Math 1231: Single-Variable Calculus 1
George Washington University   Fall 2025
Recitation 5

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Problem 1. Let $g(x) = \frac{1}{x+3}$.

(a) Write down a limit expression to compute $g'(2)$. Be careful with order of operations and parentheses!

(b) Now compute $g'(2)$.

(c) Write a limit expression to compute $g'(x)$. Again, make sure you get your order of operations right.

(d) Compute $g'(x)$.

Solution:

(a) We have

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}. \end{aligned}$$

Make sure you have $\frac{1}{5+h}$, and not $\frac{1}{5} + h$! The second thing is very different and will not give you a useful answer.

(b) We have

$$\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{5+h} - \frac{1}{5}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{5-(5+h)}{5(5+h)}\right) \\
&= \lim_{h\to 0} \frac{-h}{5h(5+h)} = \lim_{h\to 0} \frac{-1}{5(5+h)} \\
&= \frac{-1}{5(5+0)} = \frac{-1}{25}.
\end{aligned}$$

(c)

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}.$$

Again, we want to make sure that we don't write $\frac{1}{x+3} + h$ or something like that.

(d)

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)}\right) \\
&= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\
&= \frac{-1}{(x+3)^2}.
\end{aligned}$$

Problem 2. Let $a(x) = |x|$ be the absolute value function.

(a) Write down a formula for a as a piecewise function.

(b) Write down a limit expression for the derivative of a at 0.

(c) What is the limit from the right?

(d) What is the limit from the left?

(e) What does that tell you about the derivative?

Solution:

(a)
$$a(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

(b)
$$a'(0) = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

(c)
$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

(d)
$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

(e) The limits to the right and the left don't exist, so the limit doesn't exist.

Problem 3. Let $g(x) = \sqrt[3]{x}$.

(a) Write down a limit formula to compute the derivative of g at 0.

(b) What is $g'(0)$? What does this tell you?

(c) Now write down a limit formula to compute the derivative of $p(x) = \sqrt[3]{x^2}$.

(d) What is this limit? What does that tell you?

(e) Write down a limit formula to compute the derivative of g at a when $a \neq 0$.

(f) (Bonus) Can you compute this limit? What do you have to do here? (It's not obvious, but there's an algebraic trick we've mentioned that can help us.)

Solution:

(a)
$$\begin{aligned} g'(0) &= \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h} \\ &= \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0}. \end{aligned}$$

(b)

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

This is a vertical tangent line, because the limit is always $+\infty$.

(c)

$$\begin{aligned} p'(0) &= \lim_{h \to 0} \frac{p(h) - p(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h^2} - 0}{h} \\ &= \lim_{x \to 0} \frac{p(x) - p(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x^2} - 0}{x - 0}. \end{aligned}$$

(d)

$$p'(0) = \lim_{h \to 0} \frac{p(h) - p(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h^2}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

This is a *cusp*, because the limit is $\pm\infty$ rather than just $+\infty$.

(e)

$$\begin{aligned} g'(a) &= \lim_{h \to 0} \frac{g(h) - g(a)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h} \\ &= \lim_{x \to 0} \frac{g(x) - g(a)}{x - a} = \lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}. \end{aligned}$$

(f) You might recognize this as being a difference of cube roots, so we can use the difference-of-cubes formula, as a sort of generalization of multiplication by the conjugate.

$$\begin{aligned} g'(a) &= \lim_{h \to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h} \\ &= \lim_{h \to 0} \frac{(a+h) - a}{h(\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2}} \\ &= \frac{1}{\sqrt[3]{a^2} + \sqrt[3]{a^2} + \sqrt[3]{a^2}} = \frac{1}{3\sqrt[3]{a^2}}. \end{aligned}$$

Problem 4. (a) Use the product rule to differentiate $(x^2 + 1)(3x^3 - 5)$.

(b) Multiply out $(x^2 + 1)(3x^3 - 5)$ to get one big polynomial. Use our derivative rules to compute that derivative.

(c) Which process was easier?

Solution:

(a) $2x(3x^3-5)+(x^2+1)(9x^2)$.

(b) We get

$$\begin{aligned}(x^2+1)(3x^3-5) &= 3x^5-5x^2+3x^3-5\\ \frac{d}{dx}(x^2+1)(3x^3-5) &= \frac{d}{dx}3x^5-5x^2+3x^3-5\\ &= 15x^4-10x+9x^2-0.\end{aligned}$$

(c) This is a matter of personal taste, but I'd say the second derivative was easier, but took more work total when we count the work of multiplying the terms out.

Problem 5. Compute $\frac{d}{dx}\frac{x^5-7x}{4x^2+3}$.

Solution:

$$\frac{d}{dx}\frac{x^5-7x}{4x^2+3}=\frac{(5x^5-7)(4x^2+3)-(8x)(x^5-7x)}{(4x^2+3)^2}.$$

Problem 6. (a) Let $h(x)=\tan^2(x)$. Find functions f and g so that $h(x)=(f\circ g)(x)$.

(b) Compute $f'(x)$ and $g'(x)$. Use that info to compute $h'(x)$.

(c) Now let $h(x)=\tan(x^2)$. Find functions f and g so that $h(x)=(f\circ g)(x)$.

(d) Compute $f'(x)$ and $g'(x)$. Use that information to compute $h'(x)$.

Solution:

(a) We can take $f(x)=x^2$ and $g(x)=\tan(x)$.

(b) $f'(x)=2x$ and $g'(x)=\sec^2(x)$, so

$$h'(x)=f'(g(x))\cdot g'(x)=f'(\tan(x))\cdot g'(x)=2\tan(x)\cdot\sec^2(x).$$

(c) Now we have $f(x)=\tan(x)$ and $g(x)=x^2$.

(d) Now we have $f'(x)=\sec^2(x)$ and $g'(x)=2x$, so

$$h'(x)=f'(g(x))\cdot g'(x)=f'(x^2)\cdot g'(x)=\sec^2(x^2)\cdot 2x.$$