we want to use the least possible metal to construct the can—and thus build the can with the least possible surface area. We have $A = 2\pi r^2 + 2\pi r h$.

To eliminate the h, we note that the can holds one liter or 1000 cm^3, and thus $\pi r^2 h = 1000$ and $h = \frac{1000}{\pi r^2}$. (We also could have written it as one cubic decimeter, but nobody ever works in decimeters). Thus we have $A = 2\pi r^2 + \frac{2000}{r}$.

$A' = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = 0$ when $\pi r^3 = 500$, or when $r = \sqrt[3]{500/\pi}$. So this is the only critical point. Our function A has domain $(0, +\infty)$ so we can't use the extreme value theorem here. But we can see that A' is negative when $r < \sqrt[3]{500/\pi}$ and positive when $r > \sqrt[3]{500/\pi}$, so that must be a global minimum.

(Alternatively: $A'' = 4\pi + \frac{4000}{r^3}$ is always positive, so A is concave upwards everywhere, and has a unique minimum at its critical point).

But now what if the curved material for the sides costs more than the flat material for the ends, and we want to minimize cost? Say the material for the sides costs twice as much as material for the base. Then we have $C = 2\pi r^2 + \frac{4000}{r}$, and $C' = 4\pi r - \frac{4000}{r^2} = 0$ when $\pi r^3 = 1000$, when $r = 10/\sqrt[3]{\pi}$. This is the only critical point, and a similar argument to before shows it must be a global minimum.

We can break down our approach to these problems just as we did for related rates.

(a) Draw a picture of the setup.

(b) Create notation. Give names to all the quantities involved in the problem. Write down any equations that relate them.

(c) Express the quantity you want to maximize or minimize as a function of the other quantities in the problem. Rewrite it so it's a function of a single variable.

(d) Take the derivative and find the critical points.

(e) Determine the absolute maximum or minimum.

(f) Do a sanity check! Does your answer make sense?

Example 3.40. If we have 1200 cm^2 of cardboard to make a box with a square base and an open top, what is the largest possible volume of the box?

Well, we know that the total surface area of the box is $A = 1200$, and we also know that if the height of the box is h and the length of one of the base sides is b, then the area is $A = b^2 + 4bh$. So we can write $h = \frac{1200-b^2}{4b}$. We also know that the volume of the box is $V = b^2h$, so we have

$$\begin{aligned} V &= b^2h = b^2\frac{1200-b^2}{4b} \\ &= 300b - b^3/4 \\ V' &= 300 - 3b^2/4 \\ 300 &= 3b^2/4 \\ 400 &= b^2 \\ 20 &= b \end{aligned}$$

so the only critical point occurs at 20. We see that $V(20) = 400 \cdot 10 = 4000$, so this is the largest possible volume of the box. (We can see that this is the absolute maximum via the Extreme Value Theorem, and observing that $V(0) = V(\sqrt{1200}) = 0$.

Example 3.41. Suppose a man wishes to cross a 20 m river and reach a house on the other side that is 48m downstream. The man can walk at 5 m/s or swim at 3 m/s. What is the optimal path for him to take to reach the house?

The man will swim for some point on the bank of the river, and then walk the other way. Let b be a number in $[0, 48]$ representing how far he travels towards the house. Then he travels $\sqrt{400+b^2}$ meters in the river, at a speed of 3 m/s, and thus spends $\frac{1}{3}\sqrt{400+b^2}$ seconds in the river. He then spends $(48-b)/5$ seconds walking.

So total time spent is

$$\begin{aligned}
T &= \frac{\sqrt{400+b^2}}{3} + \frac{48-b}{5} \\
T' &= \frac{b}{3\sqrt{400+b^2}} - \frac{1}{5} \\
\frac{1}{5} &= \frac{b}{3\sqrt{400+b^2}} \\
3\sqrt{400+b^2} &= 5b \\
3600 + 9b^2 &= 25b^2 \\
225 &= b^2 \\
15 &= b
\end{aligned}$$

so we have a critical point at $b = 15$. On this path we have $T = 25/3+33/5 = (125+99)/15 = 224/15 \approx 14.9$ seconds.

What about the two other paths? If we head straight to the house, we travel $\sqrt{48^2+20^2} = 52$ meters at a speed of 3 m/s, for a total time of 17.3 seconds. If instead we head straight across the river to begin walking as soon as possible, we travel 20 m at 3 m/2 and then 48 m at 5 m/s, for a total time of $20/3 + 48/5 = (100+144)/15 = 244/15 \approx 16.3$ seconds. So the shortest path has us swim 25 m and deposits us 33 m from the house.

Example 3.42. A piece of wire 10 m long is going to be cut into two pieces. We will fold one piece into a square and the other into an equilateral triangle. What is the largest joint area we can enclose? What is the smallest?

Let L be the length of the wire bent into a triangle (so that $10 - L$ is the length of the wire bent into a square). Then the area of the square is $A_1 = (10-L)^2/16$. The area of the triangle is $bh/2$; the length of the base is $L/3$ and the height of the triangle is $\sin(\pi/3) \cdot L/3 = (1/2) \cdot (\sqrt{3}/2) \cdot L/3 = \sqrt{3}L/12$. So the area of the triangle is $A_2 = (1/2)(L/3)(\sqrt{3}L/6) = L^2\sqrt{3}/36$. Then we have

$$\begin{aligned}
A &= A_1 + A_2 = (100 - 20L + L^2)/16 + L^2\sqrt{3}/36 \\
A' &= -5/4 + L/8 + L\sqrt{3}/18 \\
5/4 &= L/8 + L\sqrt{3}/18 \\
90 &= 9L + 4\sqrt{3}L \\
L &= 90/(9+4\sqrt{3})
\end{aligned}$$

This is the only critical point. At that point,

$$A \approx 1.2 + 1.5 = 2.7.$$

But checking the endpoints, if we use all the wire for the square, we have area $A = 100/16 = 6.25$ and if we use all the wire for the triangle we have $A = 100\sqrt{3}/36 \approx 4.8$. So we get the biggest area when we use all the wire for the square, and the smallest if we use $90/(9+4\sqrt{3})$ m of wire for the triangle.