

Math 1231: Single-Variable Calculus 1
George Washington University Fall 2025
Recitation 3

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Problem 1. Let $h(x) = \frac{x-1}{\sqrt{5-x}-2}$.

- (a) Is this function continuous where it's defined? Where is it undefined?
- (b) We can factor an $x - 1$ out of the top, but we can't obviously factor one out of the bottom. We need to use an algebraic trick make the $x - 1$ appear. What tricks do we have that might work?
- (c) What is $\lim_{x \rightarrow 1} h(x)$?

Solution:

- (a) The function is reasonable, so it's continuous where defined. IT's undefined at $x = 1$ and also at $x > 5$.
- (b) Here we need to multiply by the conjugate. We can compute

$$\begin{aligned} \frac{x-1}{\sqrt{5-x}-2} &= \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}. \end{aligned}$$

This function is not the same as $-\sqrt{5-x}-2$, but it's very close.

(c)

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{5-x}-2} &= \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\
&= \lim_{x \rightarrow 1} -(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Problem 2. We want to compute $\lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x-3}$.

- (a) What rule do we know we need to invoke here?
- (b) What θ are we going to need to pick for this to work out, and why?
- (c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)
- (d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

- (a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.
- (b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.
- (c) We need to get a θ on the bottom as well. So we take

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x-3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x \rightarrow 3} \frac{\sin(x^2-9)(x+3)}{x^2-9} \\
&= \lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x \rightarrow 3} x+3 = 1 \cdot (3+3) = 6.
\end{aligned}$$

You have to use AIF in that first equality, because you're making the function undefined at -3 .

There are a couple different ways to make this algebra work out. You might also try

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x-3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2-9)/x^2-9 \cdot x^2-9}{x-3} \\
&= \lim_{x \rightarrow 3} \frac{x^2-9}{x-3} =_{AIF} \lim_{x \rightarrow 3} x+3 = 6,
\end{aligned}$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} = \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)(x + 3)}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x^2 - 9} \cdot \lim_{x \rightarrow 3} x + 3 = 1 \cdot (3 + 3) = 6.\end{aligned}$$

Problem 3. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

- (a) To start: what is $\lim_{x \rightarrow 0} 1/x$, and why?
- (b) Let's look at $\lim_{x \rightarrow 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?
- (c) How do we actually compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?
- (d) Now consider $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?
- (e) Compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?
- (f) Now consider $\lim_{x \rightarrow 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?
- (g) Finally, let's look at $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?
- (h) What pattern do you see from all of these?

Solution:

- (a) $\lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty$.
- (b) This looks like $\infty + \infty$ as an indeterminate form.
- (c) We see $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x} = \lim_{x \rightarrow 0} \frac{2}{x} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

- (d) We already know $\lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x \rightarrow 0} \frac{x - 1 \nearrow^{-1}}{x - x^2 \searrow_0} = \pm\infty.$$

So this is again $\infty + \infty$.

- (e) By combining fractions, we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x \rightarrow 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

So here $\infty + \infty = 0$.

- (f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x \rightarrow 0} \frac{x+1 \nearrow^1}{x^2 \searrow_0} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ “faster” than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

- (g) We compute $\lim_{x \rightarrow 0} \frac{x-3x+2 \nearrow^2}{x^2-2x \searrow_0} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x \rightarrow 0} \frac{x-2+x^2-3x+2}{x^2-2x} = \lim_{x \rightarrow 0} \frac{x^2-2x}{x^2-2x} = \lim_{x \rightarrow 0} 1 = 1.$$

So here $\infty + \infty = 1$.

- (h) In conclusion, if you know something looks like $\infty + \infty$, you don’t really know anything about it at all.