Math 1232 Midterm Solutions

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- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has eight questions, over five pages. **You should not answer all eight questions.**
  - The first two problems are three pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
  - The remaining six problems represent topics S1 through S6. You will be graded on your **best three**, with a few possible bonus points if you also do well on the others.
  - Doing three secondary topics well is much better than doing five or six poorly.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

	a	b	c
M1			
M2			
S1		S2	
S3		S4	
S5		S6	
$\sum$			/90

Problem 1 (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Compute $\displaystyle\int \frac{1}{x + x(\ln(x))^2}\,dx$.

Solution: Set $u = \ln(x)$ so $du = dx/x$. Then
$$\int \frac{1}{x + x\ln(x)^2}\,dx = \int \frac{1}{1+u^2}\,du = \arctan(u) + C = \arctan(\ln(x)) + C.$$

(b) Find the derivative of $y = (x^3 - x)^{\arctan(x)}$.

Solution:
$$\begin{aligned}
\ln|y| &= \arctan(x)\ln|x^3 - x| \\
\frac{y'}{y} &= \frac{\ln|x^3 - x|}{1+x^2} + \frac{(3x^2-1)\arctan(x)}{x^3 - x} \\
y' &= y\left(\frac{\ln|x^3 - x|}{1+x^2} + \frac{(3x^2-1)\arctan(x)}{x^3 - x}\right) \\
&= (x^3 - x)^{\arctan(x)}\left(\frac{\ln|x^3 - x|}{1+x^2} + \frac{(3x^2-1)\arctan(x)}{x^3 - x}\right).
\end{aligned}$$

(c) Compute $\displaystyle\int_0^{\ln(2)} \frac{e^x}{e^x + 3}\,dx$

Solution: Take $e^x + 3 = u$ so $du = e^x\,dx$ and then we have
$$\begin{aligned}
\int_0^{\ln(2)} \frac{e^x}{e^x+3}\,dx &= \int_4^5 \frac{1}{u}\,du \\
&= \ln(u)|_4^5 = \ln(5) - \ln(4) = \ln(5/4) \approx .223
\end{aligned}$$

Problem 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int x^2 \sin(4x)\,dx$

Solution:
$$\begin{aligned}
\int x^2\sin(4x) &= \frac{-1}{4}x^2\cos(4x) - \int \frac{-1}{2}x\cos(4x)\,dx \\
&= \frac{-1}{4}x^2\cos(4x) + \frac{1}{2}\int x\cos(4x)\,dx \\
&= \frac{-1}{4}x^2\cos(4x) + \frac{1}{2}\left(\frac{1}{4}x\sin(4x) - \int \frac{1}{4}\sin(4x)\,dx\right) \\
&= \frac{-1}{4}x^2\cos(4x) + \frac{1}{8}x\sin(4x) - \frac{1}{8}\int sin(4x)\,dx \\
&= \frac{-1}{4}x^2\cos(4x) + \frac{1}{8}x\sin(4x) + \frac{1}{32}\cos(4x) + C.
\end{aligned}$$

(b) $\displaystyle\int \frac{x-8}{(x+1)(x-2)^2}\,dx$

Solution: We use a partial fractions decomposition.

$$\begin{aligned}
\frac{x-8}{(x+1)(x-2)^2} &= \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \\
\frac{x-8}{(x-2)^2} &= A + \frac{B(x+1)}{x-2} + \frac{C(x+1)}{(x-2)^2} \\
\frac{x-8}{x+1} &= \frac{A(x-2)^2}{x+1} + B(x-2) + C
\end{aligned}$$

Plugging $x = -1$ into the middle equation gives $-1 = A$, and plugging $x = 2$ into the last equation gives $-2 = C$. So we just need to find B. But plugging in our values for A and C and setting $x = 0$ gives

$$\begin{aligned}
\frac{x-8}{(x+1)(x-2)^2} &= \frac{-1}{x+1} + \frac{B}{x-2} + \frac{-2}{(x-2)^2} \\
\frac{-8}{4} &= -1 + \frac{B}{-2} + \frac{-2}{4} \\
4 &= 2 + B + 1 \\
1 &= B.
\end{aligned}$$

Thus

$$\begin{aligned}
\int \frac{x-8}{(x+1)(x-2)^2}\,dx &= \int \frac{-1}{x+1} + \frac{1}{x-2} + \frac{-2}{(x-2)^2}\,dx \\
&= -\ln|x+1| + \ln|x-2| + \frac{2}{x-2} + C.
\end{aligned}$$

(c) $\displaystyle\int \tan^4(2x)\,dx.$

Solution:

$$\begin{aligned}
\int \tan^4(2x)\,dx &= \int (\sec^2(2x) - 1)\tan^2(2x)\,dx = \int \sec^2(2x)\tan^2(2x) - \tan^2(2x)\,dx \\
&= \int \sec^2(2x)\tan^2(2x) - \left(\sec^2(2x) - 1\right)\,dx = \int \sec^2(2x)\tan^2(2x) - \sec^2(2x) + 1\,dx \\
&= \frac{1}{6}\tan^3(2x) - \frac{1}{2}\tan(2x) + x + C. \\
&\left(= \frac{1}{6}\sec^2(2x)\tan(2x) - \frac{2}{3}\tan(2x) + x + C.\right)
\end{aligned}$$

Problem 3 (S1)**.**
Let $f(x) = x^3 + 3^x$. Find $(f^{-1})'(4)$.

Solution: We see that $f(1) = 4$, so $f^{-1}(4) = 1$. Then by the Inverse Function Theorem we have

$$\begin{aligned}
(f^{-1})'(4) &= \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(1)} \\
f'(x) &= 3x^2 + 3^x\ln(x) \\
f'(1) &= 3 \\
(f^{-1})'(4) &= \frac{1}{3}.
\end{aligned}$$

Problem 4 (S2)**.** Compute $\displaystyle\lim_{x\to 3} \frac{e^{(x^2-9)} + \cos(\pi x)}{x-3}$.

Solution:

$$\lim_{x\to 3} \frac{e^{(x^2-9)} + \cos(\pi x)^{\nearrow 0}}{x-3_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 3} \frac{2xe^{x^2-9} - \pi\sin(\pi x)^{\nearrow 6}}{1_{\searrow 1}} = 6.$$

Problem 5 (S3)**.** Approximate $\int_{-2}^{1} \frac{1}{x+3}\,dx$ with three intervals and the trapezoid rule. Give an upper bound for the error in this estimate.

Solution:

$$\begin{aligned}
\int_{-2}^{1} \frac{1}{x+3}\,dx &\approx \frac{f(-2)+f(-1)}{2} + \frac{f(-1)+f(0)}{2} + \frac{f(0)+f(1)}{2} \\
&= \frac{1/1+1/2}{2} + \frac{1/2+1/3}{2} + \frac{1/3+1/4}{2} \\
&= \frac{3}{4} + \frac{5}{12} + \frac{7}{24} = \frac{35}{24}.
\end{aligned}$$

We know that

$$E_T \le \frac{K(b-a)^3}{12n^2} = \frac{K3^3}{12\cdot 3^2} = \frac{K}{4}.$$

We know that $|f''(x)| = \left|\frac{2}{(x+3)^3}\right| = \frac{2}{(x+3)^3}$, which is maximized when $x=-2$ and thus $|f''(x)| = 2$. So the error is $E_T \le \frac{2}{4} = \frac{1}{2}$.

Problem 6 (S4)**.** Compute $\int_{1}^{5} \frac{1}{\sqrt{x-1}}\,dx$

Solution:

$$\begin{aligned}
\int_{1}^{5} \frac{1}{\sqrt{x-1}}\,dx &= \lim_{t\to 1^+} \int_{t}^{5} \frac{1}{\sqrt{x-1}}\,dx \\
&= \lim_{t\to 1^+} 2\sqrt{x-1}\,\Big|_t^5 \\
&= \lim_{t\to 1^+} 2\sqrt{4} - 2\sqrt{t-1} = 4.
\end{aligned}$$

7/10 if everything is right except they don't do the limit.

Problem 7 (S5)**.** Compute the arc length of the curve $y = \frac{1}{27}(9x^2+6)^{3/2}$ as x varies from 2 to 4.

Solution: We have $y' = x\sqrt{9x^2+6}$, and thus

$$\begin{aligned}
L &= \int_2^4 \sqrt{1+x^2(9x^2+6)}\,dx = \int_2^4 \sqrt{1+6x^2+9x^4}\,dx \\
&= \int_2^4 3x^2+1 = x^3 + x\Big|_2^4 = 64+4-8-2 = 58
\end{aligned}$$

Problem 8 (S6)**.** Find the (specific) solution to $y' = \frac{x}{4y^2}$ if $y(1) = 1$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y^2} \\
4y^2\,dy &= x\,dx \\
\int 4y^2\,dy &= \int x\,dx \\
4y^3/3 &= x^2/2 + C \\
y &= \sqrt[3]{3x^2/8 + K}
\end{aligned}$$

Plugging in $x = 1, y = 1$ gives

$$\begin{aligned}
1 &= \sqrt[3]{3/8 + K} \\
K &= 5/8 \\
y &= \sqrt[3]{3x^2/8 + 5/8}
\end{aligned}$$