§2.3 Integration by Partial fractions

Goal: integrate $\frac{\text{poly}}{\text{poly}}$

Easy / Old

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

$u = x-1$

$$\int \frac{2x}{x^2-1}\,dx = \ln|x^2-1| + C$$

$u = x^2-1$

$du = 2x\,dx$

New / Hard

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$\boxed{= \ln|x-1| - \ln|x+1| + C.}$$

$u = x^2-1$

$du = 2x\,dx$

problem

$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1}{(x-1)(x+1)} - \frac{x-1}{(x-1)(x+1)}$$

$$= \frac{2}{(x-1)(x+1)} = \frac{2}{x^2-1}.$$

Goal: Break big fraction into small fractions

Step 1: Polynomial Long division

Ex: $\dfrac{x^3+2x^2+1}{x+1}$

$= x^2+x-1+\dfrac{2}{x+1}$

r 2

$$
\begin{array}{r l}
 & x^2 + x - 1 + \dfrac{2}{x+1} \\
x+1 \,\big)\!\! & x^3+2x^2+0x+1 \\
- & x^3+x^2 \\
\hline
 & x^2+0x+1 \\
- & x^2+x \\
\hline
 & -x+1 \\
- & -x-1 \\
\hline
 & 2
\end{array}
$$

Step 1

$u = x^2+1$

$$\int \frac{x^3+1}{x^2+1}\,dx = \int x + \boxed{\frac{-x+1}{x^2+1}}\,dx = \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$\frac{\text{poly}}{\text{poly}}$! Step 1 LD

$$= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + \arctan(x) + C.$$

$$\begin{array}{r} x + \frac{-x+1}{x^2+1} \\ x^2+0x+1 \overline{\big)\, x^3+0x^2+0x+1} \\ \underline{x^3+0x^2+x} \\ -x+1 \end{array}$$

Can check easily!

$$\int \frac{x}{x^2+1}\,dx = \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$$

$u = x^2+1$

$du = 2x\,dx$

Step 1. LD

numerator degree < denominator degree

Step 2 Partial Fraction Decomposition / ABC method

$$\int \frac{3x^2-1}{x^3-x}\,dx$$

Want to write frac as sum of smaller fracs

$$\frac{3x^2-1}{x^3-x} = \frac{3x^2-1}{x(x+1)(x-1)} \overset{\text{Hope}}{=} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$\frac{3x^2-1}{x^3-x} = \frac{3x^2-1}{x(x+1)(x-1)} \overset{\text{Hope}}{=} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

Clear denominators

$$3x^2-1 = \frac{A}{\cancel{x}}(\cancel{x})(x+1)(x-1) + \frac{B}{\cancel{x+1}}x\cancel{(x+1)}(x-1) + \frac{C}{\cancel{x-1}}x(x+1)\cancel{(x-1)}$$

$$3x^2-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

Way 1: textbook

$$3x^2-1 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$$

$$= (A+B+C)x^2 + (C-B)x - A$$

$3 = A+B+C$ $\quad x^2$

$0 = C-B$ $\quad x$

$-1 = -A$ $\quad 1$

$\Rightarrow A=1,\ C=B$

$3 = 1+B+B \Rightarrow 2 = 2B \Rightarrow B=1,$

$C=1$

$$\int \frac{3x^2-1}{x^3-x}\,dx = \int \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{x-1}\,dx$$

$$= \int \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}\,dx$$

$$= \ln|x| + \ln|x+1| + \ln|x-1| + C.$$

punch line: $= \ln|x(x+1)(x-1)| + C$

$= \ln|x^3-x| + C.$

Could have done u-sub.

$$\int \frac{2x+1}{x^3+2x^2+x}\,dx = \int \frac{2x+1}{x\,(x+1)^2}\,dx$$

$$\frac{2x+1}{x\,\cancel{(x+1)^2}} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{\cancel{(x+1)^2}}$$

One fraction for each power of $x+1$

Clear denoms

$$\frac{C}{\cancel{(x+1)^2}} \times \cancel{(x+1)^2} = Cx$$

$$2x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

Way 2: plug in

-1: $-1 = 0 + 0 - C \Rightarrow C = 1$

0: $1 = A + 0 + 0 \Rightarrow A = 1$

1: $3 = 4A + 2B + C = 4 + 2B + 1$

$3 = 5 + 2B$

$-2 = 2B$

$B = -1$

$$\int \frac{2x+1}{x^3+2x^2+x}\,dx = \int \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\,dx$$

$u = x+1$

$$= \int \frac{1}{x} + \frac{-1}{x+1} + \frac{1}{(x+1)^2}\,dx$$

$A = 1, \quad C = 1, \quad B = -1$

$$\int \frac{1}{(x+1)^2}\,dx = \int \frac{1}{u^2}\,du$$
$$= \int u^{-2}\,du$$
$$= \frac{u^{-1}}{-1} + C$$

$$= \ln|x| - \ln|x+1| + (-1)(x+1)^{-1} + C$$

Check

$$\frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2} = \frac{\cancel{x^2}+2x+1}{x(x+1)^2} - \frac{\cancel{x^2}+x}{x(x+1)^2} + \frac{x}{(x+1)^2} = \frac{2x+1}{x(x+1)^2}.$$

$$\int \frac{3x-1}{x(x^2+1)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx$$

linear

quadratic

Can't factor

$3x-1 = A(x^2+1) + (Bx+C)x$

Plug in

0: $-1 = A + 0 \Rightarrow A = -1$

1: $2 = 2A + B + C$

$= -2 + B + C$

$4 = B + C$

2: $5 = 5A + 4B + 2C$

$5 = -5 + 4B + 2C$

$$\boxed{\begin{aligned} 10 &= 4B + 2C \\ 4 &= B + C \end{aligned}}$$

$C = 4 - B$

$10 = 4B + 2(4-B) = 4B + 8 - 2B$

$2 = 2B \Rightarrow B = 1$

$C = 3.$

$$\int \frac{3x-1}{x(x^2+1)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx$$

$A=-1$
$B=1$
$C=3$

$$= \int \frac{-1}{x} + \frac{x+3}{x^2+1}\,dx$$

$$= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{3}{x^2+1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C.$$

1) Can integrate

<u>any</u> $\dfrac{\text{poly}}{\text{poly}}$

2) Sometimes need to complete the square

$$\int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx$$