Math 1232 Practice Final Solutions

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- These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
- You will have 120 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made for yourself ahead of time. You must have written on the physical sheet you bring to the test in your own handwriting.
- You may not use a calculator.
- The exam has 14 problems, one on each mastery topic we've covered. The exam has 8 pages total.
- On the real final, each major topic will two questions, worth 10 points each. On this practice final I have given three questions on each major topic, for extra practice. Each secondary topic is worth 10 points.
- **You may not answer all the secondary topic questions.** You may attempt up to six secondary topics. Your four best will count towards your score on the final. You may get one or two bonus points for the fifth and sixth. **We will not grade more than six secondary topics.**
- If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Problem 1 (M1). (a) Compute $\int \frac{x}{\sqrt{4-x^4}}\,dx$.

Solution: Take $x^2 = 2u$ so $du = x\,dx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^4}}\,dx &= \int \frac{1}{\sqrt{4-4u^2}}\,du = \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2/2) + C.\end{aligned}$$

(b) Compute $\int 5^{3x}\,dx$.

Solution: Take $u = 3x$ so $du = 3\,dx$ and we get

$$\begin{aligned}\int 5^{3x}\,dx &= \frac{1}{3}\int 5^u\,du = \frac{1}{3\ln 5}5^u + C \\ &= \frac{1}{3\ln 5}5^{3x} + C.\end{aligned}$$

(c) Write a tangent line to the curve $y^2 = x^{x\cos(x)}$ at the point $(\pi/2, -1)$.

Solution: Implicit differentiation gives us

$$\begin{aligned}
2\ln(y) &= x\cos(x)\ln(x) \\
\frac{2y'}{y} &= \cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x) \\
y' &= \frac{1}{2}\left(\cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x)\right)y.
\end{aligned}$$

When $x = \pi/2, y = -1$, this gives us

$$\begin{aligned}
y' &= \frac{1}{2}\left(0\ln(\pi/2) - \pi/2\cdot 1\cdot \ln(\pi/2) + 0\right)(-1) = \frac{1}{2}\left(\pi/2\ln(\pi/2)\right) \\
&= \frac{\pi(\ln(\pi) - \ln(2))}{4}
\end{aligned}$$

and thus the tangent line has equation

$$y = \frac{\pi(\ln(\pi) - \ln(2))}{4}(x - \pi/2) - 1.$$

Problem 2 (M2)**.** Compute the following integrals:

(a) $\displaystyle\int \sin x \cos 2x \, dx$

Solution: Take $u = \cos 2x$ and $dv = \sin x\, dx$. We get $du = -2\sin 2x\, dx$ and $v = -\cos x\, dx$, and

$$\begin{aligned}
\int \sin x\cos 2x\, dx &= -\cos 2x\cos x - \int 2\sin 2x\cos x\, dx \\
&= -\cos 2x\cos x - 2\left(\int \sin 2x \cos x\, dx\right) \\
&= -\cos 2x\cos x - 2\left(\sin x\sin 2x - 2\int \sin x\cos 2x\, dx\right) \\
&= -\cos x\cos 2x - 2\sin x\sin 2x + 4\int \sin x\cos 2x\, dx \\
-3\int \sin x\cos 2x\, dx &= -\cos x\cos 2x - 2\sin x\sin 2x \\
\int \sin x\cos 2x\, dx &= \frac{1}{3}\left(\cos x\cos 2x + 2\sin x\sin 2x\right)
\end{aligned}$$

(b) $\displaystyle\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2-7}}$

Solution: We see as $\sqrt{x^2-7}$, which should make us think of trigonometric substitution, and in particular $\sqrt{7}\sec\theta = x$. (In the original version of the practice final I posted I had a typo here; see below). We work out $dx = \sqrt{7}\sec\theta\tan\theta\, d\theta$, and the bounds now range from $\sec\theta = 1$ to $\sec\theta = 2$, and thus $\theta = 0$ to $\theta = \pi/3$. Thus

$$\begin{aligned}
\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2-7}} &= \int_0^{\pi/3} \frac{\sqrt{7}\sec\theta\tan\theta\, d\theta}{\sqrt{7}\sec\theta\sqrt{7\sec^2\theta - 7}} \\
&= \int_0^{\pi/3} \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{7\tan^2\theta}} \\
&= \int_0^{\pi/3} \frac{d\theta}{\sqrt{7}} = \frac{\theta}{\sqrt{7}}\Big|_0^{\pi/3} = \frac{\pi}{3\sqrt{7}}.
\end{aligned}$$

(c) $\displaystyle\int \frac{4}{(x^2+1)(x+1)(x-1)}\, dx$

Solution: We have

$$\begin{aligned}
\frac{4}{(x^2+1)(x+1)(x-1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} \\
4 &= (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1) \\
\text{plug in1}: \quad 4 &= 0+0+4D \qquad \Rightarrow D = 1 \\
\text{plug in } -1: \quad 4 &= 0-4C+0 \qquad \Rightarrow C = -1 \\
\text{plug in } 0: \quad 4 &= -B-C+D = -B+1+1 \Rightarrow B = -2 \\
\text{plug in } 2: \quad 4 &= 3(2A+B)+5C+15D = 6A-6-5+15 = 6A+4 \Rightarrow A = 0.
\end{aligned}$$

Thus we have $A = 0, B = -2, C = -1, D = 1$, and our integral is

$$\begin{aligned}
\int \frac{4}{x^4-1}\,dx &= \int \frac{-2}{x^2+1} + \frac{1}{x-1} - \frac{1}{x+1}\,dx \\
&= -2\arctan(x) + \ln|(x-1)| - \ln|(x+1)| + C.
\end{aligned}$$

Problem 3 (M3)**.** Analyze the convergence of the following series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{3(-1)^n}{n\ln(n)}$.

Solution: We first consider the absolute value of the terms, and get the series $\sum \frac{3}{n\ln n}$. We can't compare this to $\frac{1}{n}$ because $\frac{3}{n\ln(n)} \leq \frac{1}{n}$, and being less than a divergent series doesn't tell us anything. In fact we don't have any good options to compare it to, so instead we compute

$$\begin{aligned}
\int_2^{+\infty} \frac{1}{x\ln x}\,dx &= \lim_{t\to+\infty} \int_2^t \frac{1}{x\ln x}\,dx \\
&= \lim_{t\to+\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du \\
&= \lim_{t\to+\infty} \ln u|_{\ln 2}^{\ln t} = \lim_{t\to+\infty} \ln\ln t - \ln\ln 2 = +\infty
\end{aligned}$$

so by the integral test the series does not converge absolutely.

Now we consider the original, alternating series. The terms of this series are decreasing and tend to zero, and the series is clearly alternating, so by the alternating series test the series converges. Thus the series converges conditionally.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5^n+3}{3^n-2}$.

Solution: The most obvious move is to use the Ratio Test. We compute

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{(5^{n+1}+3)/(3^{n+1}-2)}{(5^n+3)/(3^n-2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(5^{n+1}+3)(3^n-2)}{(5^n+3)(3^{n+1}-2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(5+3/5^n)(1-2/3^n)}{(1+3/5^n)(3-2/3^n)} \right| \\
&= \frac{5}{3} > 1
\end{aligned}$$

so by the ratio test this diverges.

That limit is a little gross, though. It's maybe better to just use the divergence test: we have $\lim_{n\to\infty} \frac{5^n+3}{3^n-2} = \infty$ so $\lim_{n\to\infty}(-1)^n \frac{5^n+3}{3^n-2} \neq 0$, so by the divergence test this series diverges.

We *can't* easily use the comparison test, because this series has some negative terms. The comparison test can show it doesn't converge absolutely, but can't show it diverges altogether.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3+n^2+n+1}{\sqrt{n^9}}$.

Solution: We consider the absolute value of the terms of this series. The terms in

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^3+n^2+n+1}{\sqrt{n^9}}\right| = \sum_{n=1}^{\infty} \frac{n^3+n^2+n+1}{\sqrt{n^9}}$$

are positive, so we can use the Limit Comparison Test. We have

$$\lim_{n\to\infty} \frac{(n^3+n^2+n+1)/\sqrt{n^9}}{1/\sqrt{n^3}} = \lim_{n\to\infty} \frac{n^{9/2}+n^{7/2}+n^{5/2}+n^{3/2}}{n^{9/2}} = \lim_{n\to\infty} \frac{1+1/n+1/n^2+1/n^3}{1} = 1.$$

Thus by the Limit Comparison Test, our series converges if and only if $\sum \frac{1}{n^{3/2}}$ converges. But $3/2 > 1$ so this converges, and thus our series converges (absolutely).

Problem 4 (M4). (a) Find a power series for $\frac{1}{x^3}(e^{2x^3}-1)$, and write down the first three non-zero terms explicitly.

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!} = 1 + 2x^3 + 2x^6 + \frac{4}{3}x^9 + \dots$$

$$\frac{1}{x^3}(e^{2x^3}-1) = \sum_{n=1}^{\infty} \frac{2^n x^{3n-3}}{n!} \left(= \sum_{n=0} \frac{2^{n+1}x^{3n}}{(n+1)!}\right) = 2 + 2x^3 + \frac{4}{3}x^6 + \dots$$

(b) Find a power series for $x^2\arctan(x^2)$ centered at 0.

Solution: We know that $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Thus

$$x^2\arctan(x^2) = x^2\sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+4}}{2n+1}.$$

(c) Find the degree-three Taylor polynomial for $f(x) = \frac{3}{x^3}$ centered at 3.

Solution: First we compute some derivatives:

$$f(x) = \frac{3}{x^3} \qquad f(3) = \frac{1}{9}$$

$$f'(x) = -3\frac{3}{x^4} \qquad f'(3) = -\frac{1}{9}$$

$$f''(x) = (3)(4)\frac{3}{x^5} \qquad f''(3) = \frac{4}{27}$$

$$f'''(x) = -(3)(4)(5)\frac{3}{x^6} \qquad f'''(3) = -\frac{20}{81}$$

Then the Taylor polynomial is

$$T_3(x,3) = \frac{1}{9} - \frac{1}{9}(x-3) + \frac{4/27}{2!}(x-3)^2 + \frac{-20/81}{3!}(x-3)^3.$$

Problem 5 (S1)**.** Let $g(x) = \sqrt[5]{x^9 + x^7 + x + 1}$. Find $(g^{-1})'(1)$.

Solution: We see that $g(0) = 1$, so $g^{-1}(1) = 0$. Then by the Inverse Function Theorem we have

$$(g^{-1})'(1) = \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(0)}$$

$$g'(x) = \frac{1}{5}(x^9 + x^7 + x + 1)^{-4/5}(9x^8 + 7x^6 + 1)$$

$$g'(0) = \frac{1}{5}(1)(1) = \frac{1}{5}$$

$$(g^{-1})'(1) = 5.$$

Problem 6 (S2)**.** Compute $\lim_{x\to 0} \frac{e^x - \tan(x) - 1}{x^2}$

Solution:

$$\lim_{x\to 0} \frac{e^x - \tan(x) - 1}{x^2} = \lim_{x\to 0} \frac{e^x - \sec^2(x)}{2x}$$

$$= \lim_{x\to 0} \frac{e^x - 2\sec^2(x)\tan(x)}{2} = \frac{1}{2}.$$

Problem 7 (S3)**.** How many intervals do you need with the **trapezoid** rule to approximate $\int_0^3 \frac{1}{1+x}$ to within 1/2? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$f'(x) = \frac{-1}{(1+x)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$|f''(x)| \leq \frac{2}{1^3} = 2$$

Then we have

$$|E_T| \leq \frac{2(3-0)^3}{12n^2} = \frac{9}{2n^2}$$

We want $1/2 \geq \frac{9}{2n^2}$ which implies

$$n^2 \geq 9$$

$$n \geq 3.$$

Then we have

$$T_3 = \frac{1}{2}f(0) + f(1) + f(2) + \frac{1}{2}f(3) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{35}{24} \approx 1.45833$$

Since the true answer is $\ln(4) \approx 1.38629$, this is indeed within our margin of error.

Problem 8 (S4)**.** $\int_1^{+\infty} \frac{1}{x^2 - 2x}\, dx$

Solution:

$$\begin{aligned}
\int_1^{+\infty} \frac{1}{x^2-2x}\,dx &= \lim_{t\to+\infty}\int_1^t \frac{dx}{x^2-2x} \\
&= \lim_{t\to+\infty}\int_1^2 \frac{dx}{x(x-2)} + \int_2^t \frac{dx}{x(x-2)} \\
&= \lim_{r\to 2^-}\int_1^r \frac{dx}{x(x-2)} + \lim_{s\to 2^+}\int_s^3 \frac{dx}{x(x-2)} + \lim_{t\to+\infty}\int_3^t \frac{dx}{x(x-2)}
\end{aligned}$$

The integral converges if and only if each of these three integrals converges. But let's consider the first one:

$$\begin{aligned}
\lim_{r\to 2^-}\int_1^r \frac{dx}{x(x-2)} &= \lim_{r\to 2^-}\frac{1}{2}\int_1^r \frac{1}{x-2}-\frac{1}{x}\,dx \\
&= \frac{1}{2}\lim_{r\to 2^-}(\ln|x-2|-\ln|x|)|_1^r \\
&= \frac{1}{2}\lim_{r\to 2^-}(\ln(2-r)-\ln(r)-\ln(1)-\ln(1)) \\
&= \frac{1}{2}\lim_{r\to 2^-}(\ln(2-r)-\ln(r)) = -\infty.
\end{aligned}$$

So one of the summands doesn't converge, and thus the integral as a whole diverges.

Problem 9 (S5)**.** Find the area of the surface obtained by rotating the curve $x = 1 + 2y^2$ for $1 \le y \le 2$ about the x-axis.

Solution: Recall we have the formula for surface area $A = \int 2\pi y\, ds$ when we rotate around the x-axis. We will further integrate with respect to y because everything is given as a function of y. We get $x' = 4y$, and thus $ds = \sqrt{1+16y^2}$, so

$$\begin{aligned}
SA &= \int_1^2 2\pi y\sqrt{1+16y^2}\,dy \\
&\quad u = 1+16y^2, du = 32y\,dy \\
&= \int_{17}^{65} \frac{\pi}{16}\sqrt{u}\,du \\
&= \frac{\pi}{16}\frac{2u^{3/2}}{3}|_{17}^{65} = \frac{\pi}{24}\left(65\sqrt{65}-17\sqrt{17}\right).
\end{aligned}$$

Problem 10 (S6)**.** Find the (specific) solution to $y' = x^2y^3$ if $y(0) = 1$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= x^2y^3 \\
\frac{dy}{y^3} &= x^2\,dx \\
\int \frac{dy}{y^3} &= \int x^2\,dx \\
\frac{-1}{2y^2} &= \frac{x^3}{3} + C \\
y^2 &= \frac{-1}{2x^3/3 + 2C}
\end{aligned}$$

Plugging in $x = 0, y = 1$ gives

$$1 = \frac{-1}{2C}$$
$$C = -1/2$$
$$y = \sqrt{\frac{-1}{2x^3/3 - 1}} = \sqrt{\frac{1}{1 - 2x^3/3}}.$$

Problem 11 (S7)**.** Compute $\lim_{n\to\infty} \frac{2^n n!}{(2n)!}$.

Solution: $2^n n! = 2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n$ and $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot 2n - 1 \cdot 2n$, so

$$0 \leq \frac{2^n n!}{(2n)!} = \frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} \leq \frac{1}{2n-1}$$

and since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2n-1} = 0$, by the Squeeze Theorem $\lim_{n\to\infty} \frac{2^n n!}{(2n!)} = 0$.

Alternatively, we could notice that

$$0 \leq \frac{2^n n!}{(2n!)} = \frac{2^n}{(n+1)(n+2)\dots(2n-1)n} = \frac{2}{n+1}\frac{2}{n+2}\dots\frac{2}{2n-1}\frac{2}{2n} \leq \frac{1}{n}$$

and since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$, and thus by the Squeeze Theorem $\lim_{n\to\infty} \frac{2^n n!}{(2n)!} = 0$.

Alternatively and a bit overpowered-ly, we could consider the series $\sum_{n=1}^{\infty} \frac{2^n n!}{(2n)!}$. Using the ratio test we calculate

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!/(2n+2)!}{2^n n!/2n!} \right| = \lim_{n\to\infty} \frac{2(n+1)}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

Thus by the ratio test we see the series converges; and by the divergence test, if the series converges then the sequence of terms must converge to zero.

Problem 12 (S8)**.** Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-3)^n}{(2n)^2+1}$.

Solution: We use the ratio test to find the radius of convergence. We have

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}/((2n+1)^2+1)}{(x-3)^n/((2n)^2+1)} \right| = \lim_{n\to\infty} \frac{|(x-3)|(4n^2+1)}{4n^2+4n+2} = |x-3|.$$

Thus the series converges absolutely when $|x-3| < 1$ and diverges when $|x-3| > 1$, and thus it converges absolutely on $(2, 4)$.

When $|x-3| = 1$ we have two points to check. If $x = 4$ then our series is $\sum \frac{1}{(2n)^2+1}$ which converges by the comparison test, since $\frac{1}{(2n)^2+1} < \frac{1}{n^2}$. If $x = 2$ then our series is $\sum \frac{(-1)^n}{(2n)^2+1}$ which converges by the alternating series test. Thus the real interval of convergence is $[2, 4]$.

Problem 13 (S9)**.** Use a second-degree Taylor polynomial to approximate $\sqrt[4]{82}$.

Solution: If $g(x) = \sqrt[4]{1+x}$, then by the binomial series we have $g(x) \approx 1 + \frac{x}{4} - \frac{3x^2}{32}$. Then

$$\sqrt[4]{82} = \sqrt[4]{81+1} = 3\sqrt[4]{1+1/81} \approx 3\left(1 + \frac{1}{81 \cdot 4} - \frac{3}{32 \cdot 81^2}\right)$$
$$= 3 + \frac{1}{27 \cdot 4} - \frac{1}{32 \cdot 27^2}$$
$$= 3 + \frac{1}{108} - \frac{1}{23328} = \frac{70119}{23328} \approx 3.00579.$$

Problem 14 (S10)**.** Find an equation for the tangent line to the curve defined by the polar equation $r = 2 + \sin(3\theta)$ at the point $\theta = \pi/4$.

Solution: We can use our polar equations to parametrize x and y as a function of θ:

$$\begin{aligned} x &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) \\ y &= 2\sin(\theta) + \sin(\theta)\sin(3\theta) \\ x(\pi/4) &= \sqrt{2} + 1/2 \\ y(\pi/4) &= \sqrt{2} + 1/2. \end{aligned}$$

Then we can use these parametric equations to find the derivatives of x and y:

$$\begin{aligned} \frac{dx}{d\theta} &= -2\sin(\theta) - \sin(\theta)\sin(3\theta) + 3\cos(\theta)\cos(3\theta) \\ &= \sqrt{2} + 1/2 - 3/2 = \sqrt{2} - 1 \\ \frac{dy}{d\theta} &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) + 3\sin(\theta)\cos(3\theta) \\ &= -\sqrt{2} - 1/2 - 3/2 = -\sqrt{2} - 2. \end{aligned}$$

Now we have two choices. First, we can write down a parametric equation for the tangent line. With the slopes we have, this is straightforward. We get

$$\begin{aligned} \vec{r}(t) &= \left(\sqrt{2} + 1/2, \sqrt{2} + 1/2\right) + t\left(\sqrt{2} - 1, -\sqrt{2} - 2\right) \\ &= \left(\sqrt{2} + 1/2 + t\left(\sqrt{2} - 1\right), \sqrt{2} + 1/2 + t\left(-\sqrt{2} - 2\right)\right) \end{aligned}$$

Alternatively, we can use our parametric derivatives to find the carterian derivative of y with respect to x:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos(\theta) + \cos(\theta)\sin(3\theta) + 3\sin(\theta)\cos(3\theta)}{-2\sin(\theta) - \sin(\theta)\sin(3\theta) + 3\cos(\theta)\cos(3\theta)} \\ &= \frac{\sqrt{2} + 1/2 - 3/2}{-\sqrt{2} - 1/2 - 3/2} = \frac{\sqrt{2} - 1}{-2 - \sqrt{2}}. \end{aligned}$$

And now that we have a slope, we can compute the implicit cartesian equation of this tangent line:

$$y - (\sqrt{2} + 1/2) = \frac{1 - \sqrt{2}}{2 + \sqrt{2}}(x - (\sqrt{2} + 1/2)).$$