

Math 1231: Single-Variable Calculus 1
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Recitation 2

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In class, we looked at the following question: Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

Clearly, our sides need to be between $\sqrt{15}$ and $\sqrt{17}$ but that doesn't tell us anything useful. So instead we made the following argument: We can use an absolute value to describe the way we think about errors. in particular, what we want here is

$$|s^2 - 16| < \varepsilon = 1, \tag{1}$$

and factoring the left hand side gives $|s - 4| \cdot |s + 4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s + 4$ might be bigger than 8, but it's certainly less than 9, or 10. Then we just need to solve

$$|s - 4| \cdot |s + 4| < 10|s - 4| < 1 \tag{2}$$

$$|s - 4| < .1 \tag{3}$$

$$-.1 < s - 4 < .1 \tag{4}$$

$$3.9 < s - 4 < 4.1. \tag{5}$$

Thus $\delta = .1$ and s should be $4 \pm .1$.

This is a tricky argument! But I want you to try to think through it now.

Problem 1. Let's suppose instead we want to make a square platform with area 25 square meters, plus or minus 1.

- (a) Write down the analogue of inequality (1) for this new problem. Can you explain in words what this inequality says about your error?
- (b) We can factor the left-hand side of this inequality into two factors. If our input is close to 5, one of these terms will be small, and the other will be large. Which one will be large, and about how large will that be?
- (c) This should let you write down an inequality like the one in (2). What is it?
- (d) Figure out δ such that $s = 5 \pm \delta$ will keep us in our error bounds.
- (e) Check your answer: square $5 + \delta$ and $5 - \delta$ and see whether the answers fall within your error margin.
- (f) Could you use a larger δ than the one you found in part (4)?

Solution:

- (a) $|s^2 - 25| < 1$. This says that the error between our output s^2 and our target 25 is less than one.
- (b) We get $|s - 5| \cdot |s + 5| < 1$. The $|s - 5|$ term should be small since we want s close to 5; the $|s + 5|$ term will be large, and it should be approximately 10 since $s \approx 5$.
- (c) $s + 5 \approx 10$ so we can say $s + 5 < 11$. So we get $|s - 5| \cdot |s + 5| < 11|s - 5| < 1$.
- (d) Then we need $|s - 5| < 1/11$ which gives us $\delta = 1/11$.
- (e) $(54/11)^2 \approx 24.0992$ and $(56/11)^2 \approx 25.9174$ so $\delta = 1/11$ is in fact an acceptable amount of error in the input.
- (f) We see that $4.9^2 = 24.01$ keeps us within our error margin; but $5.1^2 = 26.01$ does not. So $\delta = 1/10$ is too big. However, we could take something like $\delta = .095$, which is bigger than $1/11 \approx .091$. Then $4.905^2 = 24.059$ and $5.095^2 = 25.959$ both stay within our error margin.

The largest *possible* δ that works is $\sqrt{26} - 5 \approx .099$. But it's hard to figure that out without already knowing the value of $\sqrt{26}$.

Problem 2. Redo problem 1 with $\varepsilon = .1$. You'll notice that you can do this pretty quickly, since you already did the hard part. If we change ε again, it should be easy to find a new δ .

Problem 3. Let $f(x) = 5x + 2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \rightarrow 2} f(x)$.

- (a) If x is about 2, what should $f(x)$ be?
- (b) Write down expressions using absolute value for the input and output errors.
- (c) If we want $\varepsilon = 1$, what does δ need to be?
- (d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

Solution:

- (a) $f(x) \approx 12$.
- (b) Output error is $|f(x) - 12|$ or $|5x + 2 - 12|$, which we can simplify to $|5x - 10|$. Input error is $|x - 2|$.
- (c) We want $|5x - 10| < 1$, and dividing by 5 gives $|x - 2| < 1/5$. So we'd need $\delta = 1/5$.
- (d) We want $|5x - 10| < \varepsilon$, and dividing by 5 gives $|x - 2| < \varepsilon/5$. So we'd need $\delta = \varepsilon/5$.

Problem 4. Let $g(x) = x^2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \rightarrow 0} g(x)$.

- (a) If x is about 0, what should $g(x)$ be?
- (b) Write down expressions using absolute value for the input and output errors.
- (c) If we want $\varepsilon = 1$, what does δ need to be? What about $\varepsilon = 1/4$?
- (d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

Solution:

- (a) $g(x) \approx 0$.
- (b) Output error is $|g(x) - 0|$ or $|x^2 - 0|$, which we can simplify to x^2 . Input error is $|x - 0|$, which we can simplify to $|x|$.
- (c) We want $x^2 < 1$, and taking square roots gives $|x| < 1$, so we need $\delta = 1$.
If we want $x^2 < 1/4$ then taking square roots gives $|x| < 1/2$, so we need $\delta = 1/2$.
Note that in both cases the absolute value matters; the square root of x^2 is always positive, and thus equals $|x|$.
- (d) We want $x^2 < \varepsilon$, and taking square roots gives $|x| < \sqrt{\varepsilon}$. So we take $\delta = \sqrt{\varepsilon}$.