§2.3 Integration by Partial Fractions

Goal: Integrate $\frac{\text{poly}}{\text{poly}}$

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

$u = x-1$

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + C$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$$\int \frac{2}{x^2-1}\,dx = \int \frac{2}{u}\cdot\frac{du}{2x}$$ does n't work

$u = x^2-1$

$\frac{du}{dx} = 2x$

Split it apart

$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1}{(x-1)(x+1)} - \frac{(x-1)}{(x-1)(x+1)}$$

$$= \frac{2}{x^2-1}$$

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln|x-1| - \ln|x+1| + C.$$

Step 1: Polynomial Long Division

If deg num ≥ deg denom, fix by dividing

$$\frac{x^3+2x^2+1}{x+1}$$

$$= x^2+x-1+\frac{2}{x+1}$$

$$\boxed{\begin{aligned}&\frac{x^3+x^2}{x+1}+\frac{x^2+x}{x+1}-\frac{x+1}{x+1}+\frac{2}{x+1}\\ &=\frac{x^3+2x^2+1}{x+1} \checkmark\end{aligned}}$$

r 2

$$\begin{array}{r|l} & x^2+x-1+\frac{2}{x+1} \\ x+1 & x^3+2x^2+0x+1 \\ - & x^3+x^2 \\ \hline & x^2+0x \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & 2 \end{array}$$

Want $\int \frac{x^3+1}{x^2+1}\,dx = \int x + \frac{-x+1}{x^2+1}\,dx = \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

Long division

$$\begin{array}{r} x + \frac{-x+1}{x^2+1} \\ x^2+0x+1 \overline{\big)\, x^3+0x^2+0x+1} \\ x^3+0x^2+x \\ \hline -x+1 \end{array}$$

$= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + \arctan(x) + C.$

$\frac{-x+1}{x^2+1} = \frac{-x}{x^2+1} + \frac{1}{x^2+1}$

$\frac{3}{4} = \frac{2}{4} + \frac{1}{4}$

Guarantee:

deg num < deg denom

Want: deg denom ≤ 2

Break up denom

Partial Fractions decomposition / ABC method

Ex! $\int \frac{3x^2-1}{x^3-x}\,dx$ $\qquad \frac{3x^2-1}{x^3-x} = \frac{3x^2-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

Clear denom

$$3x^2-1 = \frac{A}{\cancel{x}}\cancel{x}(x+1)(x-1) + \frac{B}{\cancel{x+1}}x\cancel{(x+1)}(x-1) + \frac{C}{\cancel{x-1}}x(x+1)\cancel{(x-1)}$$

$A(x^2-1)$ $\qquad B(x^2-x)$

$$3x^2-1 = A(x+1)(x-1) + B(x)(x-1) + C\,x(x+1)$$

Method 1: text book

$$3x^2-1 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$$

$$= (A+B+C)x^2 + (C-B)x - A$$

Equate coeffs

$3 = A+B+C \Rightarrow 3 = 1 + B + B$

$0 = C-B \Rightarrow C = B$ $\qquad 2 = 2B$

$-1 = -A \Rightarrow A = 1$ $\qquad 1 = B$

$\Rightarrow C = 1$

$$\int \frac{3x^2-1}{x^3-x}\,dx = \int \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\,dx$$

$A=1$
$B=1$
$C=1$

$u = x^3 - x$
$du = 3x^2 - 1\,dx$

$$= \int \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}\,dx$$

$$= \ln|x| + \ln|x+1| + \ln|x-1| + C. \checkmark$$ Done

$$= \ln\left|x(x+1)(x-1)\right| + C.$$

Check: $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}$

$$= \frac{(x+1)(x-1)}{x(x+1)(x-1)} + \frac{x(x-1)}{x(x+1)(x-1)} + \frac{x(x+1)}{x(x+1)(x-1)}$$

$$\int \frac{2x+1}{x^3+2x^2+x}\,dx = \int \frac{2x+1}{x(x+1)^2}\,dx = \int \frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\,dx$$

$u = x+1$

$$\frac{2x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$2x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

Method 2: plug in x values

$0:\ 1 = A + 0 + 0$

$-1:\ -1 = 0 + 0 - C \Rightarrow C = 1$

$1:\ 3 = 4A + 2B + C \Rightarrow 3 = 5 + 2B$

$3 = 4 + 2B + 1 \qquad -2 = 2B$

$B = -1.$

$$= \ln|x| - \ln|x+1| + \frac{(x+1)^{-1}}{-1} + C \quad \checkmark$$

$$= \ln|x| - \ln|x+1| - \frac{1}{x+1} + C. \quad \checkmark$$

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+1} = \frac{A(x+1)}{x(x+1)} + \frac{Bx}{x(x+1)} + \frac{Cx}{x(x+1)}$$

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2}{x(x+1)^2} + \frac{Bx(x+1)}{x(x+1)^2} + \frac{Cx}{x(x+1)^2}$$

$$\frac{A}{x} + \frac{C}{(x+1)^2}$$

right denom, wrong numerator

$$\frac{A}{x} + \frac{Bx+C}{(x+1)^2}$$

also ok

$$\int \frac{3x-1}{x(x^2+1)}\,dx = \int \frac{-1}{x} + \frac{1x+3}{x^2+1}\,dx = \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{3}{x^2+1}\,dx$$

$u = x^2+1$

$$\frac{3x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad 2$$

$$= -\ln|x| + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C.$$

$$3x-1 = A(x^2+1) + (Bx+C)x$$

$0: -1 = A + 0$

$1: 2 = 2A + B + C$

$= -2 + B + C$

$4 = B + C$

$2: 5 = 5A + 4B + 2C$

$10 = 4B + 2C$

$4 = B + C$

$10 = 4B + 2C$

$C = 4 - B$

$10 = 4B + 8 - 2B$

$2 = 2B$

$B = 1, \quad C = 3$

$$x^3+1 = (x+1)(x^2-x+1)$$

maybe?

1) do long division, then ABC

2) sometimes complete the square on denom.

$$\int \frac{2x^2+16x+13}{x(x^2+6x+13)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+6x+13}\,dx$$

$$\int \frac{4}{x^2+6x+13}\,dx$$

Complete sq