

Math 1231 Practice Midterm Solutions

Instructor: Jay Daigle

- These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has eight questions, over five pages. **You should not answer all eight questions.**
 - The first two problems are two pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
 - The remaining six problems represent topics S1 through S6. You will be graded on your best three, with a few possible bonus points if you also do well on the others.
 - Doing three secondary topics well is much better than doing five or six poorly.
 - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Problem 1 (M1). Compute the following using methods we have learned in class. Show enough work to justify your answers.

- (a) Find the tangent line to $h(x) = \arcsin(e^x)$ at $\ln(1/2)$.

Solution: We have $h'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$, so $h'(\ln(1/2)) = \frac{e^{\ln 1/2}}{\sqrt{1-e^{2 \ln(1/2)}}} = \frac{1/2}{\sqrt{1-1/4}} = \frac{1}{\sqrt{3}}$. We also have $h(\ln(1/2)) = \arcsin(1/2) = \pi/6$.

Thus the equation of the tangent line is

$$y - \pi/6 = \frac{1}{\sqrt{3}}(x - \ln(1/2)).$$

- (b) $\int_1^2 \frac{e^{1/x}}{x^2} dx =$

Solution: We take $u = 1/x$ so $du = -\frac{1}{x^2} dx$. Then

$$\begin{aligned} \int_1^2 \frac{e^{1/x}}{x^2} dx &= \int_1^{1/2} -e^u du \\ &= -e^u \Big|_1^{1/2} = -e^{1/2} + e^1 = e - \sqrt{e}. \end{aligned}$$

- (c) $\int \frac{\cos(x) \sin(x)}{1 + \cos^4(x)} dx =$

Solution: We can take $u = \cos(x)$ so that $du = -\sin(x) dx$. Then

$$\int \frac{\cos(x) \sin(x)}{1 + \cos^4(x)} dx = \int \frac{-u}{1 + u^4} du$$

Then we can set $v = u^2$ so that $dv = 2u du$ and we get

$$\begin{aligned} \int \frac{-u}{1 + u^4} du &= \int \frac{-1}{2} \frac{1}{1 + v^2} dv = \frac{-1}{2} \arctan(v) + C \\ &= \frac{-1}{2} \arctan(u^2) + C = \frac{-1}{2} \arctan(\cos^2(x)) + C. \end{aligned}$$

Problem 2 (M2). Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

- (a) $\int \frac{2x + 1}{\sqrt{x^2 - 1}} dx$

Solution: Since we see $\sqrt{x^2 - 1}$ we want to try a trig substitution. (You might try $u = x^2 - 1$ first, which almost works, but doesn't quite). So we set $x = \sec \theta$ and $dx = \sec \theta \tan \theta d\theta$. We have

$$\begin{aligned} \int \frac{2x + 1}{\sqrt{x^2 - 1}} dx &= \int \frac{2 \sec \theta + 1}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\ &= \int \frac{2 \sec^2 \theta \tan \theta + \sec \theta \tan \theta}{\tan \theta} d\theta \\ &= \int 2 \sec^2 \theta + \sec \theta d\theta \\ &= 2 \tan \theta + \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

If $\sec \theta = x$ then θ is in a triangle with hypotenuse x and adjacent side 1 and thus opposite side $\sqrt{x^2 - 1}$. Thus $\tan \theta = \sqrt{x^2 - 1}$. This is good, since this formula appeared in our original question, and we see that

$$\int \frac{2x+1}{\sqrt{x^2-1}} dx = 2\sqrt{x^2-1} + \ln|x + \sqrt{x^2-1}| + C.$$

(b) $\int x \sec^2 x dx$

Solution: We use integration by parts. Take $u = x, dv = \sec^2 x dx$ so $du = dx, v = \tan x$. Then

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x + \ln|\cos x| + C.$$

(c) $\int_0^1 \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} dx$

Solution: We use a partial fractions decomposition.

$$\begin{aligned} \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} &= \frac{A}{x - 2} + \frac{Bx + C}{x^2 - x - 1} \\ 3x^2 - 6x + 1 &= A(x^2 - x - 1) + (Bx + C)(x - 2). \end{aligned}$$

Plugging in $x = 2$ gives us that $1 = A$. Plugging in $x = 0$ gives $1 = -A - 2C = -1 - 2C$ and thus $C = -1$. Then plugging in $x = 1$ gives $-2 = -A - B - C = -1 - B + 1$ and thus $B = 2$. So we have

$$\begin{aligned} \int_0^1 \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} dx &= \int_0^1 \frac{1}{x - 2} + \frac{2x - 1}{x^2 - x - 1} dx \\ &= (\ln|x - 2| + \ln|x^2 - x - 1|) \Big|_0^1 \\ &= \ln(1) + \ln(1) - \ln(2) - \ln(1) = -\ln(2). \end{aligned}$$

Problem 3 (S1). Let $f(x) = \sqrt[3]{x^5 + x^4 + x^3 + x^2 + 2x}$. Find $(f^{-1})'(4)$.

Solution: Plugging in numbers, we see that $f(2) = \sqrt[3]{32 + 16 + 8 + 4 + 4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(f^{-1})'(4) = \frac{1}{f'(2)}$. But

$$\begin{aligned} f'(x) &= \frac{1}{3} (x^5 + x^4 + x^3 + x^2 + 2x)^{-2/3} (5x^4 + 4x^3 + 3x^2 + 2x + 2) \\ f'(2) &= \frac{1}{3} (64)^{-2/3} (80 + 32 + 12 + 4 + 2) = \frac{130}{48} = \frac{65}{24}. \end{aligned}$$

Thus by the inverse function theorem we have

$$(f^{-1})'(4) = \frac{24}{65}.$$

Problem 4 (S2). Find $\lim_{x \rightarrow 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)}$.

Solution: $\lim_{x \rightarrow 0} 2 \sin(x) - \sin(2x) = 0 - 0 = 0$, and $\lim_{x \rightarrow 0} x - \sin(x) = 0$, so we can use L'Hospital's Rule.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 \sin(x) - \sin(2x)}{x - \sin(x)} &= \text{L'H} \lim_{x \rightarrow 0} \frac{2 \cos(x) - 2 \cos(2x)}{1 - \cos(x)} \\ &= \text{L'H} \lim_{x \rightarrow 0} \frac{-2 \sin(x) + 4 \sin(2x)}{\sin(x)} \\ &= \text{L'H} \lim_{x \rightarrow 0} \frac{-2 \cos(x) + 8 \cos(2x)}{\cos(x)} = \frac{6}{1} = 6. \end{aligned}$$

Problem 5 (S3). Use Simpson's rule and six intervals to estimate $\int_0^6 x^4 dx$. Give an upper bound for the error on this approximation.

Solution:

$$\begin{aligned} \int_0^6 x^4 dx &\approx \frac{1}{3} (0^4 + 4 \cdot 1^4 + 2 \cdot 2^4 + 4 \cdot 3^4 + 2 \cdot 4^4 + 4 \cdot 5^4 + 6^4) \\ &= \frac{1}{3} (0 + 4 + 32 + 324 + 512 + 2500 + 1296) = \frac{4668}{3} = 1556. \end{aligned}$$

To find the error: if $f(x) = x^4$ then $f''''(x) = 24$, so we can take $L = 24$. Then we have the formula

$$|E_S| \leq \frac{L(b-a)^5}{180n^4} = \frac{24 \cdot 6^5}{180 \cdot 6^4} = \frac{24 \cdot 6}{180} = \frac{4}{5}.$$

So the error in this approximation is less than or equal to $4/5$. If we work things out exactly, we see $\int_0^6 x^4 dx = 1555.2$, so the error is in fact $4/5$ exactly.

Problem 6 (S4). Compute $\int_1^{10} \frac{1}{\sqrt[3]{x-2}} dx$.

Solution: We must split the integral up into two parts:

$$\begin{aligned} \int_1^{10} \frac{1}{\sqrt[3]{x-2}} dx &= \int_1^2 \frac{1}{\sqrt[3]{x-2}} dx + \int_2^{10} \frac{1}{\sqrt[3]{x-2}} dx \\ &= \lim_{s \rightarrow 2^-} \int_1^s \frac{dx}{\sqrt[3]{x-2}} + \lim_{t \rightarrow 2^+} \int_t^{10} \frac{dx}{\sqrt[3]{x-2}} \\ &= \lim_{s \rightarrow 2^-} \frac{3}{2} (x-2)^{2/3} \Big|_1^s + \lim_{t \rightarrow 2^+} \frac{3}{2} (x-2)^{2/3} \Big|_t^{10} \\ &= \left(\lim_{s \rightarrow 2^-} \frac{3(s-2)^{2/3}}{2} - \frac{3}{2} \right) + \left(\lim_{t \rightarrow 2^+} \frac{3 \cdot 8^{2/3}}{2} - \frac{3(t-2)^{2/3}}{2} \right) \\ &= \frac{3}{2} \cdot 0 - \frac{3}{2} + \frac{12}{2} - \frac{3}{2} \cdot 0 = \frac{9}{2}. \end{aligned}$$

Problem 7 (S5). Find the surface area of the surface obtained by rotating $y = \sqrt{5+4x}$ for $-1 \leq x \leq 1$ about the x -axis.

Solution: We have $y' = \frac{1}{2}(5 + 4x)^{-1/2} \cdot 4 = \frac{2}{\sqrt{5+4x}}$, so $ds = \sqrt{1 + \frac{4}{5+4x}} dx$. Then

$$\begin{aligned} A &= \int_{-1}^1 2\pi y ds = 2\pi \int_{-1}^1 \sqrt{5 + 4x} \sqrt{1 + \frac{4}{5 + 4x}} dx \\ &= 2\pi \int_{-1}^1 \sqrt{5 + 4x + 4} dx = 2\pi \int_{-1}^1 \sqrt{9 + 4x} dx \\ &= 2\pi \left(\frac{2}{3}(9 + 4x)^{3/2} \cdot \frac{1}{4} \right) \Big|_{-1}^1 = 2\pi \left(\frac{1}{6} 13\sqrt{13} - \frac{1}{6} 5\sqrt{5} \right) = \frac{\pi}{3} (13\sqrt{13} - 5\sqrt{5}). \end{aligned}$$

Problem 8 (S6). Find a (specific) solution to the initial value problem $y'/x - y = 1$ if $y(0) = 3$

Solution:

$$\begin{aligned} y'/x &= 1 + y \\ \frac{dy}{1 + y} &= x dx \\ \ln |1 + y| x^2/2 + C & \\ 1 + y &= e^{x^2/2} e^C \\ y &= K e^{x^2/2} - 1 \\ 3 &= K - 1 \Rightarrow K = 4 \\ y &= 4e^{x^2/2} - 1. \end{aligned}$$