3 Applications of the Integral

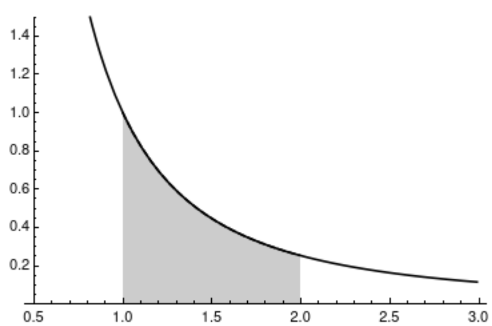
Now that we've learned some techniques for doing more complicated integrals, we want to use them to accomplish something. Why do we want to compute integrals, and what can they do for us?

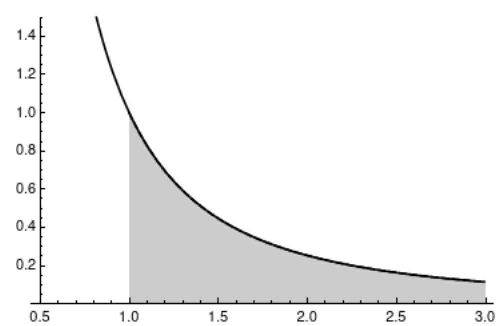
3.1 Improper Integrals and Unbounded Area

Recall that integrals were originally defined to compute areas. But so far we've only looked at the areas of regions that are, in some sense, "bounded": we may need calculus to find the exact area of the region, but we know the area is finite (and thus is a number)because we can draw a big circle around the whole shape. But sometimes we want to find the area of shapes that extend infinitely in one direction.

Example 3.1 (Motivating Example)**.** First off, we can ask: what is the area of the region bounded by the x-axis, the lines $x = 1$ and $x = 2$, and the curve $y = 1/x^2$? This is the sort of question we asked a lot in calculus 1. We can compute this area as

$$\int_1^2 \frac{dx}{x^2} = \frac{-1}{x}\Big|_1^2 = \frac{-1}{2} - \frac{-1}{1} = \frac{1}{2}.$$





But we can ask a trickier question. What is the area of the region bounded solely by the x-axis, the line $x = 1$, and the curve $y = 1/x^2$? Notice this region doesn't have any boundary at all on the right edge.

At first we don't know what to do, since our integrals are only defined on finite intervals. But we imagine the "remaining" area of the region must get smaller and smaller as x gets bigger and bigger. So what happens if we take the integral of a big chunk of the region?

If we look at the region bounded by $x = 1$ and $x = N$, for some number N, we get the area

$$A_N = \int_1^N \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^N = \frac{-1}{N} - \frac{-1}{1} = 1 - \frac{1}{N}.$$

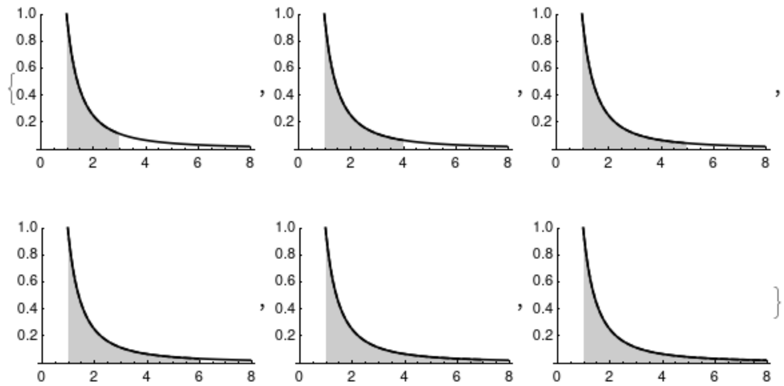


Figure 3.1: A sequence of finite, proper integrals covering increasing amounts of the area under the curve.

As N gets very large, we see that this area approaches 1, so we conclude the area of the whole region is 1.

There are two different ways for regions to be unbounded; it's entirely possible for both to happen at once, but we can always separate them and deal with them separately. We call such integrals *improper integrals*.

3.1.1 Improper integrals to ∞

The first situation is the situation in our motivating example, where we have to integrate over an "infinitely wide region."

Definition 3.2. If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then we define the *improper integral*

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

provided this limit exists.

If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists.

We say these integrals are *convergent* if the limit exists and *divergent* if the limit does not exist.

If both integrals are convergent, we write

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx.$$

In this case it doesn't matter which a we use: either both integrals will converge for every a, or one won't converge for any a. If one or both integrals don't converge, then $\int_{-\infty}^{+\infty} f(x)\,dx$ doesn't have a clear meaning and we should avoid writing it.

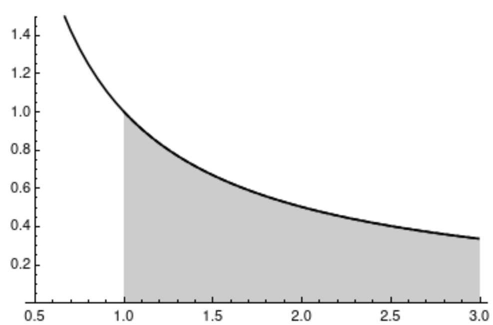
Remark 3.3. In this language, example 3.1 calculated that

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = 1.$$

Example 3.4. What is $\int_1^{+\infty} \frac{1}{x}\,dx$?

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x}\,dx &= \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} (\ln|x|)\big|_1^t \\ &= \lim_{t\to+\infty} (\ln|t| - \ln|1|) = \lim_{t\to+\infty} \ln|t| = +\infty.\end{aligned}$$

Thus this integral is divergent. Geometrically, this means that the area under this curve is in fact infinite.



Remark 3.5. It turns out that $\int_1^{+\infty} x^r\,dx$ is convergent whenever $r < -1$ and divergent whenever $r \geq -1$. This is worked out in your textbook, and will have important consequences in section 4.3.

Example 3.6. What is $\int_{-\infty}^{\pi} \sin(x)\,dx$?

We write this as a limit:

$$\begin{aligned}\int_{-\infty}^{\pi} \sin(x)\,dx &= \lim_{t\to-\infty} \int_t^{\pi} \sin(x)\,dx \\ &= \lim_{t\to-\infty} (-\cos(x))\big|_t^{\pi} \\ &= \lim_{t\to-\infty} (-\cos(\pi) - (-\cos(t))) = \lim_{t\to-\infty} \cos(t) + 1.\end{aligned}$$

This limit does not exist (because $\cos(x)$ is periodic), so the integral is divergent. Note that in this case the (net) area isn't infinite; it just isn't well-defined.

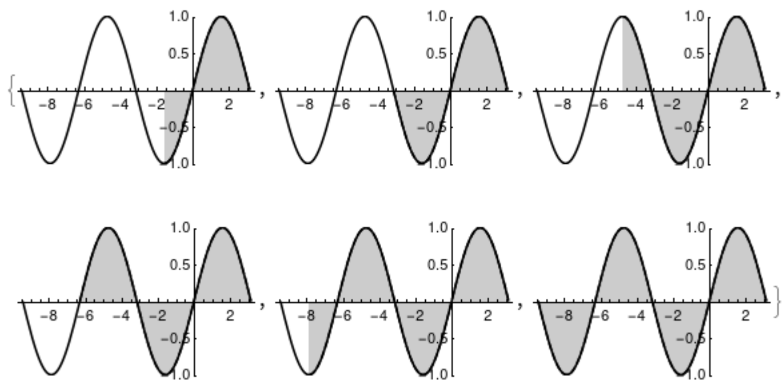
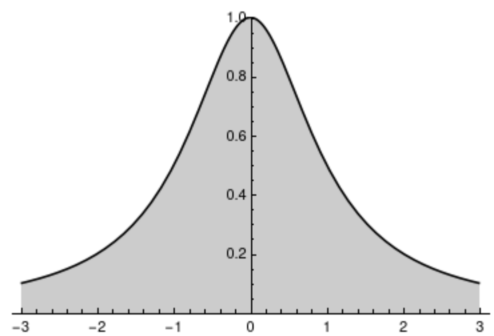


Figure 3.2: A sequence of finite integrals of $\cos(x)$ Notice that the net area is neither always increasing, nor always decreasing, nor trending to a specific number.

Example 3.7.

What is $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$?

First think about what you expect to happen. The graph of this function peaks in the middle at $(0,1)$, and trails off to zero as x gets large or small. So it's plausible that this integral is finite. It is certainly positive.



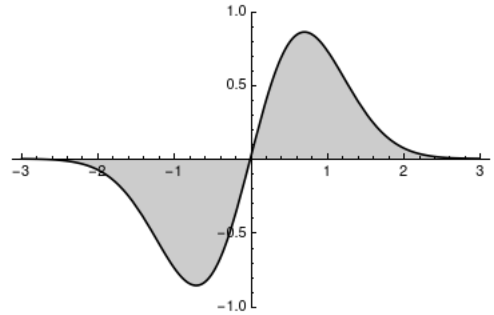
We can pick any a we want, and it's convenient to pick $a = 0$ so things are symmetrical.

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} + \lim_{s\to+\infty} \int_0^s \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \arctan(x)|_t^0 + \lim_{s\to+\infty} \arctan(x)|_0^s \\
&= \lim_{t\to-\infty} (\arctan(0) - \arctan(t)) + \lim_{s\to+\infty} (\arctan(s) - \arctan(0)) \\
&= -\lim_{t\to-\infty} \arctan(t) + \lim_{s\to+\infty} \arctan(s) = -(-\pi/2) + \pi/2 = \pi.
\end{aligned}
$$

Both partial integrals are convergent, so the total integral is convergent and the area under the curve is π.

Example 3.8. $\int_{-\infty}^{+\infty} 2xe^{-x^2}\,dx$.

We can look at a graph, and see that the areas on either side appear to balance each other out; we might expect the integral to be zero. But it's important to notice that this *only* works if each side has a finite area—the way we defined things, we can't have two infinite regions balancing each other out.



(It's possible to handle that case sensibly but you have to be a lot more careful definiting exactly what limit you're taking; in general it's a $\infty - \infty$ indeterminate form.)

We again split this into two integrals. We will also set $u = x^2, du = 2x\,dx$.

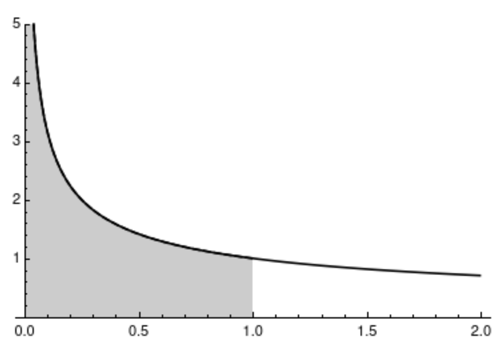
$$\begin{aligned}
\int_{-\infty}^{+\infty} 2xe^{-x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 2xe^{-x^2}\,dx + \lim_{s\to+\infty}\int_0^s 2xe^{-x^2}\,dx \\
&= \lim_{t\to-\infty}\int_{t^2}^0 e^{-u}\,du + \lim_{s\to+\infty}\int_0^{s^2} e^{-u}\,du \\
&= \lim_{t\to-\infty} -e^{-u}|_{t^2}^0 + \lim_{s\to+\infty} -e^{-u}|_0^{s^2} \\
&= \lim_{t\to-\infty} -e^0 - (-e^{-t^2}) + \lim_{s\to+\infty} -e^{-s^2} - (-e^0) = -1 - 0 + 0 + 1 = 0.
\end{aligned}$$

3.1.2 Improper integrals of discontinuous functions

There's a completely separate type of problem, where our x-values are bounded but our function behaves badly somewhere in that interval. Generally the issue comes up when our region is infinite in the *vertical* direction.

Example 3.9 (Motivating Example)**.** What is the area under $f(x) = 1/\sqrt{x}$ between $x = 0$ and $x = 1$?

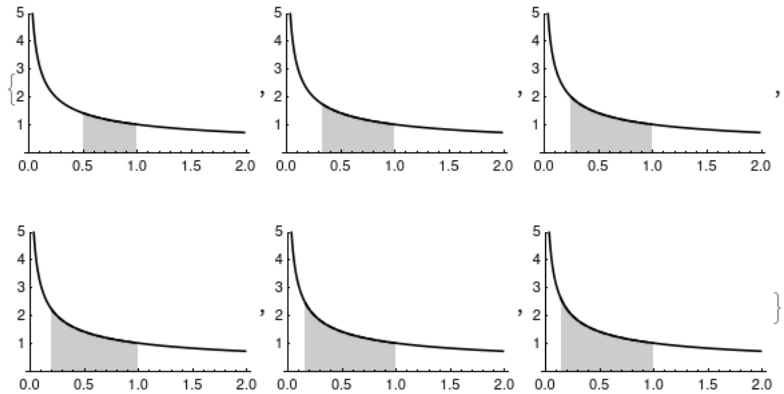
We can draw a clear picture of the region we want to study: But we can't use our normal



integral here because f isn't well-defined at 0. But we can find the area of almost all of the region, just as we did before. If ε is a small number, we have

$$\int_\varepsilon^1 x^{-1/2}\,dx = 2x^{1/2}|_\varepsilon^1 = 2(1-\sqrt{\varepsilon}).$$

It's easy to calculuate that $\lim_{\varepsilon\to 0} 2(1-\sqrt{\varepsilon}) = 2$, so we say the area of this region is 2.



Definition 3.10. If f is continuous on $[a,b)$ but discontinuous at b, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

provided the limit exists (and is finite).

If f is continuous on $(a,b]$ but discontinuous at a, we define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

provided the limit exists.

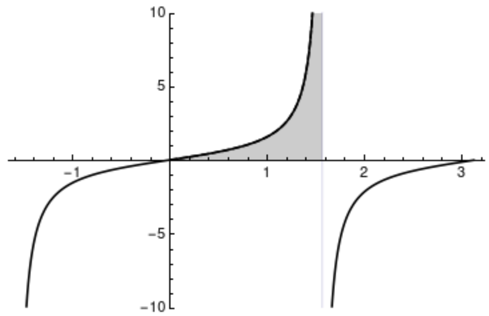
Again, the improper integral $\int_a^b f(x)\,dx$ is convergent if the limit exists, and divergent if it does not.

If f has a discontinuity at c for $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Remark 3.11. In this language, example 3.9 calculated that

$$\int_0^1 x^{-1/2}\,dx = 2.$$



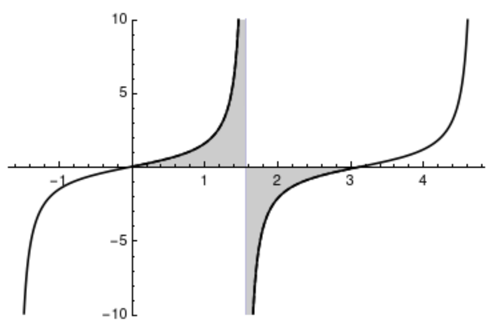
Example 3.12. What is $\int_0^{\pi/2} \tan x\, dx$?

$\tan(\pi/2)$ is not well defined. So we write

$$\begin{aligned}\int_0^{\pi/2} \tan x\, dx &= \lim_{t \to \pi/2^-} \int_0^t \tan x\, dx \\ &= \lim_{t \to \pi/2^-} \ln|\sec x||_0^t \\ &= \lim_{t \to \pi/2^-} \ln|\sec t| - \ln|1| = +\infty\end{aligned}$$

since $\lim_{t \to \pi/2^-} \sec t = +\infty$. So the integral is divergent.

Example 3.13 (Warning Example)**.** What is $\int_0^{\pi} \tan x\, dx$?



Again, it looks like the two infinities balance each other out, and there is a way to make a careful argument that justifies that impression. But we can't do it with the tools we have; this is again an $\infty - \infty$ indeterminate form.

If we're sloppy, we might reason as follows:

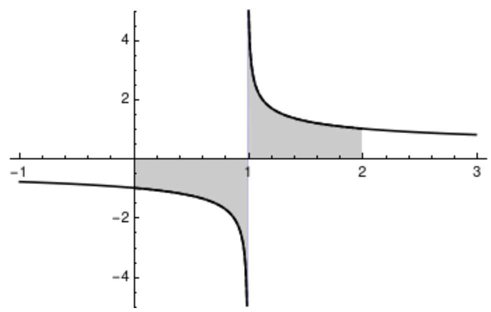
$$\int_0^{\pi} \tan x\, dx = \ln|\sec x||_0^{\pi} = \ln|-1| - \ln|1| = 0.$$

This is false because there is a discontinuity in the middle. We would need to split this integral into

$$\int_0^{\pi/2} \tan x\, dx + \int_{\pi/2}^{\pi} \tan x\, dx$$

and we already saw that the first integral is divergent (as is the second), so the whole integral is also divergent.

Example 3.14. What is $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$?



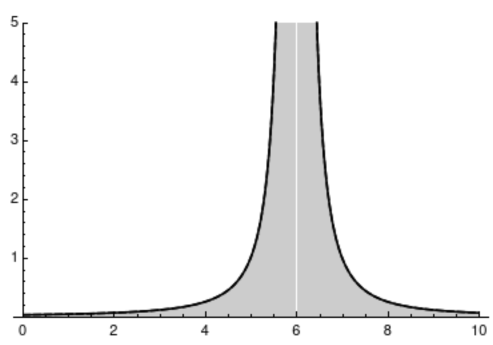
This is an improper integral since $\frac{1}{\sqrt[3]{x-1}}$ isn't defined at 1. Just like in example 3.13, it looks like the positive and negative components should balance each other.

To compute, we break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to1^+}\int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to1^-}\left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_0^t - \lim_{s\to1^+}\left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_s^2 \\
&= \lim_{t\to1^-}\left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\cdot 1\right) + \lim_{s\to1^+}\left(\frac{3}{2}\cdot 1 - \frac{3}{2}(s-1)^{2/3}\right) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

And finally, sometimes we can have both varieties of impropriety in the same question.

Example 3.15. $\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx$.



Looking carefully, we see this is improper in two ways. First, the upper bound of the integral is $+\infty$, as with the examples we saw in section 3.1.1. And second, the function has a vertical asymptote at $x = 6$. Thus we wind up needing to split this integral into *three* components:

$$\begin{aligned}\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx &= \int_0^6 \frac{dx}{(x-6)^2} + \int_6^7 \frac{dx}{(x-6)^2} + \int_7^{+\infty} \frac{dx}{(x-6)^2} \\ &= \lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} + \lim_{s\to 6^+} \int_s^7 \frac{dx}{(x-6)^2} + \lim_{t\to+\infty} \int_7^t \frac{dx}{(x-6)^2}.\end{aligned}$$

In order for our original integral to converge, we need all three of these to converge. But we see that

$$\begin{aligned}\lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} &= \lim_{r\to 6^-} \left(\frac{-1}{x-6}\right)\Bigg|_0^r \\ &= \lim_{r\to 6^-} \frac{1}{6-r} - \frac{1}{6} = +\infty\end{aligned}$$

diverges, so the whole integral diverges.

3.1.3 The Comparison Test for Improper Integrals

Sometimes we don't care much what the area of a region is; we only want to know if it's finite or not. (This will come up again in section 4.3, but also comes up in many other applications.) In those cases this theorem is enough:

Theorem 3.16. *Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then:*

- *If $\int_a^{+\infty} f(x)\,dx$ is convergent then $\int_a^{+\infty} g(x)\,dx$ is convergent.*
- *If $\int_a^{+\infty} g(x)\,dx$ is divergent then $\int_a^{+\infty} f(x)$ is divergent.*

This basically tells us that if the area under $f(x)$ is finite, then any area it contains must be finite; and if the area under $g(x)$ is infinite, any area containing it must be infinite.

Example 3.17. $\int_1^{+\infty} x^r\,dx$ is convergent if $r \leq -2$ since $x^r \leq x^{-2}$ on $[1, +\infty)$, and we know $\int_1^{+\infty} x^{-2}$ converges from example 3.1.

$\int_1^{+\infty} x^r$ is divergent if $r \geq -1$ since $x^r \geq x^{-1}$ on $[1, +\infty)$ and we know that $\int_1^{+\infty} x^{-1}$ diverges from example 3.4.

Example 3.18. Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

We will find this slightly easier if we split this up into two integrals.

It's clear that $\int_0^1 e^{-x^2}\,dx$ converges because it is a finite proper integral of a continuous function. So we just have to show that $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

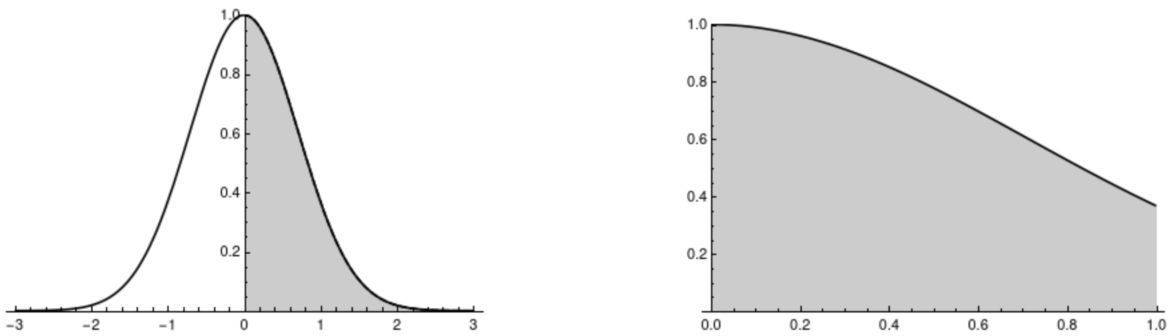


Figure 3.3: Left: the area computed by our improper integral $\int_0^{+\infty} e^{-x^2}$. Right: the proper integral $\int_0^1 e^{-x^2}$.

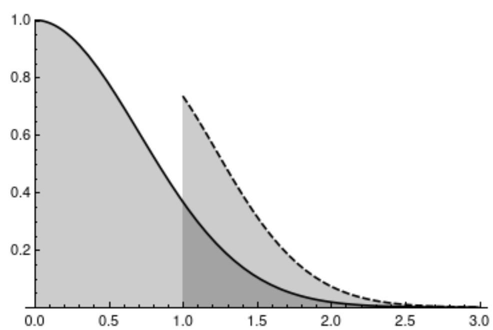


Figure 3.4: The function $2xe^{-x^2}$ is not always larger than e^{-x^2}. But when $x > 1$, the function $2xe^{-x^2}$ is in fact larger ,so we can use it with the comparison test.

But for every x in $[1, +\infty)$, we know that $e^{-x^2} < 2xe^{-x^2}$, and we've shown that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges in example 3.8. Thus by the comparison test, $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

It turns out that $\int_0^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$; we can prove this using some clever tricks from complex analysis. But we don't really have the tools to compute it in this course.

3.2 Geometric Applications

We can also use the integral to answer some other fun little geometry questions.

3.2.1 Arc Length

One question we can ask is: given a curve, how long is it? If the "curve" is a straight line, this is easy. We can just use the Pythagorean Theorem.

Example 3.19. A line with endpoints $(1, 2)$ and $(4, 6)$ has an x-coordinate distance of 3, and a y-coordinate distance of 4. So the total length of this line segment is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

If our curve is not a straight line, we have more trouble. But we can solve our problem by combining ideas from differential and integral calculus. Differential calculus tells us that, if we have a small window, we can approximate a function with a straight line—and we know how to compute the lengths of those. Integral calculus says that we can break a problem up into a bunch of pieces, solve each piece, and then bring them back together.

Suppose we have the graph of some function $f(x)$, as x runs from a to b. We can make a bad estimate of the lenth of this curve by just looking at the distance between the points $(a, f(a))$ and $(b, f(b))$. But if a and b are really close together, this estimate is actually pretty good; the function can't "curve away" from this line segment too much.

So what we can do is take the curve and split it up into a bunch of points that are close together, and try to approximate the length of each segment.

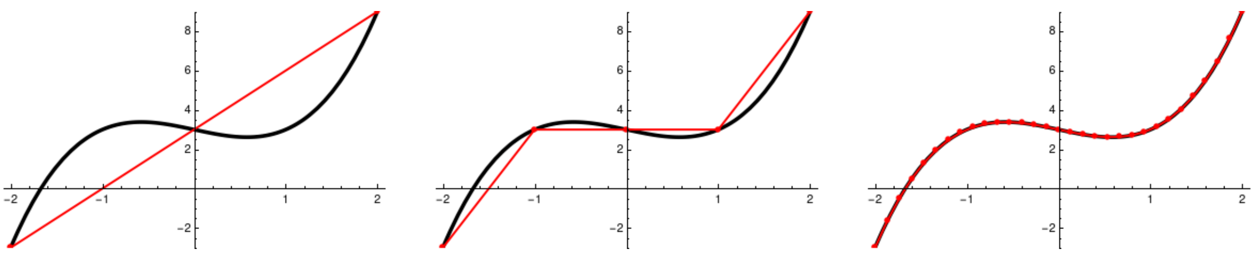


Figure 3.5: The curve $y = x^3 - x + 3$ approximated by one, four, and thirty segments

For finitely many segments, we could just work out the exact points individually. So we'd get something like $\sqrt{\Delta x^2 + (f(a + \Delta x) - f(a))^2}$ for the length of our first line segment, and a similar formula for each other line segment. In the middle picture above, we have $\Delta x = 1$ so our total formula would look like

$$\sqrt{1 + (f(-1) - f(-2))^2} + \sqrt{1 + (f(0) - f(-1))^2} + \sqrt{1 + (f(1) - f(0))^2} + \sqrt{1 + (f(2) - f(1))^2}.$$

This totally works to get an approximate value, and is the equivalent of the approach to integration we took in section 2.4.

But we want to take a limit as the number of pieces goes to infinity and Δx goes to 0. So we need some sort of nice formula. And this is where differential calculus comes to the rescue.

We know that near a point a, the function $f(x)$ is approximately equal to $f(a) + f'(a)(x - a)$. If we split the interval $[a, b]$ up into n sub-intervals, so that the ith interval goes from x_{i-1} to x_i, then the left-hand endpoint is $(x_{i-1}, f(x_{i-1}))$. The right-hand endpoint is *approximately*

$f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1})$, so the length of the line between them is approximately

$$\begin{aligned}\sqrt{(x_i - x_{i-1})^2 + \Big(\big(f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1})\big) - f(x_{i-1})\Big)^2}\\ = \sqrt{(x_i - x_{i-1})^2 + \Big(f'(x_{i-1})(x_i - x_{i-1})\Big)^2}\\ = \sqrt{(x_i - x_{i-1})^2(1 + f'(x_{i-1})^2)} = \Delta x\sqrt{1 + f'(x_{i-1})^2}.\end{aligned}$$

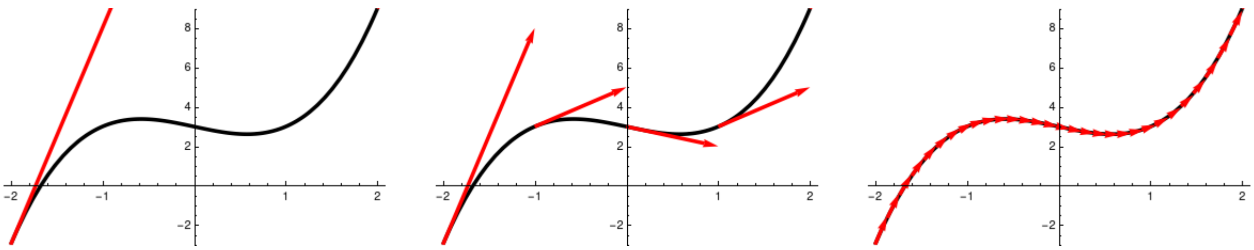


Figure 3.6: This derivative approximation is quite bad when $n = 1$, but as n gets larger we begin to get good approximations of the original curve.

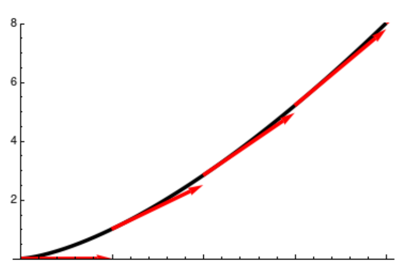
If we add up all these lengths, we get the formula

$$L \approx \sum_{i=1}^{n} \Delta x\sqrt{1 + f'(x_{i-1})^2}.$$

But this should look very familiar because this is just a Riemann sum—where the function is $\sqrt{1 + f'(x)^2}$. Thus we can take the limit as n goes to infinity and Δx goes to zero, and we get an exact formula for the length of the curve:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{1}$$

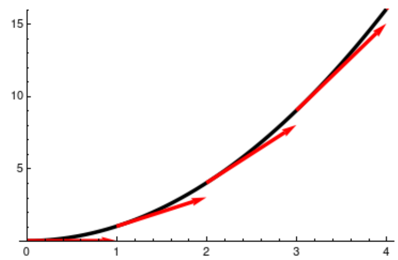
Example 3.20. Let's take the curve $y^2 = x^3$ and find the arc length of the curve between $(0, 0)$ and $(4, 8)$.



We have that $y = \sqrt{x^3}$ on this curve, so $y' = \frac{3}{2}x^{1/2}$. Then

$$\begin{aligned} L &= \int_0^4 \sqrt{1+(y')^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \frac{2}{3}\cdot\frac{4}{9}(1+9/4x)^{3/2}\Big|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right). \end{aligned}$$

Example 3.21. Let $f(x) = x^2$. Let's find the arc length between $x = 0$ and $x = 4$.



We have

$$L = \int_0^4 \sqrt{1+(2x)^2}\,dx$$

This looks a lot like a trig sub integral. We can set $2x = \tan\theta$, so $dx = \frac{1}{2}\sec^2\theta\,d\theta$. When $x = 0$ we have $\tan\theta = 0$ so $\theta = 0$, and when $x = 4$ we have $\tan\theta = 8$ so $\theta = \arctan(8)$. This gives us

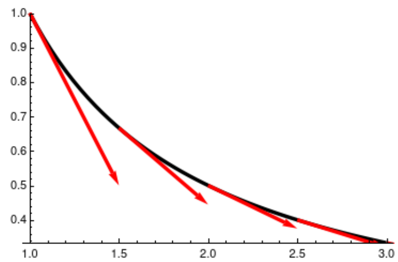
$$\begin{aligned} L &= \int_0^4 \frac{1}{2}\sqrt{1+(2x)^2}\,dx = \int_0^{\arctan 8} \frac{1}{2}\sqrt{1+\tan^2(\theta)}\sec^2\theta\,d\theta \\ &= \int_0^{\arctan 8} \frac{1}{2}\sec^3\theta\,d\theta \end{aligned}$$

and at this point I...give up on this integral. You can look it up, or you can plug it into a computer algebra package. We get

$$\frac{1}{4}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\Big|_0^{\arctan 8} \approx 16.819.$$

Often arc length calculations will give you pretty nasty integrals, and that's fine. The main goal here is to be able to set up these integrals—and, more importantly, to understand *why* this integral answers our question. On a quiz or a test, I will either pick questions that happen to give reasonable integrals, or tell you to just set up the integral and not actually compute it out.

Somtimes it's as easy—or easier—to integrate with respect to y.



Example 3.22 (Recitation)**.** Consider the graph of the hyperbola $xy = 1$ as y varies from 1 to 3. What is the arc length of this curve?

We could view this as a function of x: $y = 1/x$, so $y' = -1/x^2$, and then

$$L = \int_{1/3}^{1} \sqrt{1 + 1/x^4}\, dx \approx 2.14662.$$

Alternatively, we could view it as a function of y: $x = 1/y$, so $x' = -1/y^2$, and we have

$$L = \int_{1}^{3} \sqrt{1 + 1/y^4}\, dy \approx 2.14662.$$

Which approach is more convenient depends on what you're doing.

3.2.2 Surface Area

We can also kick this up a dimension. In Calc I, we computed the area under a curve, and then we computed the volume of a solid of revolution. Now we've computed the length of a curve; we can also compute the surface area of a surface produced by revolving a curve an axis. This is really just the area of the *outside* of the shapes we studied in Calculus I.

There are a couple of ways to think about this formula, but they both get you to essentially the same place. We can imagine cutitng the surface into little strips, and then pretending these strips are small cylinders. The radius of the cylinder is given by the height of the function; the height of the cylinder is given by the *arc length* of the bit of the function inside the band. (When the function is steeper, the average radius can be the same, but the width of our imaginary band is much greater, as we see in figure 3.7.)

Thus the area of one band will be the circumference of the circle, which is $2\pi f(x)$, times the width of the band, which from section 3.2.1 we know is approximately $\sqrt{1 + f'(x)^2}dx$. Thus the surface area of a surface of revolution is

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx. \qquad (2)$$

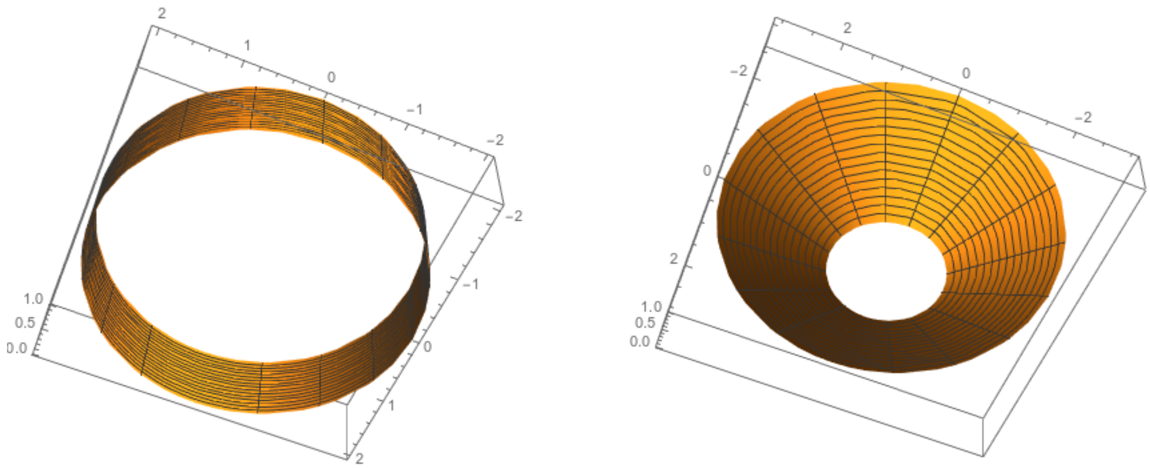


Figure 3.7: Both bands have the same x-axis thickness and the same average radius, but the right one has way more surface area

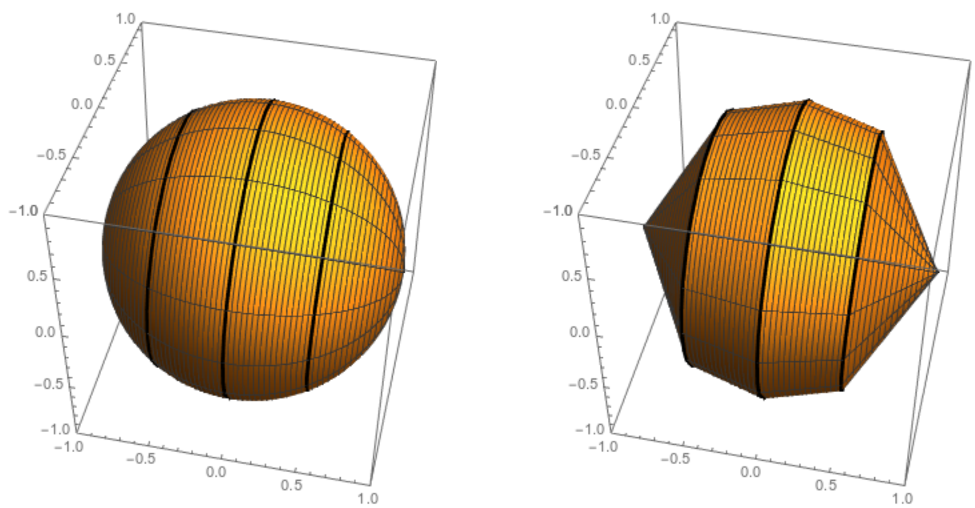


Figure 3.8: We can split the sphere into strips, and approximate each strip by pretending it's made of straight lines.

Example 3.23. What is the surface area of a sphere of radius 1? We can look at this as taking the curve $\sqrt{1-x^2}$ on $[-1,1]$ and revolving it around the x axis. Since $f'(x) = \frac{-x}{\sqrt{1-x^2}}$, we get

$$\begin{aligned} SA &= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} 1\,dx = 4\pi. \end{aligned}$$

But we could also compute the area of a part of the sphere, say the band in the middle. Then we'd have

$$\begin{aligned} SA &= \int_{-1/2}^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} 1\,dx = 2\pi. \end{aligned}$$

So the middle half of the sphere has exactly half the surface area of the whole sphere!

Example 3.24. Let $f(x) = \sqrt[3]{3x}$. Take the portion of the graph where $0 \le y \le 2$ and rotate it around the y axis. What is the surface area? (See figure 3.9.)

This one will be easiser, for multiple reasons, to view as a function of y. So we have $y = \sqrt[3]{3x}$ and thus $x = y^3/3$. Then $x' = y^2$, and we have

$$\begin{aligned} SA &= \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy \\ &= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy \\ &= \frac{2\pi}{3}\frac{2}{12}(1+y^4)^{3/2}\Big|_0^2 \\ &= \frac{\pi}{9}\left(17^{3/2}-1\right) \approx 24.118. \end{aligned}$$

And we can finish up with my favorite application/paradox, combining area, surface area, and improper integrals.

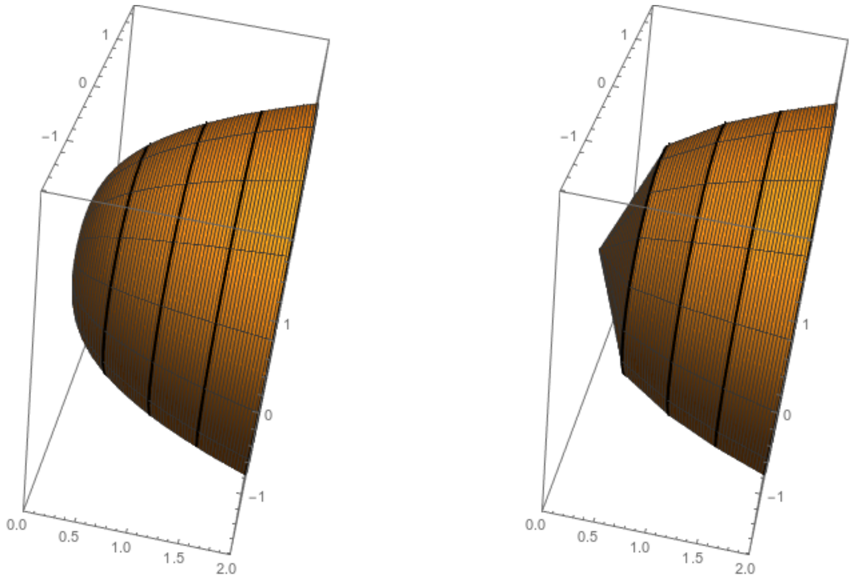


Figure 3.9: The graph of $y = \sqrt[3]{3x}$ rotated around the x-axis

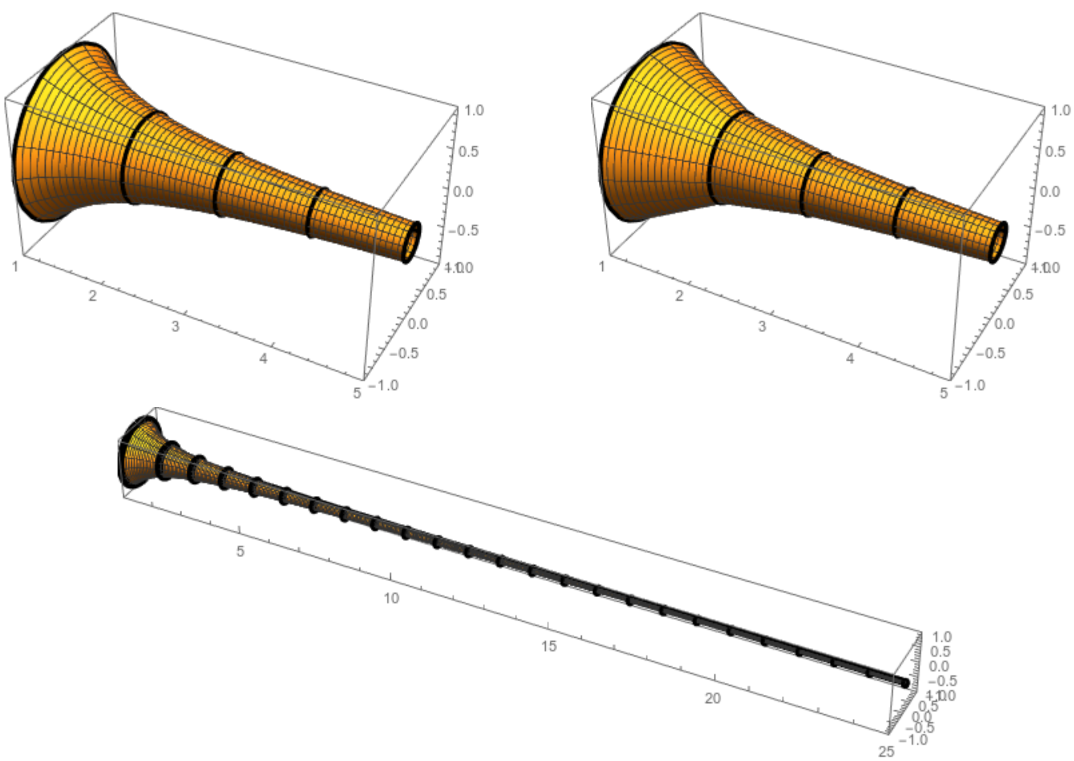


Figure 3.10: Gabriel's Trumpet

Example 3.25 (Gabriel's Trumpet/Infinite Paint Can)**.** Consider a trumpet-shaped container, given by taking the curve $y = 1/x$ and rotating around the x-axis, for $x \geq 1$. We're going to imagine this as a giant, oddly-shaped paint can. (See figure 3.10.)

We can work out the volume of this shape fairly easily, using cross-sections. If we take cross-sections perpendicular to the x-axis, each cross section is a circle of radius $1/x$. The area of this circle will be $\frac{\pi}{x^2}$ and thus the total volume will be

$$\begin{aligned}\int_1^\infty \frac{\pi}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\pi}{x^2}\,dx \\ &= \lim_{t\to\infty}\frac{-\pi}{x}\Big|_1^t = \frac{-\pi}{t} - \frac{-\pi}{1} = \pi.\end{aligned}$$

Thus the volume of our paint can is π; the can can hold π gallons of paint.

But now let's imagine painting the paint can. How much paint would we need to cover it? What's the surface area of the can?

We can do our surface area setup. We have $f(x) = 1/x$ so that $f'(x) = -1/x^2$. Then the surface area is

$$\begin{aligned}\int_1^\infty \frac{2\pi}{x}\sqrt{1+(-1/x^2)^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{2\pi}{x}\sqrt{1+1/x^4}\,dx \\ &\geq \lim_{t\to\infty} 2\pi\int_1^t \frac{1}{x}\,dx \\ &= 2\pi\lim_{x\to\infty}\ln|x|\big|_1^t \\ &= \lim_{t\to\infty} 2\pi(\ln|t|-1) = \infty.\end{aligned}$$

So the surface area of the paint can is infinite! You can fill the entire can with π gallons of paint, but it would take an infinite amount of paint to cover the interior of the paint can.

3.3 Differential Equations

There are two fundamentally different ways to think about what an integral is doing: as Riemann sums, or as antiderivatives. In calculus 1 and in the previous sections, our applications focused almost entirely on the Riemann sum aspect of things: we use integrals to split a question up into small pieces, approximating the answer on each piece, and then adding those approximations back together.

In this section we will look at the other side of the coin: how can we use *antiderivatives* to answer physical or practical questions? In this case, the questions will usually involve a derivative, so that the answer involves computing an antiderivative.

Definition 3.26. A *differential equation* is an equation that relates the derivatives of a function to the function itself.

Example 3.27.

- $y' = 2x$ is a differential equation; we can see that $y = x^2$ is a solution.

- $y' - 2y = 4\cos(t) - 8\sin(t)$ has a solution $y = 3e^{2t} + 4\sin(t)$ because

$$y' - 2y = 6e^{2t} + 4\cos(t) - \left(6e^{2t} + 8\sin(t)\right) = 4\cos(t) - 8\sin(t).$$

- $x^2y'' + xy' + x^2y = 0$ is *Bessel's differential equation of order 0* and has a solution called J_0 or *Bessel's Function of order 0*, as seen in figure 3.11. It is important in many applications, in particular studying heat flow.

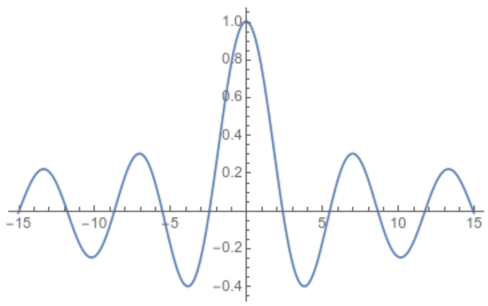


Figure 3.11: The graph of the Bessel function of order 0 J_0

Example 3.28. Confirm that $f(x) = x^2 + x + 1$ satisfies $2f(x) - xf'(x) = x + 2$.

We compute $f'(x) = 2x + 1$, so $2f(x) - xf'(x) = 2x^2 + 2x + 2 - (2x^2 + x) = x + 2$.

In a real way, a differential equations are the combination of high school algebra with precalculus and calculus. In grade school, we learned to do simple arithmetic, like being asked to compute $3 + 5$ and calculating 8. As we got to algebra, we were asked instead to *solve equations*. We would get formulas like $3 + x = 8$ and try to figure out what x is. This is the same sort of question but backwards—instead of computing with known numbers, we have to figure out which numbers will make the calculation work.

In pre-calculus and in calculus, we have done calculuations with functions. Plug a number into this function; graph this function; take the derivative of this function. But here we are being asked to solve equations whose answers are *functions*. The question is, which function satisfies the given relationship? And if we have a candidate answer, we can test it by plugging it into the differential equation and seeing if the computation we get is correct.

3.3.1 Proportional Growth

The simplest possible (non-trivial) differential equation is probably $p'(t) = kp(t)$. This tells us that the rate of change of something we're measuring is proportional to the current level of that thing.

This often comes up in the context of population growth. If we look at, say, a breeding population of rabbits, then the number of new rabbits born each year depends on the number of rabbits that are already alive: if we start with two rabbits, we won't end the year with two million. If each rabbit on average produces three new rabbits in a year, we might approximate the derivative by saying $\frac{dp}{dt} = 3p(t)$. That is, the change in the total population of rabbits is equal to three times the current number of rabbits.

In this case, if we start a year with 100 rabbits, then we have $p'(0) = 3p(0) = 300$ so we expect to get three hundred new rabbits, and end the year with 400. The next year we will get $p'(1) = 3p(1) = 1200$, so we get 1200 new rabbits and end the year with 1600 rabbits. The derivative is different each year, but the proportional growth rate is not.

Can we find a function that satisfies $p'(t) = 3p(t)$? And so far in this course, the answer is "not really". The trivial solution will still work, actually: if we start with zero rabbits, then we will always have zero rabbits, and it is true that $0' = 3 \cdot 0$. But if we want a non-trivial solution, none of the functions we've seen so far will work here. We will see that the solutions to this differential equation look like $p(t) = Ca^t$ for some constants C and a; a depends on the breeding rate, and C is the initial population of rabbits.

But this equation describes more than just rabbit population growth. Other cases where this equation appears include:

- Interest: if you are paying 8% interest per year, then your debt increases at a rate $d'(t) = .08d(t)$. This is the question Jakob Bernoulli was studying when he discovered the number e.
- Economic growth: the economy grows by 3% a year, so we have $p'(t) = .03p(t)$.
- Radioactive decay: some fraction of your sample of uranium will decay every year, so you have $u'(t) = ku(t)$. In this case k will be negative since your amount of uranium is decreasing.
- Heat transfer: the rate at which heat flows from a hot object to a cold object is proportional to the difference in temperature, so we have $T'(t) = kT(t)$.

3.3.2 Another Perspective on Compound Interest

Suppose you invest \$100 in a bank account paying 3% interest a year. Then after t years you will have $100 \cdot (1.03)^t$ dollars in the bank account. It's easy to compute how much money you'll have after t years. For instance, after three years you will have \$109 and after 20 years you will have \$180.

Often interest is "compounded" more often, meaning that you get some fraction of it every few months. Interest that is compounded quarterly—four times a year—pays you .75% of your current balance four times a year, so after t years you will have $100 \cdot (1.0075)^{4t}$ dollars. After three years you will still have \$109, and after 20 years you will have \$182. Note that your money has increased—slightly.

We can compound more often; in general, if your interest rate is r and you compound n times a year, then your total money after t years will be

$$M = M_0(1 + \frac{r}{n})^{nt},$$

where M_0 is the amount of money you started with.

In the real economy, transactions are constantly happening and the economy is (usually) constantly growing. Jacob Bernoulli asked what would happen if your interest *compounded continuously*—that is, what happens in the limit, as n goes to $+\infty$.

$$M(t) = \lim_{n \to +\infty} M_0 \left(1 + \frac{r}{n}\right)^{nt} = M_0 \left(\lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^{n/r}\right)^{rt} = M_0 e^{rt}.$$

And this was the context in which Bernoulli found the definition for e we gave in section 1.2.1.

Remark 3.29. This setup justifies a famous rule of thumb in finance. If we want to double our money, we're solving the equation

$$\begin{aligned} 2M_0 &= M_0 e^{rt} \\ \ln(2) &= rt \\ t &= \frac{r}{\ln(2)} \approx \frac{r}{.7}. \end{aligned}$$

This gives us the useful rule of thumb that if your interest rate is r, it will take about $70/r$ years to double your investment.

3.3.3 Force and Hooke's Law

For another common example, consider the phrase "acceleration is proportional to force." Recall that acceleration is the second derivative of position. If force is itself a function of position, this translates to a differential equation, relating $f''(x)$ to $f(x)$.

Hooke's law tells us that the force a spring exerts is proportional to the displacement of the spring; that is, for any given spring there is some constant k such that $F(t) = -kx(t)$, where $x(t)$ is the function that takes in the time and outputs the x coordinate of the weight on the spring. Since $F(t) = ma(t) = mx''(t)$, this gives us the differential equation $mx''(t) = -kx(t)$ or

$$x''(t) = -\frac{k}{m}x(t).$$

For simplicity let's assume $k = m$ so we have $x''(t) = -x(t)$.

Can we find a solution for this? We can start with the really silly or "trivial" solution. If the spring starts at a neutral position, it will never move, so we'd expect $x(t)$ to be constantly 0. And indeed this solution works: $0'' = 0 = -0$, so the funcion $x(t) = 0$ is a solution to this differential equation.

Can we find a solution that involves any motion at all? We're looking for a function where $x''(t) = -x(t)$. And we actually know two of these: $x(t) = \sin(t)$ and $x(t) = \cos(t)$ both satisfy this differential equation. And this is why the equation for "simple harmonic motion" is built up out of sin and cos functions.

There are many different solutions we can use; for example, $3\sin(t) + 5\cos(t) = 17$ is a solution to this differential equation. It's easy to see that if a and b are any constants, then $x(t) = a\sin(t) + b\cos(t)$ is a solution to this differential equation. It's much less obvious, but true, that any solution to the Hooke's Law equation must have this form; even the trivial solution is given by $x(t) = 0\sin(t) + 0\cos(t)$. Thus we say the *general form of the solution* is

$$x(t) = a\sin(t) + b\cos(t).$$

To pick out the specific solution we need to know some "initial conditions", which tell us what state the weight starts in. But if we know the starting position and starting velocity of the weight, we can determine a and b and thus get an exact formula for $x(t)$.

Example 3.30. Suppose we have a weight on a spring satisfying the differential equation $x''(t) = -x(t)$. Further, suppose we know that $x(0) = 3$ and $x'(0) = -1$: that is, at time 0 the weight is three units to the right of neutral, and is moving to the left with speed 1.

We know that $x(t) = a\sin(t) + b\cos(t)$ for some numbers a and b, and thus we have the equations

$$3 = x(0) = a\sin(0) + b\cos(0) = b$$
$$-1 = x'(0) = a\cos(0) - b\sin(0) = a$$

so our equation is in fact $x(t) = -\sin(t) + 3\cos(t)$, as shown in figure 3.16

Remark 3.31. Can you find a solution to $x''(t) = -4x(t)$?

Can you find a solution to $x''(t) = 4x(t)$?

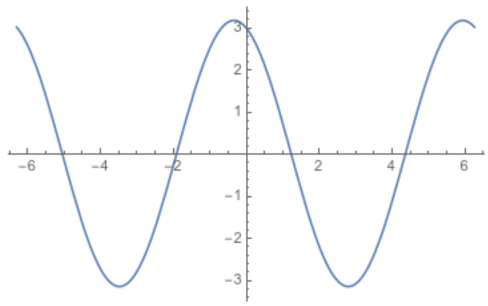


Figure 3.12: A graph of the solution to our differential equation. Notice how when the input is zero, the output is equal to three, but decreasing.

3.3.4 Evans price change model

Economists often use systems of differential equations to describe how the economy changes over time.

If there is a shortage of some good, which means that more people want to buy than sell, the price will tend to increase so that fewer people want to buy, more people want to sell, and the market clears. But the price doesn't change immediately. The Evans model says that the price change is proportional to the size of the shortage: $\frac{dp}{dt} = k(D - S)$, where D is the quantity demanded and S is the quantity supplied. So if the shortage is bigger, the price will increase faster.

So far, this looks sort of like exponential growth. But it's importantly different, because the size of the shortage is not the same thing as the price! We need to ask how demand depends on price. A simple model says that $D(p) = a - bp$ and $S(p) = r + sp$, where a is the amount demanded when the price is zero and r is the (probably negative) amount supplied when the price is zero. Then $-b = \frac{dD}{dp}$ and $s = \frac{dS}{dp}$ are the elasticities of demand and supply.

Plugging this back into the original model gives

$$p'(t) = k\Big(a - bp(t) - r - sp(t)\Big) = k\Big((a - r) - (b + s)p(t)\Big).$$

From this we can see that the trivial solution where the price is zero doesn't actually work here. And that makes sense, because if the price is zero you expect more people to want to buy than to sell. We also notice that it doesn't matter what the demand or supply elasticities are individually; it only matters what their sum is. We can use this equation to estimate the way the price will change over time.

Example 3.32. Let's assume $k = 1$, and take $a = 80, r = -20, b = s = 5$. That is, we have

$$D(p) = 80 - 5p$$
$$S(p) = -20 + 5p$$
$$p'(t) = D(p) - S(p) \qquad = 100 - 10p(t)$$

Then we can check that a solution is $p(t) = 10 + 5e^{-10t}$, in which case the price will evolve following the path in figure 3.13.

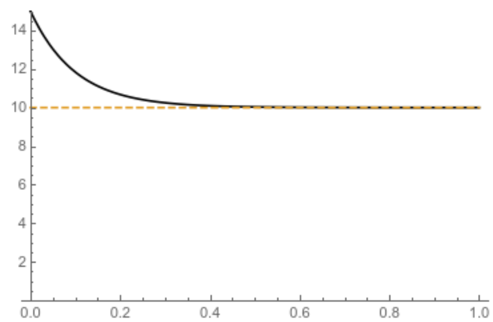


Figure 3.13: A graph of the function $p(t) = 10 + 5e^{-10t}$ solving the Evans Price Equation in example 3.32.

There is a rich and powerful theory for solving differential equations. We won't really be studying it in depth in this course; we would need a bunch of tools we don't have yet, including Taylor series (from section 5) and linear algebra (Math 2184). But we can learn about a few basic ideas.

3.3.5 Initial Value Problems

The harder part of solving differential equations is finding the general form of a solution. A given differential equation will usually have infinitely many solutions, as we saw in section 3.3.3 with the solutions $a\sin(t) + b\cos(t)$ to the equation $x''(t) = -x(t)$. This tells us the general shape of the solution, but doesn't give us an actual solution.

As we discussed, the specific solution depends on where things start. On the Hooke's Law fall system, if your fall starts at neutral then it will never move; if it starts extremely displaced then it will oscillate wildly. So to know the position over time we need to know where the system starts, known as the *initial conditions*. Finding a specific solution, given some specific conditions, is called an *initial value problem* or *boundary value problem*.

Example 3.33. Suppose we have a Hooke's Law system with $m = k$, so that we get the differential equation $x''(t) = -x(t)$. We said earlier that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

Suppose now we start with the weight stationary and displaced by 1 meter. Since this is the starting conditions, this is at time 0, so this means that $x(0) = 1$ and $x'(0) = 0$. Now we have enough information to figure out a and b and find a specific solution to describe the path of our fall.

Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$, as graphed in

And as we think about it, this answer makes some sense: there's no reason for the fall to ever displace further than one meter, and so that's exactly what we see here.

Sometimes instead of an initial value problem we have a boundary value problem. In a boundary value problem you get position values at different times, rather than position and velocity at the same time.

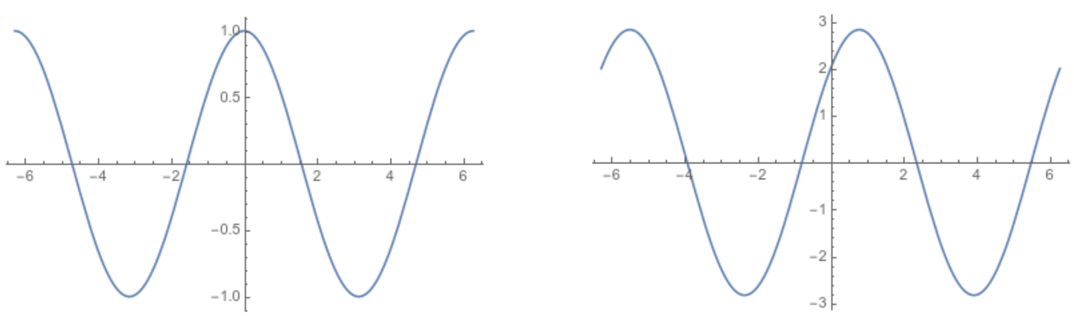


Figure 3.14: Left: the solution to example 3.33. Right: the solution to example 3.34

Example 3.34. Suppose we have a Hooke's Law setup with $m = k$, so $x(t) = a\sin(t) +$

$b\cos(t)$. Suppose we know that $x(0) = 2$ and $x(\pi/4) = \sqrt{8}$. Then we know that

$$\begin{aligned} a\sin(0) + b\cos(0) &= 2 \\ b &= 2 \\ a\sin(\pi/4) + b\cos(\pi/4) &= \sqrt{8} \\ a\sqrt{2}/2 + 2\cdot\sqrt{2}/2 &= \sqrt{8} \\ a/2 + 1 &= 2 \\ a &= 2. \end{aligned}$$

Thus we have that $x(t) = 2\sin(t) + 2\cos(t)$.

Notice that in either of these cases, we need to take only two measurements to know exactly what happens at every possible time. This is because our differential equation, coming from a physical law, severely constrains what our answers can possibly look like; we only need a bit more information to have it nailed down precisely, one measurement for each constant.

Of course, in the real world, measurements come with errors so we need to take more than two. But we can get a lot of information from our differential equation telling us what sort of relationships to look for.

Example 3.35. Suppose $f(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, and we have $f(0) = 0, f'(0) = 1, f''(0) = 2$. What can we say about $f(x)$?

We see that $f(0) = c = 0$, $f'(x) = 2ax + b$ so $f'(0) = b = 1$, and $f''(x) = 2a$ so $f''(0) = 2a = 2$. Thus $a = b = 1$ and $c = 0$, so $f(x) = x^2 + x$.

Example 3.36. Suppose $g(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, with $g(1) = 2, g'(2) = 3, g''(3) = 4$. What can we say about g?

We have $g(1) = a + b + c$. $g'(x) = 2ax + b$ so $g'(2) = 4a + b = 3$, and $g''(x) = 2a$ so $g''(3) = 2a = 4$. Thus we have $a = 2$. Going back to g' we see that $8 + b = 3$ so $b = -5$. Then plugging into g we have $2 - 5 + c = 2$ so $c = 5$. Thus $g(x) = x^2 - 5x + 5$.

3.4 Separable differential equations

Even ordinary differential equations are hard to solve; in general solving them involves using power series (see section 5) and Fourier series (which we won't really discuss in this course). But there is one fairly easy type of equation that we do have the tools to solve.

Definition 3.37. A *separable differential equation* is a differential equation that can be written

$$\frac{dy}{dx} = g(x)f(y)$$

for some functions g and f.

We call these separable because we can separate the variables, by putting all the ys on one side and all the xs on the other. Heuristically, we divide by $f(y)$ and "multiply by dx": this gives us

$$\frac{dy}{f(y)} = g(x)\,dx$$

and we can now integrate both sides. We can justify this via the chain rule: if

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

then

$$\begin{aligned} \frac{d}{dx}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right) \\ \frac{dy}{dx}\cdot\frac{d}{dy}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right) \\ \frac{dy}{dx}\cdot\frac{1}{f(y)} &= g(x). \end{aligned}$$

Alternatively, we can justify it with u-substitution. If $\frac{dy}{dx} = g(x)f(y)$, then

$$\begin{aligned} \frac{1}{f(y)}\frac{dy}{dx} &= g(x) \\ \int \frac{1}{f(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \int \frac{1}{f(u)}\,du &= \int g(x)\,dx. \end{aligned}$$

Example 3.38. Solve $y' = x/y$ for the initial value $y(0) = 2$.

We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \\ \int y\,dy &= \int x\,dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y &= \pm\sqrt{x^2 + 2C} \end{aligned}$$

This gives us our general solution; now we just need to find the specific solution.

our initial condition is $y(0) = 2$, so we have $\pm\sqrt{0 + 2C} = 2$ and thus $C = 2$. This gives $y = \pm\sqrt{x^2 + 4}$. Since our square root must be positive, we get a specific solution $y = \sqrt{x^2 + 4}$.

(Note that if our initial condition were negative, say $y(0) = -2$, then we'd have a negative square root instead, and $y = -\sqrt{x^2 + 4}$.)

Remark 3.39. If we want, we could have replaced the $2C$ with a C without losing anything. Alternatively we could set $K = 2C$ to get the same effect but make sure we don't confuse ourselves.

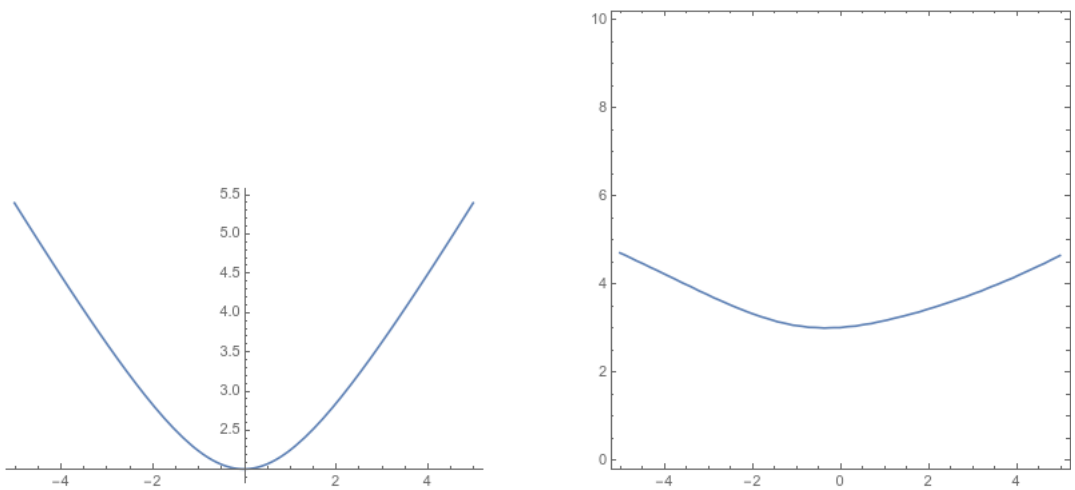


Figure 3.15: Left: the solution to example 3.38. Right: the solution to example 3.40

Example 3.40. Solve $y' = \frac{3x + \cos x}{2y + y^2}$ for the initial condition $y(0) = 3$.

We have

$$\frac{dy}{dx} = \frac{3x + \cos x}{2y + y^2}$$

$$\int 2y + y^2\, dy = \int 3x + \cos x\, dx$$

$$y^2 + \frac{y^3}{3} = \frac{3x^2}{2} + \sin x + C.$$

This gives y as a function of x implicitly, but there's no immediately obvious way to write it explicitly as a function of x. We can, however, work out the constant; we have $3^2 + 3^3/3 = 0 + 0 + C$ and thus $C = 18$. So our solution is

$$y^2 + \frac{y^3}{3} = \frac{3x^2}{2} + \sin x + 18.$$

Example 3.41. Find the general solution to $(y^2 + xy^2)y' = 1$.

We have

$$\begin{aligned}\int y^2 dy &= \int \frac{dx}{1+x} \\ \frac{y^3}{3} &= \ln|1+x| + C \\ y &= \sqrt[3]{3\ln|1+x| + 3C}.\end{aligned}$$

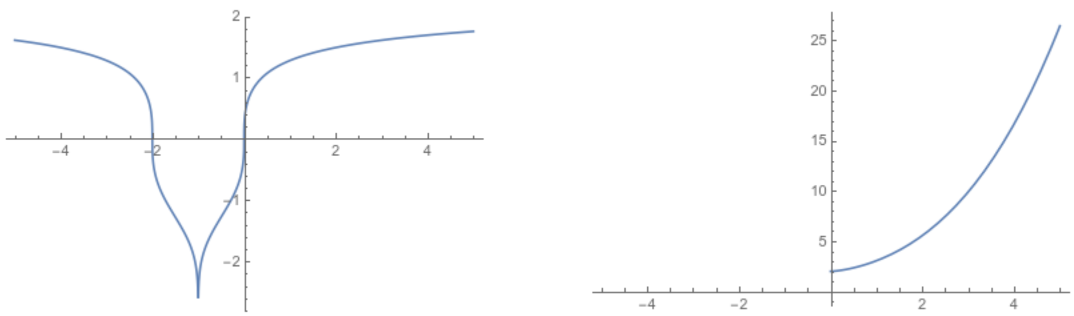


Figure 3.16: Left: solution to example 3.41 for $C = 0$. Right: the solution to example 3.42.

Example 3.42. Find the specific solution to $P' = \sqrt{Pt}$ with $P(0) = 2$.

We have

$$\begin{aligned}\int \frac{dP}{\sqrt{P}} &= \int \sqrt{t}\, dt \\ 2\sqrt{P} &= \frac{2}{3}t^{3/2} + C \\ P &= \left(t^{3/2}/3 + C/2\right)^2\end{aligned}$$

and since $P(1) = 2$ we have $2 = (0 + C/2)^2$ and thus $C = 2\sqrt{2}$. So the specific solution is

$$P = \left(\frac{t^{3/2}}{3} + \sqrt{2}\right)^2.$$

3.5 Some common separable differential equations

3.5.1 Mixing Problems

In section 3.3.1 we looked at proportional or exponential growth. A slightly more complicated variant on exponential growth occurs when there is constant growth and exponential decay at the same time (or vice versa).

Example 3.43. Suppose we have a tank containing 10 kg of salt dissolved in 1000 L of water. We can pump in a brine solution of .02 kg of salt per liter of water at 10L per minute, while ten L of solution drains out of the tank each minute. How much salt is in the tank after twenty minutes?

Let y be the amount of salt in the tank. Then we have $y(0) = 10$, and we have $y' = .2 - \frac{y}{100}$, since each minute the tank is gaining .1 kg of salt and losing a hundredth of its salt content.

We can easily rewrite this as

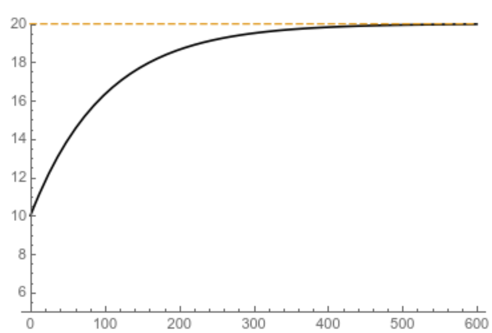
$$\begin{aligned} \frac{dy}{dt} &= \frac{20-y}{100} \\ \int \frac{dy}{20-y} &= \int \frac{dt}{100} \\ \ln|20-y| \cdot (-1) &= \frac{t}{100} + C \end{aligned}$$

Since $y(0) = 10$ so we have $-\ln(10) = C$, so we get

$$\begin{aligned} -\ln|20-y| &= \frac{t}{100} - \ln(10) \\ |20-y| &= 10e^{-t/100} \\ 20-y &= 10e^{-t/100} \\ y &= 20 - 10e^{-t/100}. \end{aligned}$$

Thus after 20 minutes we have

$$y = 20 - 10e^{-1/5} \approx 11.81kg.$$



You might recognize this formula: this is mathematically the same as the Evans price change model we saw in section 3.3.4. However, the "physics" giving rise to the equation here is somewhat different.

3.5.2 Logistic Growth

We saw that in a simple model of population growth, the population will grow exponentially. But this model assumes that there are no real resource constraints; the population will keep growing larger and larger as time goes on. In reality, they will eventually run out of space or food or some other resource.

A simple but important model for this is the model of *logistic growth*. Let M be the *carrying capacity*, i.e. the maximum population. Then when our population is small, we want growth roughly proportional to the size of our population, as before. But as the population gets closer to M the rate of growth gets closer to 0; a simple equation that captures this is the *logistic differential equation* first developed by Pierre-François Verhulst in the 1840s:

$$\frac{dy}{dt} = ky(M-y).$$

We can see that if M is much bigger than y, this is approximately $\frac{dy}{dt} = kMy$ and thus is exponential growth. But if y is very close to M, we have $\frac{dy}{dt} \approx ky \cdot 0$.

The equation is separable, so we can write

$$\begin{aligned} \int k\,dt &= \int \frac{dy}{y(M-y)} \\ &= \int \frac{1}{M}\left(\frac{1}{y} - \frac{1}{M-y}\right) dy \\ &= \frac{1}{M}\left(\int \frac{dy}{y} - \frac{dy}{M-y}\right) \\ kt + C &= \frac{1}{M}\left(\ln|y| - \ln|M-y|\right). \end{aligned}$$

Since $0 < y < M$ both y and $M - y$ are positive, so this gives

$$M(kt + C) = \ln\left(\frac{y}{M-y}\right)$$

and thus

$$\frac{y}{M-y} = Ae^{Mkt}.$$

Given an initial condition $y(0) = y_0$ we have

$$\frac{y}{M-y} = \frac{y_0}{M-y_0} e^{Mkt}$$

and solving for y to write y as a function of t gives us

$$\begin{aligned}
y &= (M-y)\frac{y_0}{M-y_0}e^{Mkt} \\
&= \frac{My_0}{M-y_0}e^{Mkt} - \frac{yy_0}{M-y_0}e^{Mkt} \\
y\left(1+\frac{y_0}{M-y_0}e^{Mkt}\right) &= \frac{My_0}{M-y_0}e^{Mkt} \\
y &= \frac{\frac{My_0}{M-y_0}e^{Mkt}}{1+\frac{y_0}{M-y_0}e^{Mkt}} \\
&= \frac{My_0e^{Mkt}}{M-y_0+y_0e^{Mkt}} \\
&= \frac{My_0}{(M-y_0)e^{-Mkt}+y_0}.
\end{aligned}$$

We can see that, as expected, as $t \to +\infty$ we have $y \to M$.

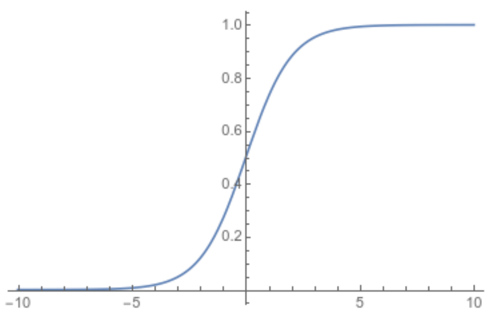


Figure 3.17: The characteristic S-curve of logistic growth. Here we take $M = 1, k = 1, y_0 = .5$.

Example 3.44 (Global Population)**.** The total population of the world was 3 billion people in 1960, and 4 billion in about 1975. We can try to model this with a simple exponential growth curve: Setting $t = 0$ to be 1960 and fitting this to our model, we have: $Ce^{k0} = 3$ and $Ce^{15k} = 4$. Thus we must have $C = 3$, and then $e^{15k} = 4/3$ implies that $15k = \ln(4/3)$ and so $k = \ln(4/3)/15$. If we want to estimate global population in 2020, this gives us

$$P(60) = 3 \cdot e^{60 \cdot \ln(4/3)/15} = 3 \cdot e^{4\ln(4/3)} = 3 \cdot (4/3)^4 \approx 9.48.$$

(Actual estimates put it at 7.8 billion, because population growth has been leveling off).

Now let's use our model to estimate when the population will reach 12 billion. We want

$$\begin{aligned}12 &= 3 \cdot e^{t \ln(4/3)/15} \\ 4 &= \left(\frac{4}{3}\right)^{t/15} \\ \log_{4/3} 4 &= t/15 \\ 15\frac{\ln(4)}{\ln(4/3)} &= t \\ 72.3 &\approx t\end{aligned}$$

So our model predicts that the world's population will reach 12 billion in about 2032.

Now let's use a more sophisticated logistic growth model. Suppose

$$\frac{dP}{dt} = kP(M - P)$$

and thus

$$P = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}.$$

We know that $P_0 = P(0) = 3$ if 0 corresponds to 1960. We also have $P(15) = 4$ and $P(60) = 7.8$. We need to solve for M and for k. Using the fact that $P(15) = 4$ we get have

$$\begin{aligned}4 &= \frac{M \cdot 3}{(M-3)e^{-Mk \cdot 15} + 3} \\ 3M &= (4M - 12)e^{-15Mk} + 12 \\ e^{-15Mk} &= \frac{3M - 12}{4M - 12} \\ -15Mk &= \ln\left|\frac{3M - 12}{4M - 12}\right| \\ k &= \frac{-1}{15M} \ln\left|\frac{3M - 12}{4M - 12}\right|.\end{aligned}$$

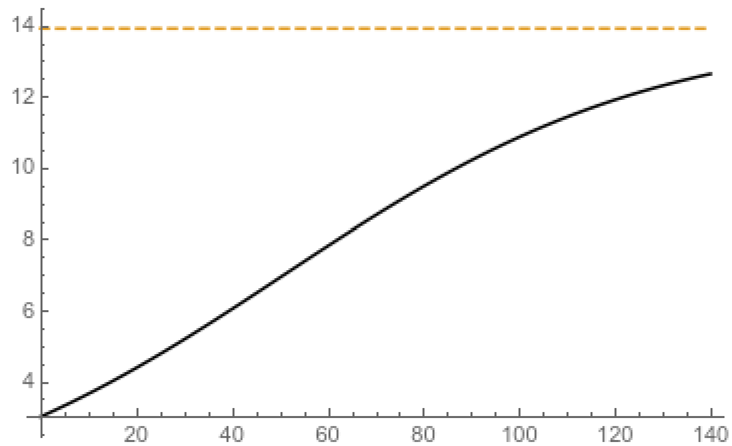
We now need to use the fact that $P(60) = 7.8$ to find actual values for k and M. Solving by hand would be terribly annoying, but I asked Mathematica and it told me that if $P(60) = 7.8$ then $M \approx 13.9$ and $k \approx .0018$ Plugging this in gives us

$$P(t) = \frac{39.6}{10.2e^{-.026t} + 3}$$

which tells us $P(0) = 3, P(15) = 4, P(54) = 7.2$, and $P(60) = 7.7$ in line with current projections. Using this equation we can ask again when population will reach 12 billion; we see this happens when $t = 122$, or in about 2082.

The UN currently projects a population of about 11.2 billion in 2100, which is a bit less than we would predict. Our logistic model is much better than the exponential model, but not quite as good as you can do with professional demographers.

Figures 3.18 and 3.19 compre graphs of our model with the UN data, provided by Our World in Data. If you want to learn more about this topic, check out the Future Population Growth page at `https://ourworldindata.org/future-population-growth`.



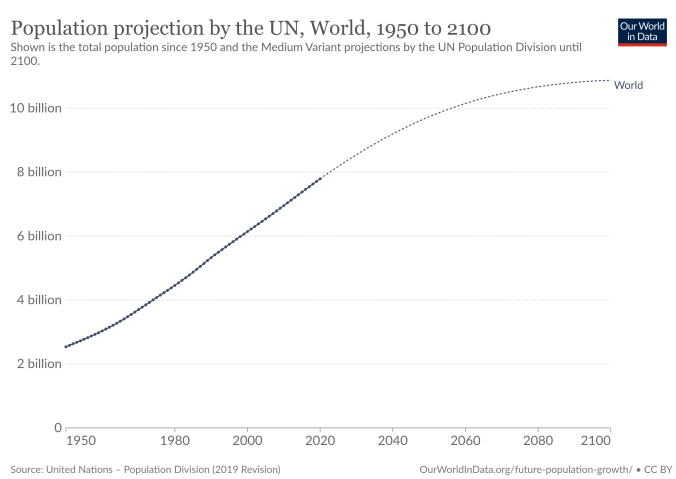


Figure 3.18: Top: our population projection through 2100. Bottom: the UN's projection.

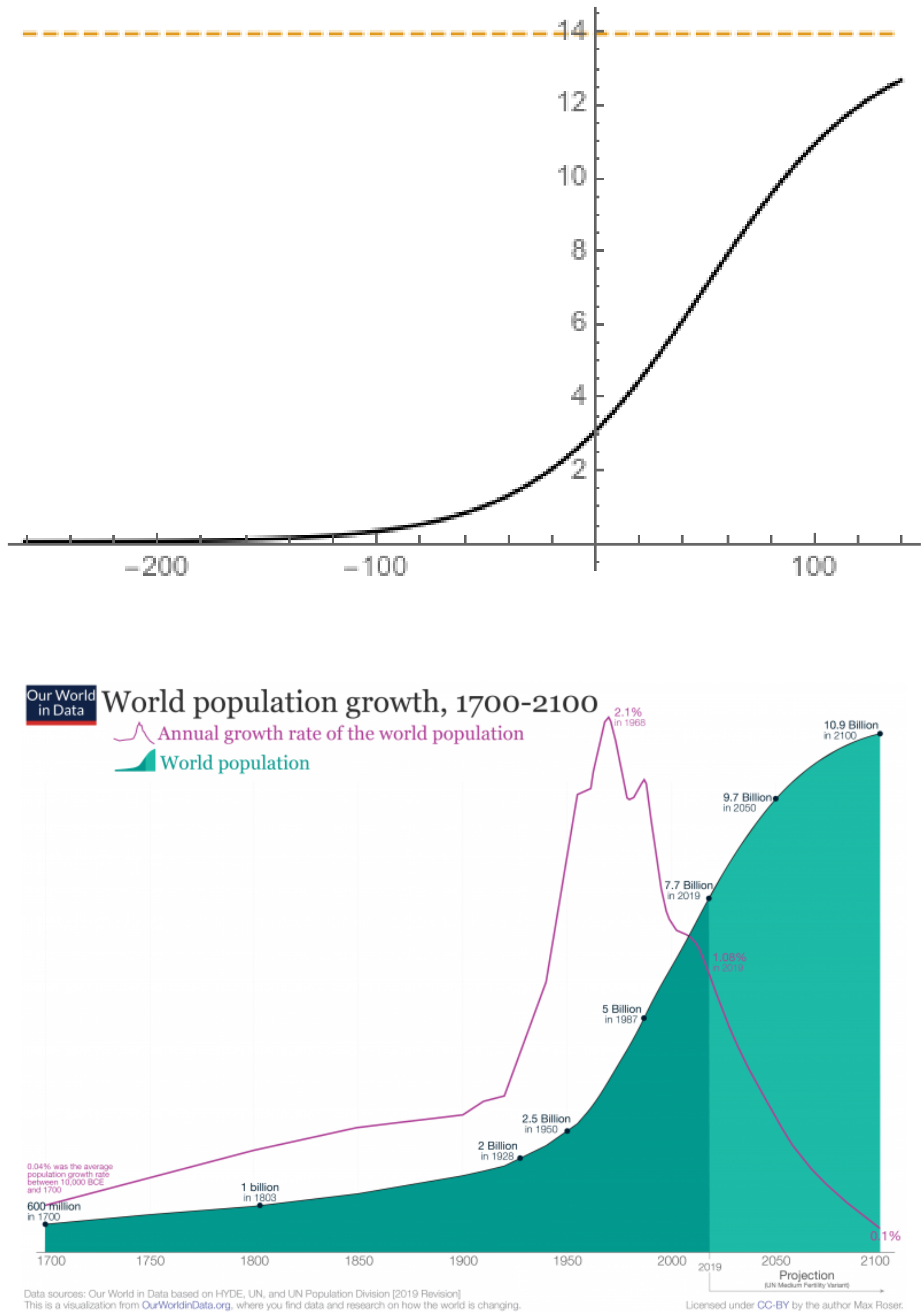


Figure 3.19: Top: our estimate of the population from 1700 through 2100. Bottom: Our World in Data's estimate (with graph of the estimated derivative).