Math 1231: Single-Variable Calculus 1

George Washington University   Fall 2025

Recitation 4

Jay Daigle

Friday September 19, 2025

Problem 1. Let $f(x) = x^3$. We want to find a formula for the derivative of this function at any given point.

(a) Write down a formula for $f'(a)$ using the $h \to 0$ limit formulation. What does the numerator mean? What does the denominator mean?

(b) Use your formula from part (a) to compute the derivative.

(c) Now write down a formula for $f'(a)$ using the $x \to a$ limit formulation. Does this look easier or harder than the formula from part (a), and why? What does the numerator mean? What does the denominator mean?

(d) Use the formula from part (c) to compute the derivative. You should get the same answer you got in part (b).

(e) Which method was faster? Which method was easier?

Solution:

(a)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h}$$

The top is the difference between two output values; the bottom is the difference between the corresponding inputs. You can think of the bottom as "change the input by a bit" and the top as the difference between the two outputs.

(b)

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &=^{AIF} \lim_{h\to 0} 3a^2 + 3ah + h^2 = 3a^2. \end{aligned}$$

(c)

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^3 - a^3}{x - a}$$

Again the top is the difference in outputs and the bottom is the difference in inputs. Here we can see two specific inputs, x and a, on the bottom; the top is the two corresponding outputs.

(d)

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} \\ &=^{AIF} \lim_{x\to a} x^2 + ax + a^2 = 3a^2. \end{aligned}$$

Notice we use the difference of cubes formula from section 1.1 of the notes.

(e) To my eyes, at least, the $h \to 0$ method is more straightforward, but the $x \to a$ method is faster *if you know the trick*. If you look at it and immediately see that $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$, then the $x \to a$ method works very quickly. But if you don't know or remember that fact, it's hard to figure out what to do at all; you just get stuck.

In contrast, the $h \to 0$ method takes more algebra and work and writing and time, but less cleverness and thinking. If you just multiply everything out and cancel out the obvious stuff, it works out. When I don't know what I'm doing, I default to the $h \to 0$ version.

Problem 2. Let $g(x) = \frac{1}{x+3}$.

(a) Write down a limit expression to compute $g'(2)$. Be careful with order of operations and parentheses!

(b) Now compute $g'(2)$.

(c) Write a limit expression to compute $g'(x)$. Again, make sure you get your order of operations right.

(d) Compute $g'(x)$.

Solution:

(a) We have

$$\begin{aligned} g'(2) &= \lim_{h \to 0} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}. \end{aligned}$$

Make sure you have $\frac{1}{5+h}$, and not $\frac{1}{5} + h$! The second thing is very different and will not give you a useful answer.

(b) We have

$$\begin{aligned} g'(2) &= \lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{5+h} - \frac{1}{5} \right) = \lim_{h \to 0} \frac{1}{h} \left(\frac{5 - (5+h)}{5(5+h)} \right) \\ &= \lim_{h \to 0} \frac{-h}{5h(5+h)} = \lim_{h \to 0} \frac{-1}{5(5+h)} \\ &= \frac{-1}{5(5+0)} = \frac{-1}{25}. \end{aligned}$$

(c)

$$g'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}.$$

Again, we want to make sure that we don't write $\frac{1}{x+3} + h$ or something like that.

(d)

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)}\right) \\
&= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\
&= \frac{-1}{(x+3)^2}.
\end{aligned}$$

Problem 3. Let $a(x) = |x|$ be the absolute value function.

(a) Write down a formula for a as a piecewise function.

(b) Write down a limit expression for the derivative of a at 0.

(c) What is the limit from the right?

(d) What is the limit from the left?

(e) What does that tell you about the derivative?

Solution:

(a)

$$a(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

(b)

$$a'(0) = \lim_{h\to 0} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

(c)

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$

(d)

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$$

(e) The limits to the right and the left don't exist, so the limit doesn't exist.

Problem 4 (Bonus)**.** Let $g(x) = \sqrt[3]{x}$.

(a) Write down a limit formula to compute the derivative of g at 0.

(b) What is $g'(0)$? What does this tell you?

(c) Now write down a limit formula to compute the derivative of $p(x) = \sqrt[3]{x^2}$.

(d) What is this limit? What does that tell you?

(e) Write down a limit formula to compute the derivative of g at a when $a \neq 0$.

(f) Can you compute this limit? What do you have to do here? (It's not obvious, but there's an algebraic trick from Day 1 that can help us.)

Solution:

(a)

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h} - 0}{h} \\ &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x} - 0}{x - 0}. \end{aligned}$$

(b)

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

This is a vertical tangent line, because the limit is always $+\infty$.

(c)

$$\begin{aligned} p'(0) &= \lim_{h\to 0} \frac{p(h) - p(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h^2} - 0}{h} \\ &= \lim_{x\to 0} \frac{p(x) - p(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x^2} - 0}{x - 0}. \end{aligned}$$

(d)

$$p'(0) = \lim_{h\to 0} \frac{p(h) - p(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h^2}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

This is a *cusp*, because the limit is $\pm\infty$ rather than just $+\infty$.

(e)

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(h)-g(a)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{a+h}-\sqrt[3]{a}}{h} \\ &= \lim_{x\to 0} \frac{g(x)-g(a)}{x-a} = \lim_{x\to 0} \frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}. \end{aligned}$$

(f) You might recognize this as being a difference of cube roots, so we can use the difference-of-cubes formula, as a sort of generalization of multiplication by the conjugate.

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{\sqrt[3]{a+h}-\sqrt[3]{a}}{h} \\ &= \lim_{h\to 0} \frac{(a+h)-a}{h(\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2}} \\ &= \frac{1}{\sqrt[3]{a^2}+\sqrt[3]{a^2}+\sqrt[3]{a^2}} = \frac{1}{3\sqrt[3]{a^2}}. \end{aligned}$$

Problem 5 (Bonus)**.** Let $f(x) = \sqrt{x^2-4}$.

(a) Set up a limit expression to calculate $f'(x)$. Do you think $h \to 0$ or $x \to a$ will be easier here?

(b) Compute $f'(x)$.

(c) Where is f differentiable? Where is it not differentiable?

Solution:

(a) We could say

$$\begin{aligned} f'(x) &= \lim_{x\to a} \frac{\sqrt{x^2-4}-\sqrt{a^2-4}}{x-a} \\ f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-4}-\sqrt{x^2-4}}{h}. \end{aligned}$$

The first would in fact work, but the second looks easier in every way to me.

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-4} - \sqrt{x^2-4}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 4 - (x^2-4)}{h(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \lim_{h\to 0} \frac{2x + h}{(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2-4}}.
\end{aligned}$$

(c) This derivative is defined for $x < -2$ and for $x > 2$, but not in between those two numbers. Thus we see that f is differentiable on $(-\infty, -2) \cup (2, +\infty)$.

We can use the limit definition of the derivative to compute derivatives, but it's not very efficient. In class we learned some rules for computing them more easily:

- (Constants) $\frac{d}{dx} c = 0$
- (Identity) $\frac{d}{dx} x = 1$
- (Scalar products) $\frac{d}{dx} cf(x) = cf'(x)$
- (Sum Rule) $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
- (Power Rule) $\frac{d}{dx} x^n = nx^{n-1}$
- (Product Rule) $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
- (Quotient Rule) $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

Problem 6. Let's find the derivative of $3x^2 + 5x$.

(a) First just plug 1 into this function. Think about what operations you do in which order. Which operation do you do last?

(b) What rule should we invoke first? Use that rule to write this as a combination of two smaller derivative problems.

(c) What rule do we use next? We need to use it in two places.

(d) Finish off computing this derivative. What rules did you use?

Solution:

(a) When we plug in 1 we get $3(1)^2 + 5(1) = 3 + 5 = 8$. The last thing we did was add two numbers.

(b) We need to use the sum rule first. We get
$$\frac{d}{dx}3x^2 + 5 = (3x^2)' + (5x)'.$$

(c) The scalar product rule gives us
$$(3x^2)' + (5x)' = 3(x^2)' + 5(x)'$$

(d)
$$3(x^2)' + 5(x)' = 3(2x) + 5.$$
This uses the power rule and the identity rule. (Technically we could just say the power rule if we really wanted.)

Problem 7. (a) Use the product rule to differentiate $(x^2 + 1)(3x^3 - 5)$.

(b) Multiply out $(x^2 + 1)(3x^3 - 5)$ to get one big polynomial. Use our derivative rules to compute that derivative.

(c) Which process was easier?

Solution:

(a) $2x(3x^3 - 5) + (x^2 + 1)(9x^2)$.

(b) We get
$$\begin{aligned}(x^2 + 1)(3x^3 - 5) &= 3x^5 - 5x^2 + 3x^3 - 5\\ \frac{d}{dx}(x^2 + 1)(3x^3 - 5) &= \frac{d}{dx}3x^5 - 5x^2 + 3x^3 - 5\\ &= 15x^4 - 10x + 9x^2 - 0.\end{aligned}$$

(c) This is a matter of personal taste, but I'd say the second derivative was easier, but took more work total when we count the work of multiplying the terms out.

Problem 8. Compute $\dfrac{d}{dx}\dfrac{x^5 - 7x}{4x^2 + 3}$.

Solution:
$$\frac{d}{dx}\frac{x^5 - 7x}{4x^2 + 3} = \frac{(5x^5 - 7)(4x^2 + 3) - (8x)(x^5 - 7x)}{(4x^2 + 3)^2}.$$