Math 1231: Single-Variable Calculus I

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Jay Daigle

Contents

5 Integration

5.1 The Area Problem

For the next month, we will primarily be occupied by the question of *area*.

What is area? This actually gets a little fuzzy. We know how to compute the area of a rectangle: base times height. From that fact, and drawing a quick picture, we know the area of triangle: $\frac{1}{2}bh$, since it's half a rectangle.

We also know the area of a circle. But how? What about an ellipse? Or something funny-looking and squiggly? What does "area" mean, exactly, in these cases?

To measure the area of a shape, we can try filling it up with small squares or rectangles—we know how to measure those. (Similar principle: if you need to measure the length of something curved, run a string along it, straighten it out, measure the string. This idea will reappear in Calculus 2.)

We're going to make our lives easier, and assume our shape has one straight side. (This isn't as strict a condition as it seems; we can always cut our shape in half. We'll talk more about that in section 6.1). In fact, let's look at shapes that are given by graphs of functions.

We want to find the area of the shape "under" the graph. For right now we'll assume the function is always positive, so we get an actual area of an actual shape. (We'll relax that assumption very soon).

When we were trying to get areas earlier, we used a lot of rectangles. We can fill this area with rectangles in a bunch of different ways. But one particular way turns out to work very well, which is to have a bunch of tall skinny rectangles.

So what's the area of these rectangles? If a rectangle goes from a to b, then its width is $b-a$. How tall is it? That depends on where we put the top. There are a few things we can do, but the easiest is to make one of the top corners lie exactly on the graph. If we pick the right corner, then the width is $(b-a)f(b)$.

Example 5.1. Let's find the area under the curve $y = x^2$, between 0 and 1. If we use just one rectangle, with width 1, then we get either 0 or 1. This is true, but not super helpful.

Let's try two rectangles. They each are $\frac{1}{2}$ wide. If we line up the right-hand corners, then the area of thefirst one is $\frac{1}{2} \cdot \frac{1}{2}^2 = \frac{1}{8}$, and the area of the second one is $\frac{1}{2} \cdot 1^2 = \frac{1}{2}$. We get a total area of $\frac{5}{8}$.

What if we used the left-hand corners instead? Then the first rectangle is $\frac{1}{2} \cdot 0^2 = 0$ and the second is $\frac{1}{2} \cdot \frac{1}{2}^2 = \frac{1}{8}$. So the "true" area is somewhere between $\frac{1}{8}$ and $\frac{5}{8}$.

Let's get skinnier. If we use four rectangles, then with the right-hand point, we get

$$A_R \approx \frac{1}{4}\cdot\frac{1}{4}^2 + \frac{1}{4}\cdot\frac{1}{2}^2 + \frac{1}{4}\cdot\frac{3}{4}^2 + \frac{1}{4}\cdot 1^2 = \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} = \frac{30}{64} = \frac{15}{32},$$

and if we line up the left-hand point instead, we get

$$A_L \approx \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\frac{1}{4}^2 + \frac{1}{4}\cdot\frac{1}{2}^2 + \frac{1}{4}\cdot\frac{3}{4}^2 = 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} = \frac{14}{64} = \frac{7}{32}.$$

So the "true" area is between $\frac{7}{32}$ and $\frac{15}{32}$.

Notice that as we draw more rectangles, these numbers are getting closer. If we use 8 rectangles, we see the area is between $\frac{35}{128}$ and $\frac{51}{128}$, and if we use 64 we find that the area is between .326 and .341.

You can probably guess what happens as the number of rectangles gets very big, but let's work it out. If we have n rectangles, then each one has width $1/n$, and if we use the right-hand approximation then each rectangle has height $\left(\frac{i}{n}\right)^2$. So we have

$$\begin{aligned} R_n &= \frac{1}{n}\cdot\frac{1}{n}^2 + \frac{1}{n}\cdot\frac{2}{n}^2 + \cdots + \frac{1}{n}\cdot\frac{n}{n}^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}. \end{aligned}$$

(We had to use a "sum of squares" formula to get to the third line; feel free to check it on your own, but don't worry about it too much.)

What happens to R_n as n gets large? From what we learned about limits in section 1.4, we can compute that this limit is $\frac{1}{3}$.

We can generalize this process to define exactly what we mean by the area under a curve.

Definition 5.2. We define the area under a curve to be the limit of the sums of the areas of these rectangles. We write

$$A = \lim_{n\to+\infty} R_n = \lim_{n\to+\infty}\left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right).$$

Here n is the number of rectangles, and Δx is the width of each rectangle. Thus $\Delta x = \frac{L}{n}$ where L is the length of our shape.

Example 5.3. Estimate the area under the curve of $f(x) = 2x$ between $x = 1$ and $x = 4$, using three rectangles and using six rectangles. Try using both right endpoints and left endpoints. Is it what you expected?

$$\begin{aligned}
R_3 &= \frac{3}{3}(4+6+8) = 18. \\
L_3 &= \frac{3}{3}(2+4+6) = 12. \\
R_6 &= \frac{3}{6}(3+4+5+6+7+8) = 16.5. \\
L_6 &= \frac{3}{6}(2+3+4+5+6+7) = 13.5.
\end{aligned}$$

What if the number of rectangles goes to infinity? We have

$$\begin{aligned}
R_n &= \frac{3}{n}f(1+3/n) + \frac{3}{n}f(1+2\cdot 3/n) + \cdots + \frac{3}{n}f(1+n\cdot 3/n) \\
&= \frac{3}{n}\left(2 + 2\frac{3}{n} + 2 + 4\frac{3}{n} + \cdots + 2 + 2n\frac{3}{n}\right) \\
&= \frac{3}{n}(2+\cdots+2) + \frac{3}{n}\left(2\frac{3}{n} + 4\frac{3}{n} + \cdots + 2n\frac{3}{n}\right) \\
&= 6 + \frac{18}{n^2}(1+2+\cdots+n) \\
&= 6 + \frac{18}{n^2}\frac{n(n+1)}{2} = 6 + 9\frac{n+1}{n}.
\end{aligned}$$

We check that this formula still works for 3 and 6. Then we take the limit:

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} 6 + 9\frac{n+1}{n} = 6 + 9\lim_{n\to+\infty}\frac{1+\frac{1}{n}}{1} = 15.$$

This makes sense, since using the area formula for triangles we get an area of 15. (It's a 4×8 triangle minus a 1×2 triangle).

5.2 Riemann Sums and The Definite Integral

5.2.1 Summation notation

For the next couple weeks we'll be writing a lot of sums, and we'd like to have notation to talk about this.

We write $\sum_{i=1}^n a_i$ for $a_1 + a_2 + \cdots + a_n$ to be the sum of a bunch of things. We can index the sums other ways—and in particular, sometimes it's helpful to start from 0 instead of from 1.

You'll learn a lot more about sums in Calculus 2, but for right now, here are a few useful facts, which should all make sense if you just think about what you know about "addition".

$$\begin{aligned}
\sum_{i=1}^{n} c &= c + c + \cdots + c = nc \\
\sum_{1=1}^{n} ca_i &= ca_1 + ca_2 + \cdots + ca_n \\
&= c(a_1 + a_2 + \cdots + a_n) = c \sum_{i=1}^{n} a_i \\
\sum_{i=1}^{n} (a_i \pm b_i) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\
&= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = \left(\sum_{i=1}^{n} a_i \right) \pm \left(\sum_{i=1}^{n} b_i \right)
\end{aligned}$$

There are also three more formulas we're going to want to know, which involve adding up specific numbers.

Let's start by thinking about the sum $\sum_{i=1}^{n} i$. This just means add up the first n numbers $1 + 2 + \cdots + n$, so for instance if $n = 4$ this is $1 + 2 + 3 + 4 = 10$.

But we want what's called a "closed-form formula" for this: that is, instead of telling someone to just add up all the numbers, we want an easy formula we can just plug n into to get the answer. And for this specific case, there's an easy formula that comes with a fun story.

One of the most famous mathematicians ever was Carl Friedrich Guass, who lived from 1777 to 1855. And the (somewhat apocryphal) story goes that when he was a small child, his schoolteacher assigned his class to add up all the numbers from 1 to 100, assuming that would take everyone a fair amount of time and the poor teacher could get a break. But this is just our problem, with $n = 100$; and Gauss immediately gave the answer

What he (apocryphally) realized is that if you add all these numbers twice, but in opposite orders, you get a nice pattern. $1 + 100 = 101$, and then $2 + 99 = 101$, and then $3 + 98 = 101$, and so on; so if you add all the numbers twice, you get 100 copies of 101, or $100 \cdot 101$. But then that's twice the real answer, so we want $\frac{1}{2}(100 \cdot 101 = 50 \cdot 101 = 5050$. Generalizing that formula we get

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

It's much harder to explain where the other two formulas here come from, but we're going to need them. (And we could make a longer list if we wanted; but it's not going to be relevant to our journey.)

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

5.2.2 Riemann Sums

Now we want to use this to put notation on the work we did in subsection 5.1, on finding areas. There's going to be a *ton* of notation, but don't worry too much: the project for the rest of this unit is to get rid of as much of it as possible.

Suppose f is a function defined on a closed interval $[a, b]$. We divide $[a, b]$ into n smaller subintervals by picking points $a = x_0 < x_1 < \cdots < x_n = b$. We get a collection of subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, which we call a *partition* P of $[a, b]$. We will also sometimes use Δx_i to refer to the length $x_i - x_{i-1}$ of the ith subinterval in our partition.

For each subinterval, we can pick a *sample point* x_i^* in the interval. We could use the left endpoints or the right endpoints, as we did last class, or we could pick others; for most of our purposes in this class it doesn't really matter. (In lab next week we'll talk about what to do when it does matter).

Definition 5.4. The *Riemann sum* associated to a partion P and a function f on an interval $[a, b]$ is given by

$$R(P, f) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n.$$

We can think about taking the limit as our partition gets very small—as we use more and more rectangles and the width of each gets close to 0. We define

Definition 5.5. If f is a function defined on $[a, b]$, the *definite integral of f from a to b* is

$$\int_a^b f(x)\, dx = \lim_{P \to 0} R(P, f) = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

if the limit exists. If the limit exists, we say f is *integrable* on $[a, b]$. (otherwise, f is not integrable).

We say a is the *lower limit* of the integral, b is the *upper limit*, and $f(x)$ is the *integrand*.

Remark 5.6. It's important to note that while there are xs inside or "under" the integral sign, after the integral is computed there are no xs left. The x is a "dummy variable" or a "parameter." We'd get the exact same answer if we calculated $\int_a^b f(t)\,dt$ or $\int_a^b f(🦇)\,d🦇$ or $\int_a^b f(thisisavariable)\,dthisisavariable$.

5.2.3 Signed Area

A very important note: the Riemann sum can give a negative number! That's kind of weird, because area can't be a negative number. So the Riemann sum doesn't literally find the area; when the function is positive it contributes positively to the area, but when the function is negative it counts negatively.

Definition 5.7. The *signed area* under a graph is the area below the graph but above the x-axis, minus the area below the x-axis and above the graph.

You can think of this as the "net area". If a rectangle with a positive height has a positive area, then a rectangle with a negative height has a negative area.

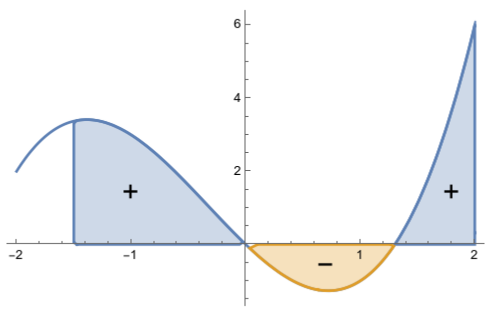

Figure 5.1: If we integrate this function from -1.5 to 2, the blue area counts positively, but the orange area counts negatively.

5.2.4 Simplifying our approach

In our definition, we took the limit over "all" partitions. This is hard to work with in practice, since there are a lot of partitions. (There are infinitely many partitions of $[0, 1]$, for instance, where $x_1 = .99999$. These are in fact partitions but they aren't incredibly helpful).

But if a function is integrable, we can always do our calculations using any collection of partitions that gets small. In particular there's one nice partition we will often use:

Theorem 5.8. *If f is integrable on $[a,b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^n f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^n f\left(a + (b-a)\frac{i}{n}\right)\frac{b-a}{n}.$$

In some sense, the dx corresponds to the Δx and the $f(x)$ corresponds to the $f(x_i^*)$. This can be made rigorous, but probably won't be in this course.

Example 5.9.

$$\begin{aligned}
\int_3^5 x^2\,dx &= \lim_{n\to+\infty} \sum_{i=1}^n \left(3 + \frac{2i}{n}\right)^2 \frac{2}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^n \left(9 + \frac{12i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^n \frac{18}{n} + \frac{24i}{n^2} + \frac{8i^2}{n^3} \\
&= \lim_{n\to+\infty} \left(\sum_{i=1}^n \frac{18}{n} + \sum_{i=1}^n \frac{24i}{n^2} + \sum_{i=1}^n \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to+\infty} \left(\frac{18}{n}\sum_{i=1}^n 1 + \frac{24}{n^2}\sum_{i=1}^n i + \frac{8}{n^3}\sum_{i=1}^n i^2\right) \\
&= \lim_{n\to+\infty} \left(\frac{18}{n}\cdot n + \frac{24}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{(n)(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to+\infty} \left(18 + 12\frac{n(n+1)}{n^2} + \frac{4}{3}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= 18 + 12 + \frac{8}{3} = \frac{98}{3} \approx 32.7.
\end{aligned}$$

5.3 Computing Integrals and The Fundamental Theorem of Calculus Part 1

We want to find a way to compute integrals *without* doing Riemann sums. We can start by taking the same approach with took with derivatives: work out some basic rules that let us manipulate integrals.

Proposition 5.10 (Properties of the Integral)**.** *The following equations are true whenever they make sense, for real numbers a, b, c and functions f, g.*

- $\int_a^b c\,dx = c(b-a)$.
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.
- $\int_a^b cf(x)\,dx = c\int_z^b f(x)\,dx$.
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

Remark 5.11. These properties are derivable from the corresponding properties of sums.

Remark 5.12. Note that while addition and scalar multiplication behave nicely, we didn't make any statements about multiplication or division, because integrals don't actually behave nicely with respect to multiplication. (We call operations like this "linear," and we study them in Math 2184 or 2185).

In Calculus 2, you will return to the idea of "the integral of the product of two functions" when you study integration by parts. But we won't quite get to that in this course.

Example 5.13. Compute $\int_1^0 2 + 3x^2 + 4x^3\,dx$.

We should remember that we've computed $\int_0^1 x^2\,dx = 1/3$ already: that was our very first example 3.21. And in recitation you should compute that $\int_0^1 x^3\,dx = 1/4$.

And with that information and the integral properties we just learned, we can work out this entire integral:

$$\begin{aligned}\int_1^0 2 + 3x^2 + 4x^3\,dx &= -\int_0^1 2 + 3x^2 + 4x^3\,dx \\ &= -\int_0^1 2 - \int_0^1 3x^2 - \int_0^1 4x^3\,dx \\ &= -\int_0^1 2 - 3\int_0^1 x^2 - 4\int_0^1 x^3\,dx \\ &= -2 - 3(1/3) - 4(1/4) = 4.\end{aligned}$$

Example 5.14. If $\int_1^5 f(x)\,dx = 3$ and $\int_3^5 f(x)\,dx = 2$, then

$$\int_1^3 f(x)\,dx = 1 = \int_1^5 f(x)\,dx + \int_5^3 f(x)\,dx = \int_1^5 f(x)\,dx - \int_3^5 f(x)\,dx = 3 - 2 = 1.$$

Unfortunately, this doesn't take us very far: we could only do these problems since we'd already done some specific Riemann sums already. (Or, alternately, because I just told you what some of the integrals were.) We need a different approach.

We can start by making some general statements about how integrals relate to each other:

Proposition 5.15 (Comparison Propreties of the Integral). *These properties only work when $a < b$. If we have a case where $a > b$ then we can always rewrite the integral before using them.*

- *If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$.*
- *If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.*
- *If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*

Example 5.16. We've used these implicitly before, when e.g. we said that $0 \leq \int_0^1 x^2 \leq 1$.

Referencing our earlier example, we know that $9 \leq x^2 \leq 25$ on $[3,5]$, so we have $18 \leq \int_3^5 x^2\,dx \leq 50$. Indeed, we calculated that $\int_3^5 x^2\,dx \approx 33$.

Suppose we want to know about $\int_0^\pi \sin(x)\,dx$. We know that $0 \leq \sin(x) \leq 1$ on $[0,\pi]$, so we see that $0 \leq \int_0^\pi \sin(x)\,dx \leq \pi$. (In fact, the integral is equal to 2, but we don't yet have the tools to calculate that).

For derivatives, this basic approach took us pretty far. But integrals are more complex and this isn't going to get us all the way there. Instead, we're going to do something very mathematician-y, that sounds dumb at first. We're going to make things more abstract, and harder now, in order to make things easier later.

From the perspective of section 5.2, the definite integral $\int_a^b f(t)\,dt$ is always a number, as long as f is integrable.(Technically the integral is a function from the set of integrable functions to the set of real numbers, but we don't need to worry about that in this class). In fact the integral is just "the area of a shape I just described," so it should always be a number. If I asked you for the area of a shape you shouldn't ever tell me $y = x^2$, for instance.

But we can use the integral to define a function (in the same way that we can have the function "input a number x and return the area of a square with side length x"—that is, $f(x) = x^2$). In particular, we want to consider functions of the form

$$F(x) = \int_a^x f(t)\,dt \tag{4}$$

where a is some fixed constant, and x is a variable. So our function is "put in a number x, and output the number $\int_a^x f(t)\,dt$, which is the area of some shape, determined by x."

Now that we have a function, there are a bunch of questions we can ask about it. What is its domain? Is it continuous? Is it differentiable?

The domain of $F(x) = \int_a^x f(t)\,dt$ is all x so that f is integrable on $[a,x]$; this answer isn't terribly satisfying, since it boils down to "The domain of F is the domain of F." It's not possible to do better without knowing something about f. But if we impose a fairly mild condition, we can say a bit more:

Theorem 5.17. *If f is continuous on $[a, b]$, or if it is continuous except for finitely many jump discontinuities, then f is integrable on $[a, b]$.*

Sketch of proof. If f has finitely many jump discontinuities, we can pick our partition to chop it up into a finite collection of continuous functions. So we just have to worry about continuous functions.

For any partition, you can always pick a "biggest" sample point in each interval, and a "smallest." The first will give you an upper bound to the integral, and the second will give you a lower bound. If the function is continuous, we can show that those two sums will always get closer together, and every other possible sum will be between the two; so all possible sums converge to the same integral. □

Example 5.18. $f(x) = x^n$ is integrable, as is $|x|$ and $\sqrt[n]{x}$ on any interval on which it is defined. The Heaviside (step) function is integrable. $1/x$ is not integrable on $[0, 1]$. The characteristic function of the rationals is not integrable (At least, not until grad school, when they change the definitions on you).

We can see a bit more. It's not too hard to show that F is continuous on its domain. Geometrically, changing x a little bit will change $F(x)$ by about the height of the function times the change in input; if the change in input is small, the change in output will also be small. Algebraically:

$$\begin{aligned}\lim_{x\to b} F(x) - F(b) &= \lim_{x\to b} \int_a^x f(t)\,dt - \int_a^b f(t)\,dt \\ &= \lim_{x\to b} \int_a^x f(t)\,dt + \int_b^a f(t)\,dt \\ &= \lim_{x\to b} \int_b^x f(t)\,dt.\end{aligned}$$

If x and b are close enough we can always find m, M such that $m \leq f(t) \leq m$ on $[x, b]$, so we get

$$\begin{aligned}\lim_{x\to b} m(x-b) \leq \lim_{x\to b} \int_b^x f(t)\,dt &\leq \lim_{x\to b} M(x-b) \\ 0 \leq \lim_{x\to b} \int_b^x f(t)\,dt &\leq 0 \\ 0 = \lim_{x\to b} \int_b^x f(t)\,dt&.\end{aligned}$$

The question of differentiability is a little trickier, but significantly more important. Intuitively and geometrically, we can simply look at pictures and ask how much the area under a curve changes if we widen our x-values a bit. After drawing some pictures we conclude that the area should change by "about" the height of the curve on one end.

We can in fact prove this fact. It's important enough for us to give it a silly name:

Theorem 5.19 (The Fundamental Theorem of Calculus, Part 1)**.** *Suppose f is continuous on $[a, b]$, and set*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $\frac{d}{dx}F(x) = f(x)$ for $a < x < b$.

Remark 5.20. As we'll discuss shortly, this theorem is the key to calculuating integrals. Note that it only applies to continuous functions. But if we have a function that's continuous in pieces, we can just split it up into separate integrals, and we see it has the correct derivative on each piece.

Proof. We want to capture our geometric intuitions. Recall that by definition, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

(This calculuation should look similar to the one above for continuity.) Let's assume for now that $h > 0$. By the extreme value theorem, f has an absolute minimum m and an absolute maximum M on $[x, x+h]$, and further we can write $f(u) = m$ and $f(v) = M$ for u, v in $[x, x+h]$. Then

$$\begin{aligned} f(u)h \le \int_x^{x+h} f(t)\,dt \le f(v)h \\ f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v). \end{aligned}$$

As $h \to 0$, the numbers u and v must get closer together, and in fact closer to x, and so by continuity $\lim_{h\to 0} f(u) = \lim_{h\to 0} f(v) = f(x)$. So we have $F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$ as desired. □

Example 5.21.

- If $F(x) = \int_a^x \sqrt{x^3+1}\,dt$ then $F'(x) = \sqrt{x^3+1}$.
- If $G(x) = \int_a^x \sin(\pi t)\cos(\pi t)\,dt$ then $G'(x) = \sin(\pi x)\cos(\pi x)$.
- If $H(x) = \int_a^{x^3} \sqrt{1+t}\,dt$ then we have to be careful. We can write $H(x) = H_1(x^3)$ where $H_1(x) = \int_a^x \sqrt{1+t}\,dt$. So by the chain rule, we have $H'(x) = \sqrt{1+x^3} \cdot 3x^2$.

5.4 Computing Integrals and the FTC 2

We still haven't quite figured out how to compute integrals without going back to the Riemann sum formulation. But we're almost there!

The Fundamental Theorem of Calculus tells us that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. But it isn't the only function with this property. We can give this a name:

Definition 5.22. If $F'(x) = f(x)$, we call F an *antiderivative* of f.

Example 5.23. $\frac{1}{3}x^3$ is an antiderivative of x^2.

$\sin(x)$ is an antiderivative of $\cos(x)$.

7 is an antiderivative of 0.

So $\int_a^x f(t)\,dt$ is an antiderivative of f. Further, we know a lot about what antiderivatives look like:

Proposition 5.24. *If $F'(x) = G'(x)$ for all x, then $F(x) = G(x) + C$ for some constant C.*

Proof. Differentiation is additive, so $(F - G)'(x) = F'(x) - G'(x) = 0$. But since the derivative is the rate of change, any function with zero derivative is constant. (We proved this in proposition 3.19 in section 3.3, using the Mean Value Theorem.) Thus $(F-G)(x) = C$ for some constant C, and so $F(x) = G(x) + C$. □

This proposition incredibly useful, because it means *any* function whose derivative is $f(x)$ is "almost" the same as $\int_a^x f(t)\,dt$. We have some sort of constant hanging around, which we need to get rid of; it turns out that this constant is essentially related to the a, the lower limit of integration.

Theorem 5.25 (Fundamental Theorem of Calculus, Part 2)**.** *Suppose f is continuous on $[a,b]$, and F is any antiderivative of f. Then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Proof. Since $F(x)$ and $\int_a^x f(t)\,dt$ are both antiderivatives of $f(x)$, we know that $F(x) = \int_a^x f(t)\,dt + C$ for some constant C. Then

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - \left(\int_a^a f(t)\,dt + C\right) = \int_a^b f(t)\,dt + C - 0 - C = \int_a^b f(t)\,dt.$$

□

Example 5.26. What is $\int_1^3 3x^2\,dx$?

We can see that $F(x) = x^3$ is an antiderivative of $3x^2$. (It's not the only one, but that's okay.) So $\int_1^3 3x^2\,dx = F(3) - F(1) = 27 - 1 = 26$.

What if we'd picked, say, $G(x) = x^3 + 5$? Then we'd have $\int_1^3 3x^2\,dx = G(3) - G(1) = 32 - 6 = 26$ again.

Example 5.27. What is $\int_{\pi/4}^{3\pi/4} \cos(x)\,dx$?

We see that $\sin(x)$ is an antiderivative for $\cos(x)$. So we have

$$\int_{\pi/4}^{3\pi/4} \cos(x)\,dx = \sin(3\pi/4) - \sin(\pi/4) = \sqrt{2}/2 - \sqrt{2}/2 = 0.$$

5.4.1 Indefinite Integrals

Because antiderivatives are so important, we want a notation for them that is less awkward than having to write the word "antiderivative" over and over. Because they are so closely tied to integrals, we use notation specifically designed to confuse you about what the integral sign means.

Definition 5.28. The *indefinite integral* of a function f, written $\int f(t)\,dt$, is any antiderivative of f. That is, $\int f(t)\,dt$ refers to any function $F(x)$ such that $F'(x) = f(x)$.

The *general form of the indefinite integral* is $\int f(x)\,dx = F(x) + C$. The constant represents the fact that there are many possible antiderivatives of f.

Very Important Note: Remember the difference between the definite and indefinite integrals. The definite integral $\int_a^b f(x)\,dx$ is a number. It is the area of some region under a graph. The indefinite integral $\int f(x)\,dx$ is a collection of functions, which are all antiderivatives of f and are all the same up to a constant. They are related by

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b = F(b) - F(a).$$

In general the notation $|_a^b$ means "the value at b minus the value at a. We will use it a lot while doing integrals.

Example 5.29. We can write $\int x^5\,dx = \frac{1}{6}x^6 + C$, and $\int \sec^2(x)\,dx = \tan(x) + C$.

5.4.2 Antiderivatives, Net Change, and Linear Approximation

We can look at all of what we've done from another perspective, and connect it back to the work we did earlier on linear approximation.

Suppose we have a function F that we want to know about, but we only know about the derivative $F'(x)$. For instance, we may want to know the position of an object but only have measured the speed, or want to know the speed after measuring the acceleration. Or we want to figure out how much money we owe from a record of our annual deficits; we've seen a lot of examples of derivatives.

The example of deficit and debt makes this maybe easy to think of. Suppose you have a deficit of \$3000 one year, \$5000 the second year, and \$2000 the third year. At the end of three years, the debt has increased by \$10,000, which we get by adding the three deficits up.

This works exactly because we have a discrete set of payments, but if we don't have that we can still approximate it. Suppose that $F(t)$ gives the position of a particle at time t, and we know the velocity $F'(t)$. If we also know the starting position $F(0)$, we could estimate $F(4) \approx F(0) + F'(0)(4-0)$, but that might not be very good.

One way we could make this better is to do something like a quadratic approximation, or a Taylor series, but that gets messy. Another option is to do *multiple approximations*. Since the approximation gets worse the further x gets from a, we can try to bring it closer, and approximate in multiple steps.

Thus maybe we have

$$\begin{aligned} F(2) &\approx F(0) + F'(0)(2-0) \\ F(4) &\approx F(2) + F'(2)(4-2) \approx F(0) + F'(0)(2-0) + F'(2)(4-2). \end{aligned}$$

So if we take, say, $F'(t) = 10t$ and $F(0) = 0$, this would give us

$$\begin{aligned} F(2) &\approx 0 + 0(2-0) \\ F(4) &\approx 0 + 20(2) = 40 \end{aligned}$$

which is close-ish but not super close to the true answer of 80 (as we'll see soon).

What if we take more steps? We get

$$\begin{aligned} F(1) &\approx F(0) + F'(0)(1-0) & &\approx 0 + 0(1-0) \\ F(2) &\approx F(1) + F'(1)(2-1) & &\approx 0 + 10(2-1) = 10 \\ F(3) &\approx F(2) + F'(2)(3-2) & &\approx 10 + 20(3-2) = 30 \\ F(4) &\approx F(3) + F'(3)(4-3) & &\approx 30 + 30(4-3) = 60. \end{aligned}$$

But what is this last formula, really? It's

$$F(4) \approx F(0) + F'(0)(1-0) + F'(1)(2-1) + F'(2)(3-2) + F'(3)(4-3).$$

If we rearrange this a bit, we just get

$$F(4) - F(0) \approx F'(0)(1-0) + F'(1)(2-1) + F'(2)(3-2) + F'(3)(4-3)$$

and the right-hand side is a sum of terms that look like $F'(x_i)\Delta x_i$. So we have

$$F(4) - F(0) \approx \sum_{i=1}^{n} F'(x_i)\frac{4}{n}.$$

This is just a Riemann sum! And as we take the limit, we get an integral

$$F(4) - F(0) = \lim_{n\to\infty} \sum_{i=1}^{n} F'(x_i)\frac{4}{n} = \int_0^4 F'(x)\,dx.$$

Early on in the class, we saw that if you know the value of F and the derivative of F at 0, then you can use a linear approximation to estimate the value at any point. What we see now is that if you know the derivative of F everywhere, and the value at one point, you can find the value exactly, by taking an infinite collection of very small linear approximations.

Specifically, if you know the derivative, you can figure out the net change of F between any two values; so if you have one value, you can find any value.

Corollary 5.30 (Net Change Theorem)**.** *The integral of a rate of change is the total (net) change.*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Remark 5.31. Note that to find the value of $F(b)$ this way, we need to start by knowing $F(a)$ for some a. If we think of F as just being an antiderivative of F', the starting value is nailing down exactly the constant C.

Remark 5.32. This process of taking a large number of linear approximations is used in the real world a lot. If you have an integral that you *can't* find an exact formula for, this is very useful. It generalizes even more to solving differential equations, which are equations that specify F using a *formula* for $F'(x)$. They are more complicated than simple integrals, and you will see a little of them in calculus 2. But they are also the fundamental underpinning of most mathematical models, in the physical sciences and the social scienes.

5.4.3 Computing Integrals for the Practical Person

We've learned that computing integrals is reducible to finding antiderivatives. Now we're finally ready to practice actually computing integrals. In order to do this, we start by recalling a number of antiderivatives.

I'll list a few in these notes. There is an extensive card listing many of htese rules on page 6 of the reference in the back of Stewart, and a shorter table on page 331 in section 4.4.

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- $\int cf(x)\,dx = c\int f(x)\,dx$.
- $\int x^n\,dx = \frac{x^n}{n+1} + C$ if $n \neq -1$.
- $\int \sin(x)\,dx = -\cos(x) + C$.
- $\int \cos(x)\,dx = \sin(x) + C$.
- $\int \sec^2(x)\,dx = \tan(x) + C$.
- $\int \csc^2(x)\,dx = -\cot(x) + C$.
- $\int \sec(x)\tan(x)\,dx = \sec(x) + C$.
- $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$.

Example 5.33.

- What is $\int_1^4 x^2\,dx$? We know that $\int x^2\,dx = \frac{1}{3}x^3 + C$, so $\int_1^4 x^2\,dx = \frac{1}{3}x^3|_1^4 = \frac{1}{3}(64-1) = 21$. Note the Cs cancel each other out so it doesn't matter what they are.
- What is $\int_2^3 x + x^3\,dx$? We can work out that $\int x + x^3 = \frac{x^2}{2} + \frac{x^4}{4}$, so
$$\int_2^3 x + x^3\,dx = \frac{x^2}{2} + \frac{x^4}{4}\Big|_2^3 = \frac{9}{2} + \frac{81}{4} - \frac{4}{2} - \frac{16}{4} = \frac{99}{4} - 6 = \frac{75}{4}.$$
- Calculuate $\int_{-1}^2 |x|\,dx$. We don't really have an antiderivative of $|x|$, so the easiest way to approach this is probably to break it up into two distinct integrals.

  If $x \geq 0$ then $|x| = x$, so we have $\int_0^2 |x|\,dx = \int_0^2 x\,dx = \frac{x^2}{2}|_0^2 = 2 - 0 = 2$.

  If $x \leq 0$ then $|x| = -x$ and we have $\int_{-1}^0 |x|\,dx = \int_{-1}^0 -x\,dx = \frac{-x^2}{2}|_{-1}^0 = 0 - \frac{-1}{2} = \frac{1}{2}$.

  Thus $\int_{-1}^2 |x|\,dx = \int_{-1}^0 |x|\,dx + \int_0^2 |x|\,dx = \frac{1}{2} + 2 = \frac{5}{2}$.
- Calculate $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$. At first blush this looks hard, until you remember that $\sec'(x) = \sec(x)\tan(x)$. So we have
$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \sec(x)|_0^{\pi/4} = \sec(\pi/4) - \sec(0) = \sqrt{2} - 1.$$

- What if we want $\int_0^\pi \sec(x)\tan(x)$? This is a much bigger problem, because $\sec(x)\tan(x)$ is not continuous on $[0, \pi]$. We actually won't be able to do that one without new ideas that we won't develop in this course.

Leading question: can you do $\int 3x^2\sqrt{9+x^3}\,dx$?

5.5 Integration by Substitution

The Fundamental Theorem of Calculus is a powerful tool for computing integrals. And with functions that are obviously the derivatives of some other function, like x^2 or $\cos(x)$, it's very easy to apply. With more complicated functions it takes a bit more work.

Example 5.34. What is $\int 3x^2\sqrt{9+x^3}\,dx$?

There are two ways to approach this problem. The first is to notice that you almost have an antiderivative to $\sqrt{9+x^3}$, because $(9+x^3)^{3/2}$ has $\frac{3}{2}(9+x^3)^{1/2}\cdot 3x^2$ as its derivative. The extra $3x^2$ from the chain rule precisely matches up with the extra $3x^2$ from the problem, so we just have to correct for the constant, and we have that $\int 3x^2\sqrt{9+x^3} = \frac{2}{3}(9+x^3)^{3/2}+C$.

If that made sense, great. Whenever you can "just see" the antiderivative, you can go for it; the fact that you can check your work by taking a derivative means that you are safe. But for the cases where you can't just see the answer, we'd like to be a little more systematic in our approach.

We know how to take the antiderivative of $\sqrt{x}$. So let's try using a new variable, which we traditionally call u. We write $u = 9+x^3$ so the thing under the radical is a u. We also notice that $\frac{du}{dx} = 3x^2$; by "abuse of notation" (by which I mean we won't justify it, but just assume it works) we write $du = 3x^2dx$. Since our original integral was $\int \sqrt{9+x^3}\cdot 3x^2dx$, we can rewrite this as $\int \sqrt{u}\,du$, or just $\int u^{1/2}\,du$.

From our integral table, we know that $\int u^{1/2}\,du = \frac{2}{3}u^{3/2}+C$. Now we can replace the u with $9+x^3$ to get $\int 3x^2\sqrt{9+x^3}\,dx = \frac{2}{3}(9+x^3)^{3/2}+C$.

We can formalize this into a rule:

Proposition 5.35 (The Substitution Rule for Indefinite Integrals)**.** *If $u = g(x)$ is differentiable, and $f(x)$ is continuous on the range of g, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)du.$$

Proof. This follows from the chain rule. Let F be an antiderivative of f; then $(F(g(x)))' = F'(g(x))\cdot g'(x) = f(g(x)g'(x)$. Thus $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

I'd like to give you geometric intuition here, but it's a bit hard to communicate. In essence we're changing to a new coordinate system where the integral is easy, but it's hard to make that observation *useful* until you get to multivariable calculus. For right now, yo ushould probably think of this as a way of keeping track of algebraic manipulations. □

How do we use this? Basically, when we see a complicated integral, there are a couple things we can look for. The first is to check whether one part is a derivative of another part, in a way that could reflect a chain rule. The other is to find the most complicated chunk of the expression and replace it with a u, and see how much of our problem that solves.

Choosing the right variable to substitute is a bit of an art; I can't possibly give you a complete set of rules, but I can give you a lot of examples to model off of.

Example 5.36.

- Consider $\int x^2 \sin(x^3+3)\,dx$. We can take $u = x^3+3$, and then $du = 3x^2\,dx$ so $dx = \frac{du}{3x^2}$. So this becomes $\int \sin(u)/3\,du = -\cos(u)/3 + C = -\cos(x^3+3)/3 + C$.

- Consider $\int \sqrt{5x+2}\,dx$. It makes sense to take $u = 5x+2$, so $du = 5dx$. Then $\int \sqrt{u}/5\,du = \frac{2}{15}u^{3/2} + C = \frac{2}{15}(5x+2)^{3/2} + C$.

  Alternatively, we could take $u = \sqrt{5x+2}$. Then $du = \frac{5}{2\sqrt{5x+2}}dx$ and we get $dx = \frac{2}{5}\sqrt{5x+2} = \frac{2}{5}u$. So we have $\int \frac{2}{5}u^2\,du = \frac{2}{15}u^3 + C = \frac{2}{15}(5x+2)^{3/2} + C$.

- For a more complex example, we can look at $\int \sqrt{1+x^2}x^5\,dx$. This doesn't look like it will happen automatically, and indeed it doesn't. But we can still get rid of the complicated bit by taking $u = 1+x^2$, so $du = 2x\,dx$ or $dx = du/2x$.

  This gives us $\int \sqrt{u}x^4\frac{1}{2}du$, but what do we do with the other x^4 term? Well, if $u = 1+x^2$ that means that $x^2 = u-1$, so our integral is

$$\begin{aligned}\int \frac{1}{2}\sqrt{u}(u-1)^2\,du &= \int \frac{1}{2}\left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du \\ &= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C.\end{aligned}$$

5.5.1 Substitution and Definite Integrals

The above talked about indefinite integrals. When we have a definite integral, we can be more specific. We can use substitution in two ways: one is to do what we did above, where we substitute in a u, then integrate, then switch the us back to xs. But we can avoid switching back at all by changing the limits of integration.

Proposition 5.37 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a,b]$, and f is continuous on the range of $g(x)$, then*

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. If F is an antiderivative of f, then the left side is clearly $F(g(b)) - F(g(a))$. But the antiderivative of $f(g(x))g'(x)$ is $F(g(x)$, so the left side is also $F(g(b)) - F(g(a))$. □

Example 5.38.

- Find $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$. We take $u = g(x) = 1 + 2x^2$ so that $du = 4dx$, so $dx = du/4$, and $g(0) = 1, g(2) = 9$. We have
$$\frac{1}{4}\int_1^9 u^{-1/2}\,du = \frac{1}{4}2u^{1/2}|_1^9 = \frac{1}{2}(3-1) = 1.$$
- Find $\int_1^3 \frac{dx}{(1-2x)^2}$. Set $u = g(x) = 1 - 2x$, then $du = -2dx$ and $g(1) = -1, g(3) = -5$. So
$$\int_1^3 \frac{dx}{(1-2x)^2} = \int_{-1}^{-5} \frac{-du}{2u^2} = \frac{1}{2u}\Big|_{-1}^{-5} = \frac{1}{-10} - \frac{1}{-2} = \frac{2}{5}.$$

A nice bonus application of this is to look at symmetric functions. Since even and odd functions have nice geometric symmetries, integrals, which are about the area under the curve, should also have nice properties.

Corollary 5.39 (Integrals of Symmetric Functions)**.** *Suppose f is a continuous function on $[-a,a]$. Then*

- *If f is even, then $\int_{-a}^a f(t)\,dt = 2\int_0^a f(t)\,dt$.*
- *If f is odd, then $\int_{-a}^a f(t)\,dt = 0$.*

Proof. Intuitively this should be plausible; even functions look the same on either side of the y-axis, and so you should get the same area on both sides, while odd functions are the same but upside down, so you should get the opposite area. (Try sketching a picture of sin and cos to see this).

For either integral, notice that $\int_{-a}^a f(t)\,dt = \int_{-a}^0 f(t)\,dt + \int_0^a f(t)\,dt$. Consider the first integral, and use the substitution $u = g(t) = -t$, and thus $-du = -dt$. Then $\int_{-a}^0 f(t)\,dt = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)\,dt$.

If f is even then $f(-t) = f(t)$, so $\int_{-a}^0 f(t)\,dt = \int_0^a f(t)\,dt$. If f is odd then $f(-t) = -f(t)$ and thus $\int_{-a}^0 f(t)\,dt = -\int_0^a f(t)\,dt$. □

Example 5.40.

- $\int_{-3}^3 x^5 - x^3\,dx = 0$.
- $\int_{-2}^2 x^6 + 1\,dx = 2\int_0^2 x^6 + 1\,dx = 2(x^7/7 + x)|_0^2 = 2(128/7 + 2) = \frac{284}{7}$.

5.6 A Brief Note on How to Cheat

We've now learned how to compute basic integrals. There are a lot of integrals we haven't yet learned to compute; a prominent example is $\int \frac{1}{x} \, dx$, but there are many. In calculus 2 you will develop many other techniques of integration which allow us to integrate more difficult functions. However, as good mathematicians we're also fundamentally lazy and would prefer to avoid work when we can manage it. There are two common solutions here.

First, the back of your textbook has an extensive integral table, and even more extensive tables can be found online. It often requires minor massaging to get your integral into the form of the table, but for complex integrals the table will be much easier than figuring things out from scratch. (For instance, the table incorporates the results of trig subsitution without making you work through it explicitly).

Second, computers are very good at doing integrals. Wolfram Alpha can often integrate a function for you, as can Mathematica and other computer tools. It's dangerous to become overly reliant on these tools—it's easy to make a mistake if you don't understand what's going on, and sometimes the computer will return the answer in a less useful form. They are very good for automated computations and checking your work, however.

A final cautionary note: there are some functions that don't have a nice closed-form antiderivative. Famously, there's no way to write $\int e^{x^2} \, dx$ in terms of "elementary functions." That doesn't mean there is no antiderivative; the obvious one is $\int_0^x e^{t^2} \, dt$. But while correct, that answer isn't terribly enlightening.

We can't easily compute these definite integrals exactly, but we can approximate them using various approximation techniques (among other things, just computing a finite Riemann sum). We can also use the concept of "infinite series" to handle this sort of situation; those techniques occur towards the end of Calculus 2.