5 Power Series and Taylor Series

In this section we want to use what we've done with series in order to accomplish things. In particular, we can use series to define functions. And this allows us to work with a lot of functions that we've talked about in the past, but didn't have ways to compute. Functions like $\ln(x)$ and $\arctan(x)$ are easy to *say* but hard to compute, and the same applies for things like $\int e^{-x^2}\, dx$.

With series, we can write down formulas for these things that allow us to get good approximate answers to questions we care about. And as a bonus, we can replace a lot of the calculus we've done before with much simpler operations.

5.1 Power Series

We want to start by figuring out how to build a function out of a series. There are a few ways to do this, but the one we'll be studying is called a *power series*.

(Another important tool is the *Fourier series*, which is important to any sort of digital music or video. Unfortunately they're a little more complicated and we won't have time to talk about them. But if you're interested you can check out section 5.7, or I recommend this video by 3Blue1Brown.)

Definition 5.1. A *power series* is a series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

More generally, a *power series centered at a* is a series

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Because this series contains x, we can plug in a number for x and get a "regular" series, which either converges to a number or diverges. Note we adopt the convention $x^0 = 1$, and $(x-a)^0 = 1$, so that if we plug in $x = a$ our sum is just c_0.

We can think of a power series as an "infinitely long polynomial"; c_0 is the constant term, then c_1 is the coefficient of x^1, and in general c_n is the coefficient of x^n.

We've seen one important power series in disguise before.

Example 5.2. The series

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

is a geometric series with $a = 1$ and $r = x$. We know from our study of geometric series that this converges if $|x| < 1$ and diverges if $|x| \geq 1$, and that when it converges, the sum is $\frac{1}{1-x}$.

A first important step in understanding a power series is figuring out when it converges. That is, if we plug in a value for x, does it converge to a real number, or does it diverge? This is equivalent to asking for the domain of the function, an idea we'll return to in section 5.2.

Example 5.3. For what x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Because of the nth powers, we use the ratio test. (Note that power series always have nth powers and so you basically always use the ratio test). Since $a_n = \frac{x^n}{n!}$, we have

$$L = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim \left| \frac{x^{n+1} n!}{(n+1)! x^n} \right| = \lim \left| \frac{x}{n} \right| = 0.$$

Since $L = 0 < 1$ for any value of x, this series converges absolutely for any real number x.

Example 5.4. For what x does $\sum_{n=0}^{\infty} n(x-2)^n$ converge?

Again, we use the ratio test, as we almost always do for power series. We have $a_n = n(x-2)^n$ and $a_{n+1} = (n+1)(x-2)^{n+1}$, so

$$L = \lim_{n \to +\infty} \left| \frac{(n+1)(x-2)^{n+1}}{n(x-2)^n} \right| = \lim_{n \to +\infty} \left| \frac{n+1}{n}(x-2) \right| = |x-2|.$$

The ratio test says a series converges if $L < 1$, so this power series converges when $|x-2| < 1$. For real numbers this is when $1 < x < 3$.

When $L > 1$ the series diverges, so if $x < 1$ or $x > 3$ this series diverges. But the ratio test doesn't tell us what happens when $L = 1$, so we have to look at those cases separately.

When $x = 3$ then our series is

$$\sum_{n=0}^{\infty} n \cdot 1^n = \sum_{n=0}^{\infty} n$$

which clearly diverges by the divergence test. Similarly, if $x = 1$ our series is

$$\sum_{n=0}^{\infty} n \cdot (-1)^n$$

which is an alternating series, but again diverges by the divergence test.

Some power series will converge for all x, like that last example. And some power series will diverge for any $x \neq 0$. But most of them look like this, and converge sometimes. But that sometimes will always follow the same specific pattern.

Theorem 5.5. *If $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series, then exactly one of the following things occurs:*

- *The series converges only when $x = a$;*
- *The series converges for any real number x;*
- *There is a positive number R, called the* radius of convergence*, such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Note this tells us nothing about what happens when $|x-a| = R$; we have to check those cases individually.*

Remark 5.6. This is another explanation for the language of "absolute" and "conditional" convergence. A power series will converge everywhere on the interior of its interval of convergence. It diverges everywhere outside the interval. On the boundary of the interval it may or may not converge, depending on the specific boundary point; thus, on the boundary it converges "conditionally."

(This can all generalize to complex numbers in a really important and interesting way, but we're not going to engage with that much in this course. But in the complex case, you can replace the word "interval" with "disk" in this remark.)

Definition 5.7. The *open interval of radius r centered at c*, is

$$(c-r, c+r) = \{x : |x-c| < r\}$$

the set of all points of distance *less than* r from the center c.

The *closed interval* of radius r centered at c is

$$[c-r, c+r] = \{x : |x-c| \leq r\}$$

the set of all points of distance at most r from the center c.

Note the closed interval contains its boundary points and the open interval does not. This is important!

Example 5.8. For what real x does $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n}$ converge? What is the radius of convergence?

Guess what? We use the ratio test! We have

$$L = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(x-4)^{n+1}/(n+1)}{(x-4)^n/n}\right| = \lim \left|\frac{(x-4)n}{n+1}\right| = \lim |x-4|$$

so the power series converges for $|x-4| < 1$, and thus the radius of convergence is 1. The power series converges absolutely on $(3, 5)$.

To find the real numbers where the series converges, we have to check 3 and 5 manually. For $x = 3$ we get the series $\sum \frac{(-1)^n}{n}$ which converges by the Alternating Series Test; and for $x = 5$ we get the series $\sum \frac{1}{n}$ which diverges. Thus the series converges on $[3, 5)$ in the real numbers.

Example 5.9 (recitation)**.** The Bessel function (of order 0) is critical to any physics done in cylindrical coordinates, and thus any physics that occurs on a cylinder. We saw it in section 3.3 as the solution to the differential equation $x^2y'' + xy' + x^2y = 0$, but it can also be given by the power series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

What is the radius and interval of convergence?

We use the ratio test. We have $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, so

$$\begin{aligned} \lim \left| \frac{a_{n+1}}{a_n} \right| &= \lim \left| \frac{x^{2n+2}/2^{2n+2}((n+1)!)^2}{x^{2n}/2^{2n}(n!)^2} \right| \\ &= \lim \left| \frac{x^{2n+2}}{x^{2n}} \frac{2^{2n}}{2^{2n+2}} \frac{(n!)^2}{((n+1)!)^2} \right| \\ &= \lim \frac{|x|^2}{4(n+1)^2} = 0. \end{aligned}$$

Thus the Bessel function of order 0 converges absolutely for all real numbers x. We say the radius of convergence is ∞ and the interval is all reals, or $(-\infty, +\infty)$.

Example 5.10. What is the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n^2+n}}?$$

Ratio test.

$$\lim \left| \frac{(-2)^{n+1}x^{n+1}/\sqrt{(n+1)^2+n+1}}{(-2)^n x^n/\sqrt{n^2+n}} \right| = \lim 2|x| \frac{\sqrt{n^2+3n+2}}{\sqrt{n^2+n}} = 2|x|.$$

Thus by the ratio test the power series converges absolutely when $2|x| < 1$, or in other words when $|x| < 1/2$. The radius of convergence is $1/2$ and it converges on the open interval $(-1/2, 1/2)$. Now we need to test endpoints.

When $x = 1/2$ then the series is

$$\sum \frac{(-1)^n}{\sqrt{n^2+n}}.$$

This is an alternating series; the terms are decreasing and tend towards zero, so it converges.

When $x = -1/2$ then the series is $\sum \frac{1}{\sqrt{n^2+n}}$. We use the limit comparison test, and see that

$$\lim \frac{1/n}{1/\sqrt{n^2+n}} = \lim \sqrt{1+1/n} = 1$$

and thus $\sum \frac{1}{\sqrt{n^2+n}}$ has the same behavior as $\sum \frac{1}{n}$, and thus diverges.

So the real interval of convergence is $(-1/2, 1/2]$.

Example 5.11 (recitation)**.** What is the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x-1)^n}{7^{n+2}}?$$

Using the ratio test, we have

$$\lim \left| \frac{(n+1)^2(x-1)^{n+1}/7^{n+3}}{n^2(x-1)^n/7^{n+2}} \right| = \lim \frac{|x-1|}{7} \frac{(n+1)^2}{n^2} = \frac{|x-1|}{7}.$$

So the series converges absolutely when $|x-1| < 7$, and thus on the interval $(-6, 8)$. For the full interval we need to test the endpoints, at $x = -6$ and $x = 8$.

When $x = -6$ we have

$$\sum \frac{n^2(-7)^n}{7^{n+2}} = \sum (-1)^n \frac{n^2}{49}.$$

This is an alternating series, but the terms tend towards infinity and so by the divergence test it diverges.

Similarly, when $x = 8$ we have

$$\sum \frac{n^2 7^n}{7^{n+2}} = \sum \frac{n^2}{49}.$$

The terms tend towards infinity, so the series diverges by the divergence test.

Thus the real interval of convergence is $(-6, 8)$.

5.2 Power Series as Functions

Now that we understand how power series converge, we can see how to use them as as functions. In general, if we have a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, then we can define a function by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

The domain of the function will be the interval of convergence of the power series. If the power series converges everywhere then the domain is all real numbers.

We already know how to express at least one function as a power series: by our geometric series argument, we know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. We can use this fact to figure out how to express some other functions as power series as well.

Example 5.12. We can get new functions through composition, by plugging new formulas into the formula we already have. Thus

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n$$
$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n = \sum(-1)^n x^{5n}.$$

Further, we can see that both of these must have the same interval of convergence as the original power series. The series for $\frac{1}{1-x}$ converges when $-1 < x < 1$, and thus the series for $\frac{1}{1-x^2}$ will converge when $-1 < x^2 < 1$, and this is precisely when $-1 < x < 1$. Similarly, $-x^5$ is in the interval $(-1, 1)$ exactly when x is.

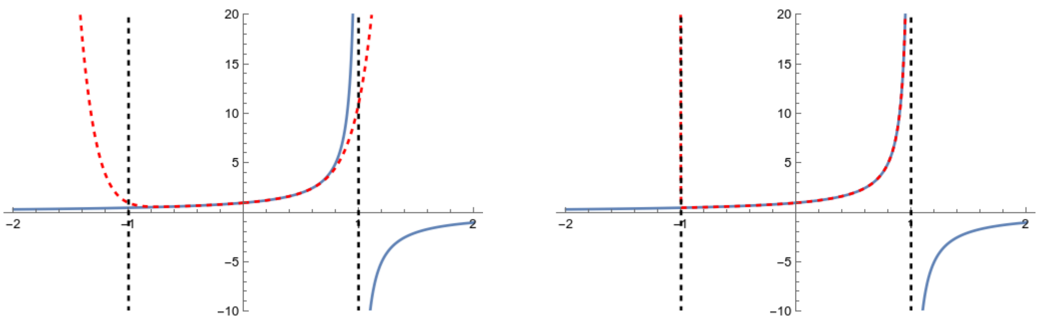

Figure 5.1: Left: the truncated power series $\sum_{n=0}^{10} x^n$. Right: The truncated power series $\sum_{n=0}^{1000} x^n$, which might as well be un-truncated

Remark 5.13. You may notice that $\frac{1}{1-x}$ has a domain bigger than $(-1, 1)$. But the *power series* only converges on that smaller interval. There's something funky going on here because power series convergence has this symmetry requirement; if it converges at 2, it must also converge at -2. Since we *have* to have a divergence at $x = 1$, we must also get divergence for $x > 1$ and $x < -1$, as we see in figure 5.1.

Sometimes we have multiple options.

Example 5.14. How can we express $\frac{1}{x-3}$ as a power series? Since we want to write the denominator as $1 - y$ for some expression y, we factor out a -3:

$$\frac{1}{x-3} = \frac{1}{-3} \cdot \frac{1}{1-x/3} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}x^n.$$

We know this will converge when $\left|\frac{x}{3}\right| < 1$, and thus when $|x| < 3$. So the interval of convergence is $(-3, 3)$.

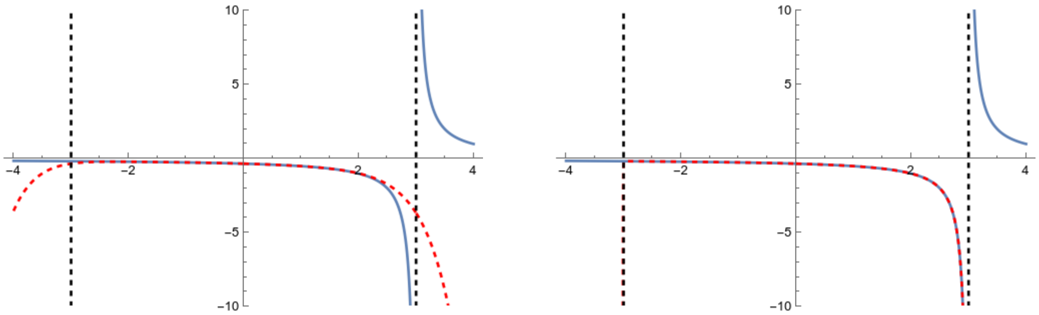

Figure 5.2: Power series for $\frac{1}{x-3}$ centered at 0.

Alternatively, we could write $\frac{1}{x-3} = \frac{1}{1-(4-x)}$. Then we have

$$\frac{1}{x-3} = \sum_{n=0}^{\infty}(4-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-4)^n.$$

This is a power series with center 4, and it converges when $|4-x| < 1$ so it has radius of convergence 1.

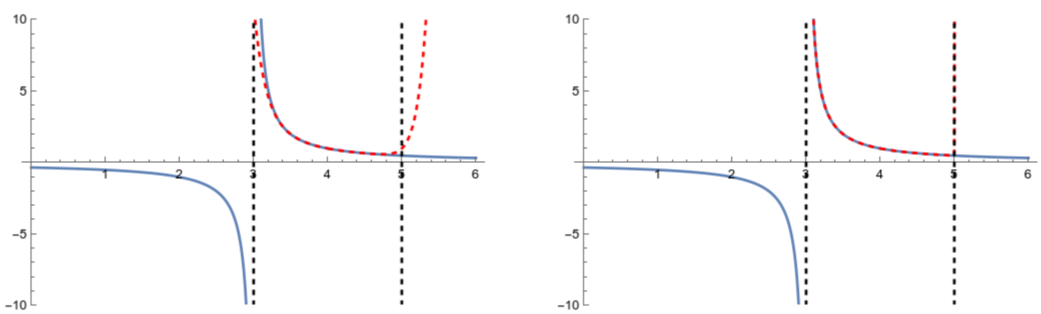

Figure 5.3: Power series for $\frac{1}{x-3}$ centered at 4. Notice how this power series has a totally distinct interval of convergence: it centers on on the other side of the asymptote, but still stops at that asymptote.

These are two different power series for the same function, but they're completely different. Each one has the largest radius of convergence it can without crossing the bad point at $x = 3$, but since they start in different places, on opposite sides of the asymptote, they have completely distinct intervals of convergence.

We can also do most basic algebra with power series.

Example 5.15. How can we express $\frac{x}{1-x}$ as a power series? This is just $x \cdot \frac{1}{1-x}$ and so

$$\frac{x}{1-x} = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}.$$

The interval of convergence is again $(-1, 1)$.

5.2.1 Calculus of power series

Recall that possibly the easiest functions for us to work with when we do calculus (or, indeed, almost anything else) are polynomials. It's easy to differentiate or integrate polynomials, and to calculate their outputs. The nice thing about power series is that they're basically fake polynomials, so they're almost as good.

Taking a derivative or integral of a single term of a power series is something we already know how to do, since each term is a polynomial (or, technically, a monomial). So for instance, we have

$$\frac{d}{dx}c_n(x-a)^n = c_n \cdot n(x-a)^{n-1}$$
$$\int c_n(x-a)^n\,dx = c_n\frac{(x-a)^{n+1}}{n+1} + C.$$

That means that we can do calculus on polynomials easily, just by working on each term. And it turns out the same thing works for power series.

Proposition 5.16. *If* $\sum c_n(x-a)^n$ *has a radius of convergence* $R > 0$*, then the function defined by* $f(x) = \sum c_n(x-a)^n$ *is differentiable on* $(a-R, a+R)$*, and we have*

- $f'(x) = \sum_{n=0}^\infty c_n \frac{d}{dx}((x-a)^n) = \sum_{n=0} nc_n(x-a)^{n-1}$.
- $\int f(x)\,dx = \sum_{n=0}^\infty \left(\int c_n(x-a)^n\,dx\right) = \sum_{n=0}^\infty c_n\frac{(x-a)^{n+1}}{n+1} + C$.

Remark 5.17. This proposition tells us that after taking the derivative or integral, our power series still has the same radius of convergence. However, convergence at the *endpoints* may change.

Now that we have this extra tool we can find power series for more functions.

Example 5.18. Since $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$, we can differentiate both sides. The derivative of the left hand side is

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}\cdot(-1) = \frac{1}{(1-x)^2}$$

and so

$$\frac{1}{(1-x)^2} = \sum_{n=1}^\infty nx^{n-1}.$$

Note that we've dropped the $n = 0$ term because the derivative of x^0 is 0, and writing $0 \cdot x^{-1}$ would be silly.

Also note that we have

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

so this is still a proper power series. For instance, we can use the ratio test to double-check that the radius of convergence is still $R = 1$.

Example 5.19. A subtler and more clever question: can we find a power series expression for $\ln(1+x)$?

We know that $\ln(1+x) = \int \frac{1}{1+x}\,dx$. We also know that $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$. Integrating gives us that

$$\begin{aligned}\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} \cdot (-1) + C \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + C.\end{aligned}$$

To find the constant C we compute $\ln(1+0)$, since plugging 0 in on the right hand side will just yield C. (If this reminds you of what we did in section 3.3.5 to solve differential equations, good!) We know that $\ln(1) = 0$, so $C = 0$, and we have

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

The radius of convergence is still 1; we can see this using the ratio test or by recalling that our original power series has radius of convergence 1.

As with the geometric series, the radius of convergence can't possibly be larger than 1, since the function $\ln(1+x)$ has an asymptote at $x = -1$, as we see in figure 5.4.

In passing, this justifies my repeated claims that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$, by plugging $x = 1$ into this formula.

Example 5.20. Find a power series for $\arctan x$.

Again, we note that $\arctan x = \int \frac{1}{1+x^2}\,dx$, and we know that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n$. Integrating gives

$$\begin{aligned}\arctan x &= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx + C \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C.\end{aligned}$$

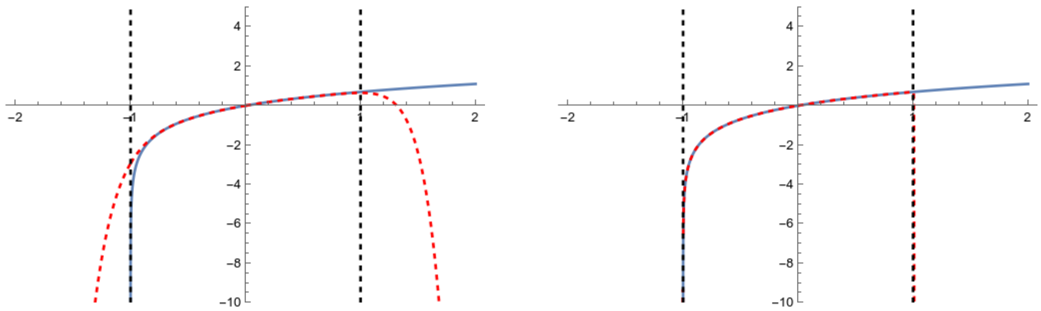

Figure 5.4: Left: the truncated power series $\sum_{n=1}^{10}(-1)^{n-1}\frac{x^n}{n}$. Right: The truncated power series $\sum_{n=1}^{1000}(-1)^{n-1}\frac{x^n}{n}$, which might as well be un-truncated

To find C we calculate $C = \arctan(0) = 0$, so we have

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$$

Again the radius of convergence is 1, as we see in figure 5.5. However, this time nothing about the function $\arctan(x)$ obviously forces this radius of convergence on us; it's purely an artifact of the way we set this up.

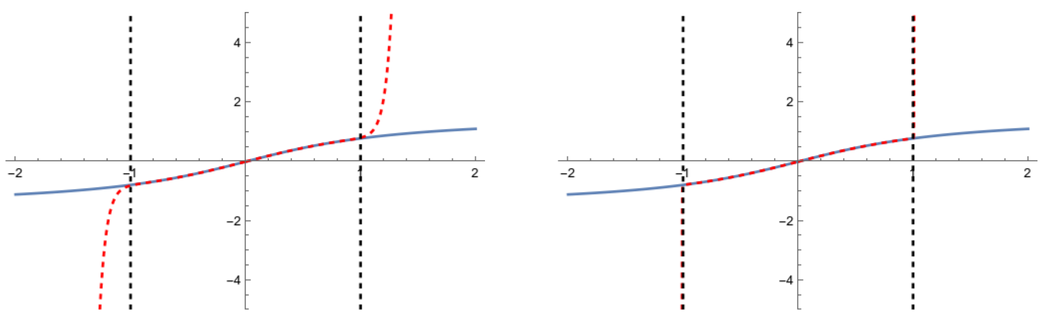

Figure 5.5: Left: the truncated power series $\sum_{n=0}^{10}(-1)^n\frac{x^{2n+1}}{2n+1}\sum_{n=1}^{10}(-1)^{n-1}\frac{x^n}{n}$. Right: The truncated power series $\sum_{n=0}^{500}(-1)^n\frac{x^{2n+1}}{2n+1}$, which might as well be un-truncated

Finally, these power series representations allow us to compute integrals that we either couldn't do or couldn't do easily before. We'll see more of this soon.

Example 5.21 (recitation)**.** What is $\int \frac{1}{1+x^6}$?

We could use a partial fractions decomposition, if we know that $1 + x^6 = (1 + x^2)(x^2 -$

$\sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$, but that's really unpleasant. Instead, we write

$$\frac{1}{1+x^6} = \sum_{n=0}^{\infty}(-x^6)^n = \sum_{n=0}^{\infty}(-1)^n x^{6n}$$
$$\int \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty}\int(-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{6n+1} + C.$$

This again converges for $|x| < 1$.

Example 5.22 (recitation)**.** What is $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$?

Again we have

$$\frac{1}{1-(x-4)^3} = \sum_{n=0}^{\infty}(x-4)^{3n}$$
$$\int_3^4 \frac{1}{1-(x-4)^3}\,dx = \sum_{n=0}^{\infty}\int(x-4)^{2n}\,dx\Big|_3^4$$
$$= \sum_{n=0}^{\infty}\frac{(x-4)^{3n+1}}{3n+1}\Big|_3^4$$
$$= 0 - \sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{3n+1}$$

which converges for any n by the Alternating Series Test.

There are many other functions we wish we could integrate but can't; the most prominent example is e^{-x^2}, but there are others. Unfortunately, we don't have a power series for e^{-x^2}, and don't have an obvious way of obtaining power series for functions except by luck. But we can fix that.

5.3 Taylor Series

In the previous section we found power series for a number of familiar functions by starting with the power series for $\frac{1}{1-x}$, and then using clever algebraic or calculus manipulations to obtain forms for our new functions. But we'd like a more systematic way of approaching the problem, that doesn't rely on cleverness and luck.

Example 5.23. One particular function we'd like a power series representation for is e^x. Let's be optimistic and assume it has one, so we can write

$$e^x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

for some collection of constants c_i.

As we saw in section 5.2, we can evaluate the power series at 0 (plug in $x = 0$) pretty easily. Plugging in $x = 0$ on the right hand side gives $c_0 + 0 + 0 + \cdots = c_0$, and so $c_0 = e^0 = 1$. But how can we determine the other constants?

Let's take the derivative of both sides. We get

$$e^x = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

and plugging in 0 for both sides gives $c_1 = e^0 = 1$. We can repeat the process; taking more derivatives gives

$$\begin{aligned} e^x &= 2c_2 + 6c_3 x + 12c_4 x^2 + \dots \\ e^x &= 6c_3 + 24c_4 x + 60c_5 x^2 + \dots \end{aligned}$$

and thus $2c_2 = 1, 6c_3 = 1$, and thus $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{6}$. Continuing this pattern, and more generally we have $c_n = \frac{1}{n!}$. Thus *if* we can represent e^x as a power series centered at 0, the power series must be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We can generalize this to any function:

Theorem 5.24. *If f has a power series representation centered at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for some sequence of constants c_n, then $c_n = \dfrac{f^{(n)}(x)}{n!}$ for each n, where $f^{(n)}(x)$ is the nth derivative of f at x.

Definition 5.25. We define the *Taylor series* of f centered at a to be

$$T_f(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

We sometimes say a Taylor series centered at 0 is a *Maclaurin series*, which we write

$$T_f(x,0) = T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

Remark 5.26. It is *not true* that every function *can* be represented as a power series. If a function is not infinitely differentiable ("smooth") then it clearly doesn't have a power series, since all power series are smooth. Thus $|x|$ doesn't have a power series expansion that includes 0.

Not even all smooth functions have power series; we'll see an example in section 5.6.1. Functions that *can* be represented by power series are called "analytic."

But if a function can be represented by a power series, that power series is the Taylor series. Our next goal is to figure out when a function is equal to its Taylor series.

Definition 5.27. We call the truncated Taylor series the nth *Taylor polynomial* of f centered at a.

$$T_{F,k}(x,a) = T_k(x,a) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark 5.28. You might notice that $T_0(x,a) = f(a)$, and $T_1(x,a) = f(a) + f'(a)(x-a)$, which might look familiar from calculus 1 as the linear approximation to f near a. The Taylor polynomials in general are an expansion of this concept; T_1 is the best linear approximation we can make to f, and T_2 is the best quadratic appoximation we can make.

In computation and in modelling we often replace a function by its Taylor polynomial to make our lives easier. We'll use this for some applications later on in section 5.5.

The Taylor polynomials are not the same thing as our original function; they're approximations. so we can ask how much error this approximation has.

Definition 5.29. We define $R_k(x,a) = f(x) - T_k(x,a)$ to be the kth *remainder* of the Taylor series.

Now we can reframe our question—when is the Taylor series is equal to the original function?—by asking when the Taylor polynomials converge to the function. So $f = T_f(x,a)$ on some interval (b,c) if and only if $\lim_{k\to+\infty} R_k(x,a) = 0$ for any x in (b,c).

Fortunately there's a way to check this, related to the Mean Value Theorem. Recall from Calculus 1 that, if f is differentiable on an interval $[a,x]$, then there is a z in that interval such that

$$f'(z)(x-a) = f(x) - f(a) = R_0(x,a).$$

We can extend this result to include more derivatives, and get:

Proposition 5.30. *If f has enough derivatives on an interval I containing a, then for any x in I, there is a number z between x and a such that*

$$R_k(x,a) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}.$$

Note that if we take $n = 0$ then we get the Mean Value Theorem.

Example 5.31. We'd like to show that the Taylor series for e^x we computed earlier actually gives us e^x. We have $f^{(k+1)}(z) = e^z$, so $R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1}$. Note that z depends on n, and z is between 0 and x, so assuming x is positive, we have

$$R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1} = \frac{x^{k+1}}{(k+1)!}e^z \leq \frac{x^{k+1}}{(k+1)!}e^x.$$

But as n goes to infinity, e^x doesn't change, and $\frac{x^{k+1}}{(k+1)!} \to 0$. So $\lim_{k\to+\infty} R_k(x,0) = 0$, and e^x is equal to its Taylor series for $x > 0$.

If x is negative, that argument doesn't quite work. But in that case we have $e^z < 1$ so we get

$$R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1} = \frac{x^{k+1}}{(k+1)!}e^z \leq \frac{x^{k+1}}{(k+1)!} \to 0.$$

Thus e^x is equal to its Taylor series for all x, and we can write

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and in particular

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots.$$

Example 5.32 (Recitation)**.** We can also ask for the Taylor series of e^x centered at another number, say $a = 1$. Each derivative is still e^x and thus e^3, and so we have

$$T(x,1) = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

Is this actually equal to e^x? We compute

$$R_k(x,1) = \frac{e^z}{(k+1)!}(x-1)^{k+1}$$

which, for any fixed x and $|z| \leq x$ goes to 0 as k goes to infinity. So we have

$$e^x = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

This is superficially different from the previous power series, but clearly the two series give the same function. The series centered at zero will be more efficient for computing with inputs near zero, and the series centered at 1 will be more efficient for inputs near 1.

This example also tells us another nice fact about e^x. Notice that if I plug 1 into this power series, I get e times what I would have gotten by plugging zero into the other power series. In general I can compute that

$$T_{x,a} = \sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!} = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

which tells me that $e^x = e^a \cdot e^{x-a}$, which is the basic arithmetic rule for multiplying exponentials.

In a much crueler course, I could define e^x by its Taylor series, and then prove (or ask you to prove!) the rule that $e^{a+b} = e^a e^b$ by doing these sorts of algebraic manipulations.

Example 5.33. Can we find a power series for $f(x) = xe^x$ centered at 0? The obvious approach is to compute the Taylor series. We have

$$\begin{aligned} f(x) &= xe^x & f(0) &= 0 \\ f'(x) &= e^x + xe^x & f'(0) &= 1 \\ f''(x) &= 2e^x + xe^x & f''(0) &= 2 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= ne^x + xe^x & f^{(n)}(0) &= n. \end{aligned}$$

Thus the Taylor series formula gives us

$$\sum_{n=0}^{\infty} \frac{n}{n!} x^n.$$

In order to remember that the constant term c_0 here is zero, I'll change the indexing to be from $n = 1$, and then I can see that $\frac{n}{n!} = \frac{1}{(n-1)!}$, and I get

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n.$$

But there's another, easier way we could approach this. We already have a power series for e^x, so we can compute

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ xe^x &= x \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}. \end{aligned}$$

And these are in fact two ways of writing the same series, after we change the indexing around.

Example 5.34. Can we find a power series for $g(x) = e^{x^2}$?

Technically we can do this with the direct Taylor series method, but that's actually quite annoying. We can compute as before

$$\begin{aligned} g(x) &= e^{x^2} & g(0) &= 1 \\ g'(x) &= 2xe^{x^2} & g'(0) &= 0 \\ g''(x) &= 2e^{x^2} + 4x^2e^{x^2} & g''(0) &= 2 \\ g'''(x) &= 12xe^{x^2} + 8x^3e^{x^2} & g'''(0) &= 0 \\ g^{(4)}(x) &= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} & g^{(4)}(0) &= 12 \end{aligned}$$

and we can kind of see a pattern here, but figuring out exactly what it is will be tricky and proving it works will be even trickier.

But there is, again, an easier way.

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n \\ e^{x^2} &= \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n}. \end{aligned}$$

5.4 Computing Taylor Series

We want to compute some more Taylor series, both to get practice at doing the computation, and to have some more Taylor series to work with. We'll warm up with some simple examples.

Example 5.35. We can compute the Taylor series of a polynomial. In fact, if we take the series centered at 0, we get back exactly what we started with.

Let $f(x) = x^3 + 3x^2 + 1$. Then we have $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for $n > 3$. Thus the Taylor series centered at 0 is

$$\begin{aligned} T_f(x, 0) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\ &= 1 + 0x + \frac{6}{2}x^2 + \frac{6}{6}x^3 = 1 + 3x^2 + x^3. \end{aligned}$$

Hopefully this is what you expected.

Probably more useful is the ability to write the Taylor series centered at a *different* point. If we take the Taylor series centered at 2, for instance, we have

$$\begin{aligned} T_f(x,2) &= f(2) + f'(2)x + \frac{f''(2)}{2}x^2 + \frac{f'''(2)}{6} \\ &= 21 + 24(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{6}(x-2)^3 \\ &= 21 + 24(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

If you multiply this out you will get your original polynomial back; but sometimes it is very useful to have a polynomial expressed in terms of $x-2$, say, instead of in terms of x.

Example 5.36. Let's consider the function $\ln x$. (We've computed a Taylor series for $\ln(1+x)$ but that's a bit awkward).

If $f(x) = \ln x$ then we have

$$\begin{aligned} f'(x) &= \frac{1}{x} & f''(x) &= \frac{-1}{x^2} \\ f'''(x) &= \frac{2}{x^3} & f^{(4)}(x) &= \frac{-6}{x^4} \\ &\dots \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{x^n} \end{aligned}$$

Thus if we wish to compute the Taylor series centered at 1, we have

$$\begin{aligned} \ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!1^n}(x-1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \end{aligned}$$

This should look familiar (see figure 5.6); it's exactly the same thing as our previous series $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$, replacing x with $x-1$.

But wait, there's more! If we want to compute $\log(5)$, for instance, that power series doesn't work. But we can pick a new center for the power series and compute things there, and these power series will have different intervals of convergence, as we see in figure 5.7.

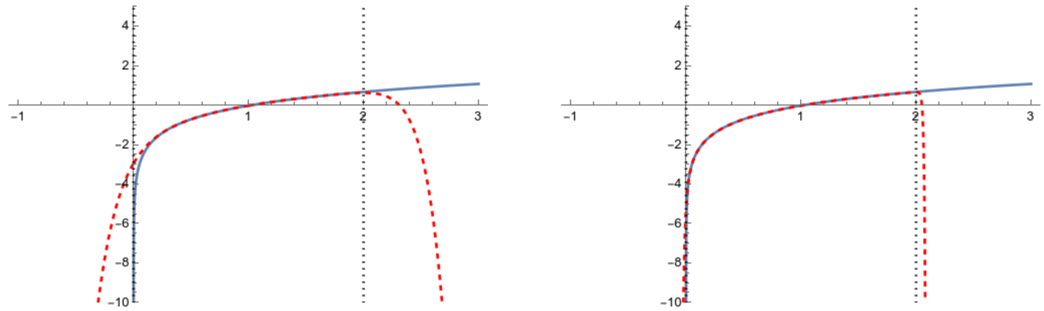

Figure 5.6: Graphs of the Taylor polynomials for $\ln(x)$ centered at 1, with 10 and 100 terms. Compare to the pictures in figure 5.4.

$$\begin{aligned}\log(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n \\ &= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!5^n}(x-5)^n \\ &= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(x-5)^n.\end{aligned}$$

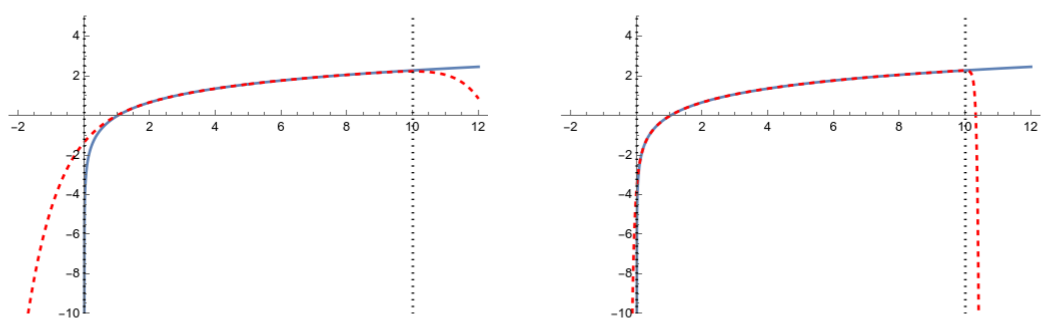

Figure 5.7: Graphs of the Taylor polynomials for $\ln(x)$ centered at 5, with 10 and 100 terms.

An application of the ratio test will show that this has radius of convergence 5, and we can see it converges conditionally on the boundary; in particular it will converge at 10 (by the alternating series test) and diverge at 0 (where it will just be the harmonic series). This behavior is in fact what we should expect: we know the series won't converge at 0 since $\log(0)$ is undefined, but we expect the series to converge everywhere it "can".

5.4.1 Trigonometry and Exponentials

While the most mysterious function we've been dealing with is e^x, we also would like to be able to compute $\sin x$ and $\cos x$.

First we'll compute the Maclaurin series for $\sin x$. Computing a few derivatives gives us

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \cos(x) & \frac{d}{dx}\sin(x)\mid_0 &= 1\\
\frac{d^2}{dx^2}\sin(x) &= -\sin(x) & \frac{d^2}{dx^2}\sin(x)\mid_0 &= 0\\
\frac{d^3}{dx^3}\sin(x) &= -\cos(x) & \frac{d^3}{dx^3}\sin(x)\mid_0 &= -1\\
\frac{d^4}{dx^4}\sin(x) &= \sin(x) & \frac{d^4}{dx^4}\sin(x)\mid_0 &= 0
\end{aligned}$$

And since this pattern will repeate, we get that $\frac{d^{2n}}{dx^{2n}}\sin x = (-1)^n \sin x$, and $\frac{d^{2n+1}}{dx^{2n+1}}(\sin x) = (-1)^n \cos x$. Since $\sin(0) = 0$ and $\cos(0) = 1$, the Maclaurin series is

$$T(x, 0) = 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0 - \frac{x^7}{7!} + \ldots \qquad = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Computing the remainder gives

$$\begin{aligned}
R_{2n}(x, 0) &= \frac{(-1)^n \cos z}{(2n+1)!}x^{2n+1}\\
|R_{2n}(x, 0)| &= \left|\frac{(-1)^n \cos z}{(2n+1)!}x^{2n+1}\right| \leq \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

since $|\cos z| \leq 1$, and this tends to zero as n tends to infinity. So $\sin x$ is equal to its Maclaurin series, and we have

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

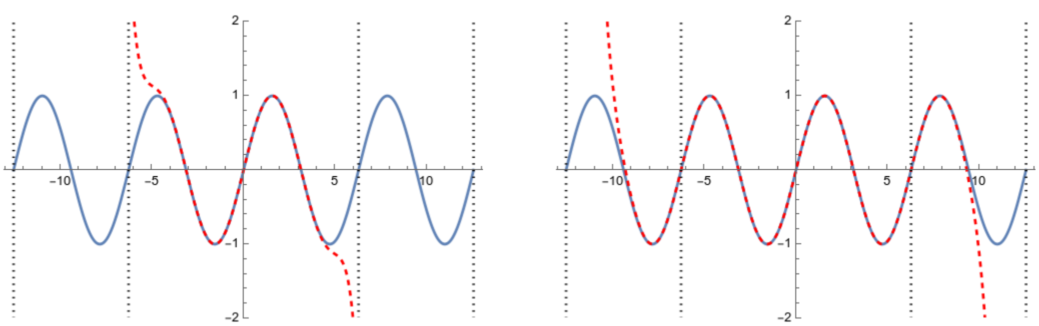

Figure 5.8: Graphs of the Taylor polynomials for $\sin(x)$ centered at 0, with 5 and 11 terms. Dashed asymptotes at multiples of 2π. Notice how quickly this converges near zero!

We also want a Maclaurin series for $\cos x$. We could compute it as we did before, but there's an easier way; $\cos x = (\sin x)'$, so

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n!} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots\end{aligned}$$

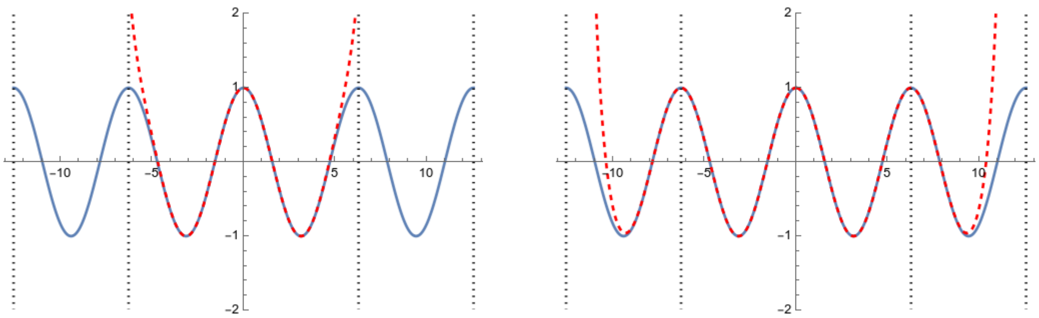

Figure 5.9: Graphs of the Taylor polynomials for $\sin(x)$ centered at 0, with 5 and 11 terms. Dashed asymptotes at multiples of 2π. Notice how quickly this converges near zero!

Though this is less important, sometimes we want to know things like the Taylor series for $x \sin x$. Again we'd rather not do this by computing derivatives, because that's hard. But we can find this Taylor series by doing power series algebra:

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ x\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)!} = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots\end{aligned}$$

5.4.2 The Binomial Series

Another important and widely applicable example is the *binomial series*, which is the Maclaurin series expansion for $f(x) = (1+x)^\alpha$. We can calculuate that

$$\begin{aligned}
f(x) &= (1+x)^\alpha & f(0) &= 1 \\
f'(x) &= \alpha(1+x)^{\alpha-1} & f'(0) &= \alpha \\
f'''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & f''(0) &= \alpha(\alpha-1) \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} & f'''(0) &= \alpha(\alpha-1)(\alpha-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n} & f^{(n)}(0) &= \alpha(\alpha-1)\dots(\alpha-n+1) \\
&= \frac{\alpha!}{(\alpha-n)!}(1+x)^{\alpha-n} & &= \frac{\alpha!}{(\alpha-n)!}
\end{aligned}$$

So we get the formula

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha!}{(\alpha-n)!n!} x^n.$$

We sometimes use the notation

$$\binom{\alpha}{n} = \frac{\alpha!}{(\alpha-n)!n!}$$

which we read "α choose n"; if α is a positive integer this represents the number of ways to choose n things out of α choices. Then we can write

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

By the ratio test, this power series converges when $|x| < 1$.

This series is called the binomial series, and is used very, very often to do numerical approximations, and especially in physics applications.

Notice that if α is a positive integer this is just the usual polynomial expansion. If α is an integer then $\binom{\alpha}{\alpha+1} = 0$, and so we get formulas like

$$(1+x)^5 = \sum_{n=0}^{5} \binom{5}{n} x^n = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Example 5.37. What is $(3+x)^3$? We need to get a formula that looks like $(1+y)^3$, and factoring out a 3^3 gives us

$$(3+x)^3 = 3^3(1+x/3)^3.$$

Then the binomial series gives us

$$\begin{aligned}(3+x)^3 &= 27\cdot(1+x/3)^3 \\ &= 27\sum_{n=0}^{\infty}\binom{3}{n}\left(\frac{x}{3}\right)^n \\ &= 27\left(1+3\cdot\frac{x}{3}+3\cdot\frac{x^2}{9}+\frac{x^3}{27}\right) \\ &= 27+27x+9x^2+x^3.\end{aligned}$$

Example 5.38. What is $\sqrt[3]{1+x^2}$? This is the binomial series with $\alpha = 1/3$. So the Binomial Series tells us:

$$\begin{aligned}\sqrt[3]{1+x^2} &= \sum_{n=0}^{\infty}\binom{1/3}{n}x^{2n} \\ &= 1+\frac{1}{3}x^2+\frac{(1/3)(-2/3)}{2!}x^4+\frac{(1/3)(-2/3)(-5/3)}{3!}x^6+\dots \\ &= 1+\frac{x^2}{3}-\frac{x^4}{9}+\frac{5x^6}{81}-\dots\end{aligned}$$

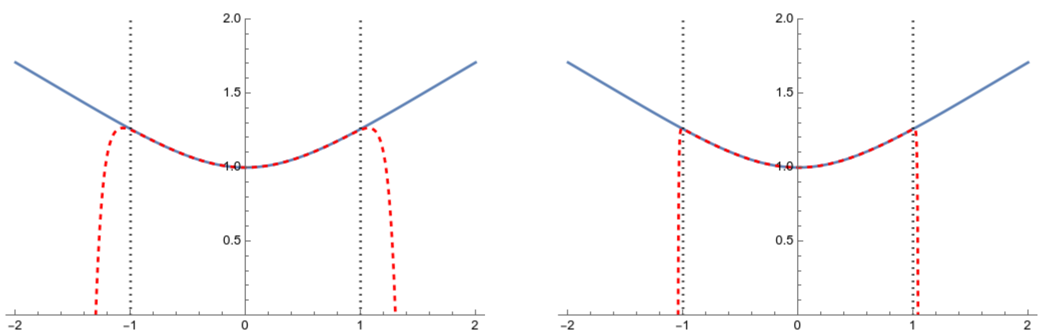

Figure 5.10: Graphs of the Taylor polynomials for $\sqrt[3]{1+x^2}$ centered at 0, with 20 and 200 terms. Again, notice how quickly this converges.

Thus we can estimate, for small x, that $\sqrt[3]{1+x^2} \approx 1+\frac{x^2}{3}$. We would approximate that $\sqrt[3]{1.3} \approx 1+\frac{.09}{3} = 1.03$. In fact Mathematica tells us that $\sqrt[3]{1.3} \approx 1.09139$, so that's pretty good.

5.5 Applications of Taylor Series

Now that we have power series representations of a bunch of functions, we can use them to calculate limits and integrals and other messy calculus things, and in general we can use them to do lots of cool things.

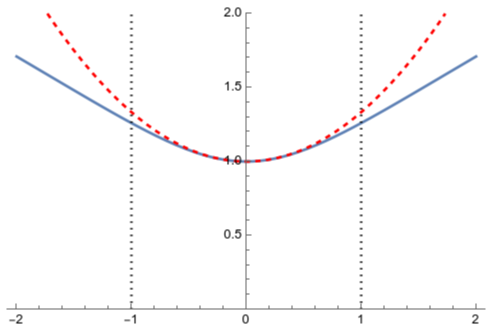

Figure 5.11: Graphs of the quadratic Taylor polynomial for $\sqrt[3]{1+x^2}$ centered at 0, which is just $1 + x^2/3$. That's already enough to get us a really good approximation.

5.5.1 Calculating constants

This is quick, but important. We all know that $\pi \approx 3.14$ and $e \approx 2.7$, but where do these numbers come from?

I've mentioned before that

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Summing the first five terms, though $n = 4$, gives

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} \approx 2.708$$

summing the first nine terms gives $\frac{109601}{40320} \approx 2.71828$.

Slightly trickier is finding π. The simplest way we have to do this is observing that $\arctan(1) = \pi/4$, and then computing

$$\pi = 4\arctan(1) = 4\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

This series converges *very slowly*. Summing through $n = 10$ gives $\pi \approx 3.23232$; summing through $n = 100$ gives $\pi \approx 3.15149$. After a few hundred terms we see 3.14 show up, and at a thousand we get $\pi \approx 3.14259$.

There are much better series for calculating numerical approximations of π. But this one was good enough for a particularly stubborn gentleman named Abraham Sharp to compute π to 71 digits in 1699. By hand.

5.5.2 Approximating functions

The primary use of Taylor series is to conduct approximate calculations of functions we can't or don't want to calculate exactly. This means that when we're doing something like trying to understand a physical or economic model or solve a differential equation, we can pretend our functions are all polynomials, which are a lot easier to analyze.

Example 5.39. What is $\sqrt[n]{1+x}$ when x is small?

When we look at this we should immediately think of the binomial series (with $\alpha = 1/n$). Thus

$$\begin{aligned} \sqrt[n]{1+x} &= \sum_{k=0}^{\infty} \binom{1/n}{k} x^k = 1 + \frac{1}{n}x + \frac{(1/n)((1-n)/n)}{2}x^2 + \dots \\ &\approx 1 + \frac{x}{n}. \end{aligned}$$

Note that this approximation works better when x is small. But we can, as a rule of thumb, approximate $\sqrt[n]{2} \approx \frac{n+1}{n}$.

Example 5.40. Approximate $\sqrt[5]{x}$ near 32 to degree two.

We have two options. The first is to use the binomial series approximation. It looks like we want to approximate $(32+x)^{1/5}$, which isn't quite the binomial series which comes from $(1+x)^\alpha$; but we can factor out a two and get somtehing that works quite well:

$$\begin{aligned} \sqrt[5]{32+x} = 2(1+\tfrac{x}{32})^{1/5} &= 2\sum_{n=0}^{\infty}\binom{1/5}{n}x^n \\ &= 2 + \frac{2}{5}(x/32) - \frac{4}{25}(x/32)^2 + \dots \end{aligned}$$

(Note this converges when $|x/32| < 1$ and thus when $-32 < x < 32$). So we can approximate

$$\sqrt[5]{32+x} \approx 2 + \frac{x}{80} - \frac{x^2}{6400}.$$

This first approach is very common, and the reason we don't much mind that the binomial series approximation is specifically centered at 1. If we want to approximate $(r+x)^\alpha$ we can view this as $r^\alpha(1+x/r)^\alpha$ and then use the binomial series approximation.

Alternately, we could compute the Taylor polynomial anew, centered at 32:

$$\begin{aligned} f(x) &= \sqrt[5]{x} & f(32) &= 2 \\ f'(x) &= \frac{1}{5}x^{-4/5} & f'(32) &= \frac{1}{80} \\ f''(x) &= \frac{-4}{25}x^{-9/5} & f''(32) &= \frac{-1}{3200} \end{aligned}$$

and thus

$$\begin{aligned} \sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(32)}{n!}x^n \\ &\approx 2 + \frac{x}{80} - \frac{x^2}{3200 \cdot 2}. \end{aligned}$$

Either way, we can estimate $\sqrt[5]{36} \approx 2 + \frac{4}{80} - \frac{16}{6400} = 2 + \frac{1}{16} - \frac{1}{800} \approx 2.06.$

5.5.3 Limits

Taylor series can make computing limits very easy. Heuristically when we calculate a limit we tend to ask "how many times" the top and bottom go to zero; we can see L'Hospital's rule as a way of calculating this. But working with Taylor series makes this idea precise.

Example 5.41. What is $\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$?

We could use L'Hospital's rule here three times, and we did that in section 1.6. But we can also approach this with Taylor series. We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$, so this is

$$\lim_{x \to 0} \frac{\frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x^3} = \lim_{x \to 0} \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \dots = \frac{1}{6}.$$

Example 5.42. What is $\lim_{x \to 0} \frac{\sin x}{x}$?

With the same trick, we have

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \\ \lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots = 1. \end{aligned}$$

Similarly, we have

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots) - 1}{x^2} = \lim_{x \to 0} \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots = \frac{1}{2}.$$

Remark 5.43. In some sense, this what L'Hôpital's Rule is "really" doing. When the top and bottom are both zero, that means the constant terms of each power series are zero. We take the derivative to shift both power series over one place and then try comparing the linear terms. Then the quadratic, etc.

Notice that this is very like how we handle all limits as $x \to \infty$. In that case, we only have to care about the highest-degree term, and we can ignore all the others. Here, if $x \to a$, we only have to care about the *lowest*-degree term of the Taylor expansion around a.

5.5.4 Integration with Taylor Series

We can also use Taylor series to make difficult integrals easy.

Example 5.44. What is the integral of $x^6 \cos x$?

We can do this with integration by parts, but it's tedious. Instead, we calculate:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$
$$x^6 \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+6}}{2n!}$$
$$\int x^6 \cos x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+7}}{(2n+7)(2n!)} + C.$$

Example 5.45 (Recitation)**.** There are some integrals that simply cannot be computed by normal means. I'v ementioned a few times that we can't represent $\int e^{-x^2}\,dx$ with "elementary" functions. But the integral is very important; any time you're dealing with, for instance, a normal distribution, the integral of e^{-x^2} is lurking in the background.

With our new techniques this is easy to handle:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$
$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} + C.$$

Thus we can compute, for instance, that

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)}\Big|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)} \approx .75.$$

There are still interesting questions in actually computing things with this; our answers are written in terms of *infinite series* and we still need ways to approximately sum those series. But this gives us a way to answer the questions at all.

Remark 5.46. A technique of complex analysis called "contour integration" tells us that $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$. (I told you π shows up everywhere for no reason). From this fact it's not too hard to show that $\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$, and thus that $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$.

This is why the standard bell curve (with mean zero and standard deviation 1) is given by the probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$; the total probability of *something* happening has to be exactly one. We can generalize this to a normal distribution with mean μ and standard deviation σ, which has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

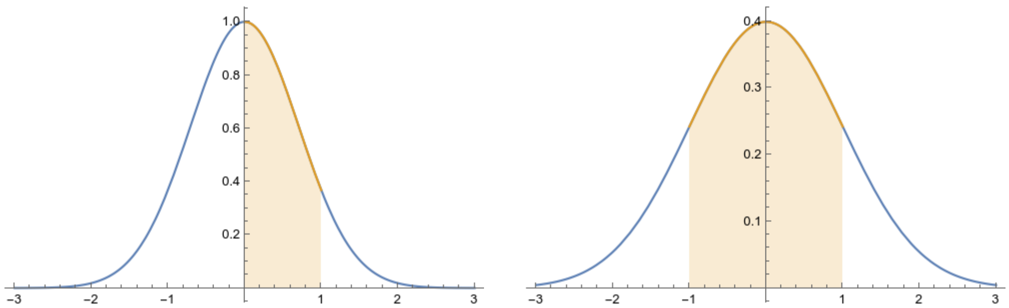

Figure 5.12: Left: $\int_0^1 e^{-x^2}\,dx$. Right: The probability of landing within one standard deviation of the mean.

This still has total integral one. But there's still no elementary antiderivative to this function, so if we want to compute, say, the probability of an output between -1 and 1 as in figure 5.12 we still need to use some sort of Taylor series argument to approximate the probability.

5.5.5 Maxima and Minima

Taylor series give us another way to think about what we're doing when we look for maxima and minima of functions. We can approximate any (reasonable) function with Taylor polynomials. So first think about what happens when we approximate with a linear function. If the linear function is sloping upwards or downwards, then our function doesn't have a maximum or a minimum. And that's basically what we said in calculus 1. At a maximum, T_1 needs to be constant.

But now think about the second-order Taylor approximation, which will be a parabola. Since the first-order Taylor approximation was constant, we know we'll be at the vertex of a parabola. But this parabola will either open up or down. If our function is approximately an upwards parabola, we have a minimum; if our function is approximately a downwards parabola, we have a maximum. And what determines if the parabola opens up or down? If the leading quadratic term has a positive coefficient, it will open up, and if the leading quadratic term has a negative coefficient, then it will open down. And that's exactly the second derivative test from calculus 1.

But now we can do more! If the second derivative is zero, then the second derivative test doesn't give us any information. And that's because the second-order Taylor polynomial T_2 is just a horizontal line: it doesn't tell us enough. But we can compute the Taylor series further out.

Example 5.47. Suppose we want to look at $f(x) = \cos(x^2)$. We see that $f'(x) = -2x\sin(x^2)$

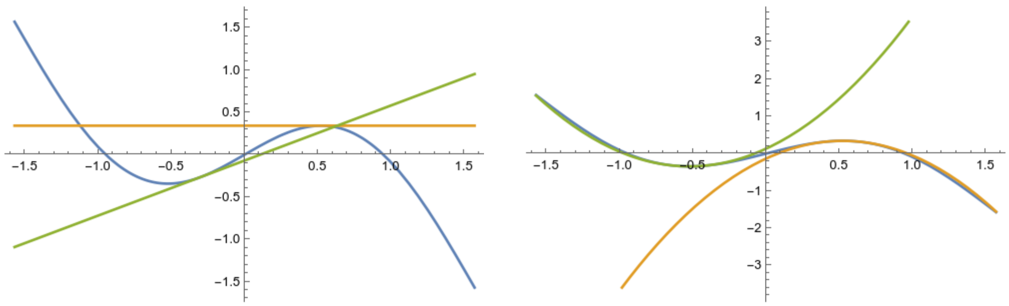

Figure 5.13: The graph of $\sin(2x) - x$. On the left we have first-order Taylor approximations at two points, one of which is a maximum. On the right we have second-order Taylor approximations at two points, which are a maximum and a minimum.

is zero when $x = 0$. (Also at other points, which we'll ignore for now.) But the second derivative is $f''(x) = -2\sin(x^2) - 4x^2\cos(x)$, which is also zero, so the second derivative test doesn't tell us anything. And a graph of $T_2(x, 0)$ indeed gives us a horizontal line.

So let's look at the Taylor series now. We know

$$\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ \cos(x^2) &= \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!} \\ &= 1 - \frac{x^4}{2} + \frac{x^8}{24} + \cdots \approx 1 - \frac{x^4}{2}. \end{aligned}$$

So near zero, we can approximate $\cos(x^2)$ with a degree-four polynomial, which opens downwards.

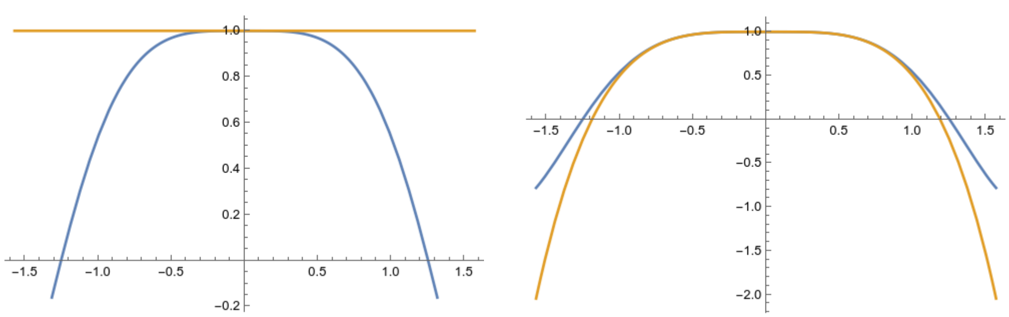

Figure 5.14: The second-order Taylor expansion on the left isn't very helpful, but the fourth-order Taylor expansion shows our function has a maximum at 0..

Example 5.48. But what about the function $g(x) = \sin(x) - x$? Again we'll find that

$g'(0) = g''(0) = 0$, so the second derivative test is useless. But we know

$$\begin{aligned}\sin(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \\ \sin(x) - x &= -\frac{x^3}{6} + \frac{x^5}{120} + \dots \approx -\frac{x^3}{6}.\end{aligned}$$

Since $\frac{-x^3}{6}$ doesn't have a maximum or a minimum at 0, $\sin(x) - x$ won't either.

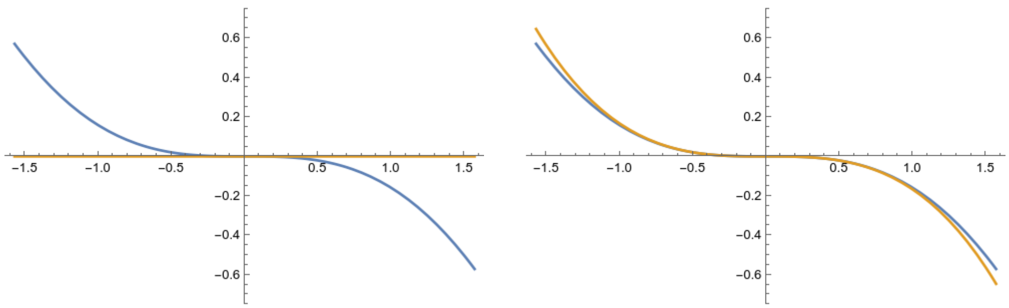

Figure 5.15: The second-order Taylor expansion on the left isn't very helpful, but the third-order Taylor expansion on the right shows that this function has no extrema at 0.

5.5.6 Physical Models

Example 5.49 (Pendulums and Clocks)**.** One place we often use Taylor approximations is in modelling physical systems, such as a pendulum.

We use pendulums in clocks (e.g. grandfather clocks) becaues they keep accurate time. The principle underlying this is the idea that a given pendulum takes the same amount of time to complete a swing regardless of the size of that swing.

This is, unfortunately, false. The angular acceleration on a pendulum (that is, how quickly it changes the angle of rotation) is given by $\alpha = -\frac{mg}{L} \sin \theta$, and thus the position of the pendulum as a function of time satisfies the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin(\theta),$$

where g is the acceleration due to gravity and L is the length of the pendulum. Solving this differential equation involves a nasty integral that doesn't have a closed-form answer, known as an "elliptic integral." (It's called this because it's also the type of integral used to calculate the circumference of an ellipse).

But you may notice that most clocks have a long pendulum that only makes small arcs. When the angle of the pendulum is small, we can use the Taylor series of sin to approximate $\sin\theta \approx \theta$, and then we have $\alpha \approx -\frac{mg}{L}\theta$, and the approximate differential equation

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta.$$

You might recognize this as the equation $y'' = -ky$ from section 3.3.3, that gave simple harmonic motion. With initial conditions $\theta(0) = \theta_0, \theta'(0) = 0$, we get the solution $\theta(t) \approx \theta_0 \cos\left(\sqrt{g/L}t\right)$, and we see that the time a pendulum takes to complete one swing is $T \approx 2\pi\sqrt{L/g}$, regardless of the initial position. (The error in this approximation causes a typical grandfather clock to lose about 15 seconds a day).

If we need to know the answer to more precision, Taylor series still provide a solution. If we add more terms to our Taylor approximation, we get the formula

$$T = 2\pi\sqrt{L/g}\left(1 + \frac{\theta_0^2}{16} + \frac{11}{3072}\theta_0^4 + \dots\right).$$

Here we see that the period of oscillation does in fact depend on the initial angle θ_0, but if this initial angle is about .1 radians and the pendulum is about a meter then the duration of a swing is about 2 seconds, with an error of roughly a millisecond.

Example 5.50 (Springs in physics)**.** More generally, if you spend more time doing physics, you'll discover that almost every system you want to study is modeled as a collection of springs.

A spring is a system governed by a simple quadratic equation. So any system governed by a quadratic equation can be treated as a spring. So if you take any system and approximate it with the second Taylor polynomial, you get something that looks like a spring.

As examples: we often model light interacting with matter by treating the atom as an electron on a spring. The strength of the chemical bond between two atoms is given by the Lennard-Jones Potential, which is $\left(\frac{R_m}{r}\right)^{12} - 2\left(\frac{R_m}{r}\right)^6$. If we take the Taylor series centered at R_m we get $-1 + 0r + \frac{72}{R_m^2}r^2 + \cdots \approx \frac{72}{R_m^2}r^2 - 1$, which is just a parabola.

As a note: we often refer to this process as "dropping higher order terms"—the constant term is the order-0 term; the linear term is the order-1 term; we have dropped every term of order higher than 2. We sometimes abbreviate even further and call them the H.O.T.

Example 5.51 (Relativity)**.** The last example we want to do concerns special relativity. Relativity includes a number of interesting phenomena that occur when your velocity is relatively large compared to the speed of light. But we know that at low velocities, special relativity should "look like" Newtonian mechanics.

Most of the relativity equations feature a variable γ, where $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$. We'd like to use the binomial expansion, so we write

$$\gamma = (1-(v/c)^2)^{-1/2} \approx 1 + \frac{-1}{2}(-(v/c)^2) = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

is the first-order Taylor approximation to γ. It should be accurate when v/c is small–that is, when our velocity is very small relative to the speed of light.

Famously, the energy of an object at rest is given by $E = mc^2$. Less famously, the energy of an object in motion is given by $E = mc^2\gamma$; when $v = 0$ then $\gamma = 1$ and we get the famous equation. But what if v is small, but non-zero? We can take the Taylor expansion from before, and get

$$E \approx mc^2\left((1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2.$$

And the second term is just the usual formula for kinetic energy.

Similarly, the formula for time dilation is $T' = T\gamma$. If we take the first-order approximation, we have $T' = T + \frac{T}{2}\frac{v^2}{c^2}$. But even better, if we take the zeroth-order approximation, we have $\gamma \approx 1$ and thus $T' \approx T$. This tells us that at low velocities, time dilation is negligible.

5.6 Bonus Taylor Series Fun

5.6.1 Failure Modes of Taylor Series

Sadly, while Taylor series are awesome, they don't always work. Consider the function defined by $f(x) = e^{-1/x^2}$ and $f(0) = 0$. This function is continuous and in fact differentiable at 0:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0.$$

We can repeat this work, and we see that $f''(0) = f'''(0) = \cdots = f^{(n)}(0) = 0$. Thus the Taylor series is

$$T_f(x,0) = \sum_{n=0}^{\infty}\frac{0}{n!}x^n = 0.$$

But clearly $f(x) \neq 0$ when $x \neq 0$, so f is equal to its Taylor series only in the trivial case when $x = 0$. So just remember: Taylor series *don't always work*.

But once we look at the graph, this makes perfect sense:

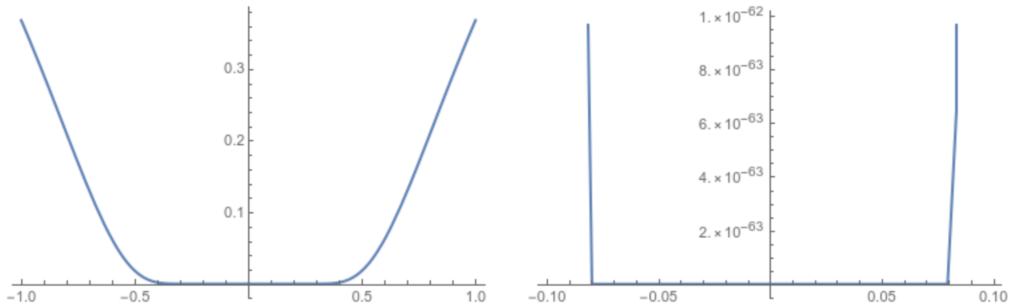

Figure 5.16: The graph of e^{-1/x^2} is absurdly flat near 0.

5.6.2 Power Series and Differential Equations

In section 3.4 we talked about solving separable differential equations, but most differential equations are not separable. There are a lot of tools we can use to solve non-separable equations, but one approach is to use power series—which basically always works.

Example 5.52. Recall the classic differential equation $y' = y$. Suppose the function $y(x)$ can be represented by a power series. Then we have

$$y = \sum_{n=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

and since these are the same function, each coefficient has to match up. So we get the system

$$\begin{aligned} a_1 &= a_0 & a_1 &= a_0 \\ 2a_2 &= a_1 & a_2 &= a_1/2 \\ 3a_3 &= a_2 & a_3 &= a_2/3 \\ &\vdots & &\vdots \end{aligned}$$

So if we know a_0, then we can figure out all the other coefficients.

How do we find a_0? Well, that's a choice. Remember any differential equation will wind up with free constants in the end. But if we take $a_0 = 1$, which seems like a reasonable choice, we then get

$$\begin{aligned}
a_1 &= a_0 = 1 \\
a_2 &= a_1/2 = 1/2 \\
a_3 &= a_2/3 = 1/6 \\
&\vdots \\
a_n &= a_{n-1}/n = 1/n! \\
y &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots
\end{aligned}$$

which is exactly the Taylor series for e^x we worked out in section 5.3.

(What happens if we choose a different a_0? That just multiplies all the coefficients by a constant, so if $a_0 = C$ then our power series gives us Ce^x. And we already know that Ce^x is the solution to this differential equation!)

Example 5.53. Solve $y'' - 3xy' + y = 0$.

We don't have any tools for this, so we use Taylor series. Assume $y = \sum_{n=0}^\infty c_n x^n$, and then we have

$$\begin{aligned}
\left(\sum_{n=0}^\infty c_n x^n\right)'' - 3x\left(\sum_{n=0}^\infty c_n x^n\right)' + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty c_n n(n-1)x^{n-2} - 3x\sum_{n=0}^\infty c_n n x^{n-1} + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty c_{n+2}(n+1)(n+2)x^n - 3\sum_{n=0}^\infty c_n n x^n + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty \left((n+1)(n+2)c_{n+2} - 3nc_n + c_n\right)x^n &= 0
\end{aligned}$$

and thus for each n we have

$$\begin{aligned}
(n+1)(n+2)c_{n+2} &= (3n-1)c_n \\
c_{n+2} &= \frac{(3n-1)c_n}{(n+1)(n+2)}
\end{aligned}$$

as our recurrence relation. As before, we see that our solution must have the form

$$\begin{aligned}
y &= \sum_{k=0}^\infty \frac{c_0((5)(11)\cdots(6k-1))}{(2k)!}x^{2k} + \frac{c_1((8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1} \\
&= c_0\sum_{k=0}^\infty \frac{(5)(11)\cdots(6k-1)}{(2k)!}x^{2k} + c_1\sum_{k=0}^\infty \frac{(8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1}.
\end{aligned}$$

Example 5.54. In section 5.1 I mentioned the *Bessel function*

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

This arises naturally as a solution to the differential equation

$$x^2y'' + xy' + x^2y = 0$$

which is used to study a lot of physics on cylinders.

5.6.3 Taylor Series and Complex Numbers

Perhaps the most surprising and important fact about the trigonometric power series is the way they combine. You'll notice that for both sin and cos, every term looks like $\frac{x^n}{n!}$ but neither series has all the terms.

Leonhard Euler, around 1740, asked himself what it means to exponentiate an imaginary number. Since the Taylor series of e^x agrees with the function everywhere on the real line, it makes sense to define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. If we plug in a purely imaginary number ix for $x \in \mathbb{R}$, we see:

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(i)^{2n}x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x). \end{aligned}$$

Thus we obtain *Euler's Formula*:

$$e^{i\theta} = cos\theta + i\sin\theta.$$

As a corollary, we get the statement Euler called the most beautiful in all mathematics: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = 1$, or

$$e^{i\pi} - 1 = 0.$$

This statement relates the five most fundamental constants in the complex plane.

But in addition to being really pretty, this also has a geometric interpretation. If we think about the point $\cos\theta + i\sin\theta$ on the complex plane, it has x-coordinate $\cos\theta$ and y-coordinate $\sin\theta$ and is thus the point on the unit circle corresponding to angle θ. (This is why the unit circle is oriented as it is!)

So in general we can represent any point on the unit circle as $e^{i\theta}$, where θ is the angle the point makes from the positive x-axis. Further, if we have *any* complex number z, we can represent it in "polar coordinates" by giving its absolute value $|z|$ and its complex argument θ. Thus for any complex number z, we have

$$z = |z|e^{i \arg(z)}.$$

Another result from this line of thinking is De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

There are some functions, unfortunately, that don't behave nearly so nicely, such as the logarithm. Unlike exp and sin and cos we can't just extend the power series for log to the whole complex plane, since it has finite radius of convergence (and an unavoidable pole at 0).

But in fact the problem is deeper. We generally want to define log to be the inverse of exp, so that $\log \exp z = z$ for every z. But notice that $\exp(0) = 1$, and also $\exp(2\pi i) = 1$. So if we want to define log on the whole complex plane we must have $\log 1 = 0$ and also $\log 1 = 2\pi i$ and this is obviously a problem, since functions can't have multiple outputs. We in fact have infinitely many numbers z with $e^z = 1$, or in fact any complex number you choose (except 0; $\log 0$ is never defined).

We solve this by choosing a "branch," which basically corresponds to which complex arguments we allow; we will typically require our arguments to be in $(-\pi, \pi]$; this is called the "principal value" of the argument. (Notice this is similar to the way we require $\arcsin x$ to be in $[-\pi, \pi]$ in section 1.5). Notice also that in this case, we aren't really happy at negative real numbers–there's a huge jump discontinuity there in the complex argument.

There is another option, which avoids this discontinuity, and which I mention mainly because it's cool. We can define what's called a *Riemann Surface*, which is a two-dimensional surface that we can think of as sitting in three-dimensional space. In this case our Riemann surface looks like a giant helical Archimedes Screw.

This surface "covers" the complex plane with infinitely many "sheets." The logarithm is a function defined on this surface; which sheet we are in tells us which "branch" the argument should be in, and other than that the logarithm is defined as it would be for the point of the complex plane "under" our surface. Thus the logarithm of a point on one sheet would be 0, and the logarithm of the point one sheet above it is $2\pi i$, and if we go up another sheet we get $4\pi i$, and so on. See figure 5.17 for a sketch of what this might look like.

This behavior also occurs with functions like $\sqrt[n]{z}$. Since every (non-zero) number has two square roots, the square root function is "doubly ramified" or a "two-fold cover" of

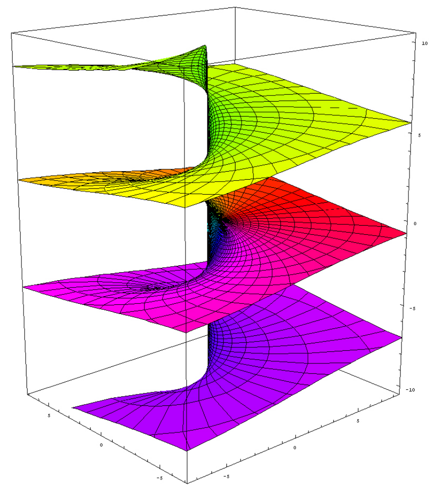

Figure 5.17: The Riemann surface corresponding to the complex logarithm

the complex plane. The nth-root function is an n-fold cover. They are, in fact, essentially the same picture as the logarithm picture, but with only finitely many sheets, which wrap around and join up. By convention, we put the discontinuity still on the negative real line. We can see an attempted picture in figure 5.18.

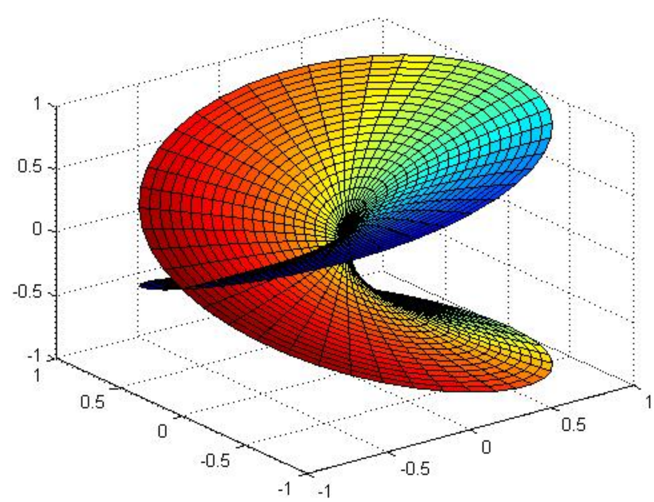

Figure 5.18: The Riemann surface corresponding to the square root function

Special attention should be paid to the idea of an "nth root of unity", where "unity" is just a fancy word for "one." We know that a number has two square roots; for instance the number one has 1 and -1 as square roots. By the same logic, any number should have n distinct nth roots.

A little creativity shows that the nth roots of unity are the numbers are $e^{2\pi ik/n}$, since

$$\left(e^{2\pi ik/n}\right)^n = e^{2\pi ik} = \left(e^{2\pi i}\right)^k = 1^k = 1.$$

These are points spaced evenly around the unit circle. They are very useful in creating functions and other operations that have certain types of "periodicity", which means they repeat every n times. (The most recognizable periodic function is $\sin(x)$, which has a period of 2 *pi*.)

The roots of unity are especially useful in conjunction with Fourier series, which give a useful way of representing a periodic function as an infinite sum of sine and cosine functions. They are an alternative tool to Taylor series that are useful in situations where Taylor series don't work terribly well.

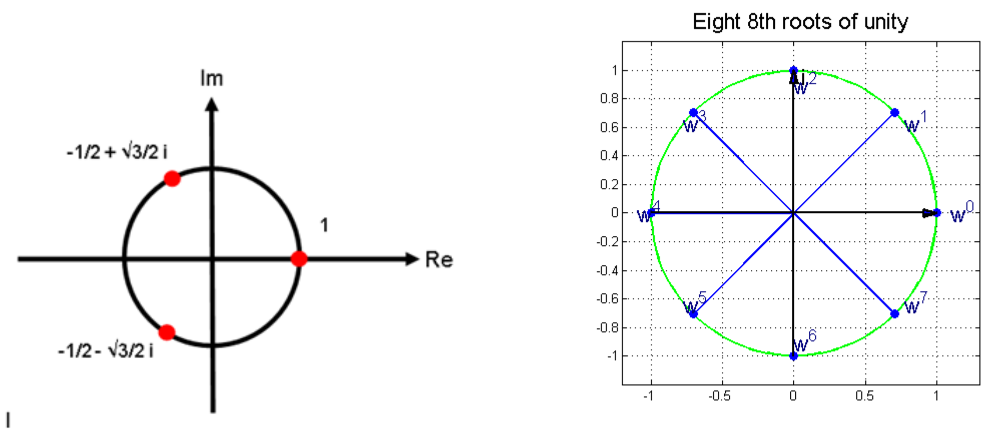

Figure 5.19: Cube Roots and Eighth Roots of Unity

5.7 Double Bonus: Real Fourier Series

We've seen that we can represent many functions as a power series, an infinite sum of multiples of powers of x. But some functions are hard to represent this way, and we want other tools. In particular we can represent a function as an infinite sum of trigonometric functions.

For this discussion we'll confine ourselves to real functions on the interval $[-\pi, \pi]$. (The same idea works for functions on any closed interval, but it's easier to talk about just this particular interval for right now. Also, π has popped up again for no reason. Hi, π!)

Definition 5.55. Let f be a function on the interval $[-\pi, \pi)$. Then the *Fourier series* of f

is given by

$$f_\infty(x) = \frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$$

where

$$\begin{aligned} C &= 2\langle f(x), 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= 2\langle f(x), \sin(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ b_n &= 2\langle f(x), \cos(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx. \end{aligned}$$

For Taylor series we could match things up by taking derivatives; to find Fourier series coefficients we instead compute integrals. That makes the computations much nastier! But the computations are *doable*, and they make sense because the integral of one term times a different term is always zero.

Before we dive into computations to prove this, let's think about why we should expect it to be true. A sin or cos function passes through a complete cycle between $-\pi$ an π, so the positive bits will exactly cancel out the negative bits. When we multiply two different sin or cos functions, they don't correlate with each other—each one passes through cycles at a different rate from the others, so the cycles don't reinforce or cancel out. Thus we'll still have exactly as much on top as we do on bottom, and the integrals should be zero.

Proof. We use the notation $\langle f, g\rangle$ to represent $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$$\begin{aligned} 2\pi\langle 1, \sin(nx)\rangle &= \int_{-\pi}^{\pi} \sin(nx)\,dx = \left.\frac{-\cos(nx)}{n}\right|_{-\pi}^{\pi} = \frac{1}{n} - \frac{1}{n} = 0. \\ \langle 1, \cos(nx)\rangle &= \int_{-\pi}^{\pi} \cos(nx)\,dx = \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi} = 0 - 0 = 0. \end{aligned}$$

The products of the sin and cos functions are a bit trickier.

$$\begin{aligned} \langle \sin(nx), \sin(mx)\rangle &= \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \\ &= -\frac{\cos(nx)}{n}\sin(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{m}{n}\cos(nx)\cos(mx)\,dx \\ &= 0 + \frac{m}{n^2}\sin(nx)\cos(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{m^2}{n^2}\sin(nx)\sin(mx)\,dx \\ 0 &= \left(\frac{m^2}{n^2} - 1\right)\int_{-\pi}^{\pi} \sin(nx)\sin(mx) \end{aligned}$$

As long as $\frac{m^2}{n^2} \neq 1$ this implies that $\langle \sin(nx)\sin(mx)\rangle = 0$. For positive integers m, n, this holds whenever $m \neq n$. In contrast

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(nx)\,dx &= \int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx \\ &= \left(\frac{x}{2} - \frac{\sin(2nx)}{4}\right)\Bigg|_{-\pi}^{\pi} = \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \pi. \\ \langle \sin(nx), \sin(nx)\rangle &= \frac{1}{2\pi}\cdot\pi = \frac{1}{2}.\end{aligned}$$

Similar arguments work for $\langle \cos(nx), \sin(mx)\rangle$ and $\langle \cos(nx), \cos(mx)\rangle$. □

Theorem 5.56. *Suppose f is a continous function with continuous derivative, except for finitely many points, on $[-\pi, \pi)$. Then $f(x)$ is equal to its Fourier series except for at finitely many points.*

Notice that unlike in the case of Taylor series, this always works. every *continuous function is (essentially) equal to its Fourier series.*

What does this mean? It means that if we have a function on $[-\pi, \pi)$ then we can look at it as being composed of a bunch of different "waves" of different frequencies, and the coefficients tell us how large each wave is. (The constant term tells us the average value around which the waves are oscillating). Further, a Fourier series is always a periodic function on the whole real line. So any periodic function can be viewed as a Fourier series, and this technology allows us to see it as composed of many smaller simpler waves. We'll return to the physics and geometry of this soon.

Example 5.57. Let $f(x) : [-\pi, \pi) \to \mathbb{R}$ be given by $f(x) = x$. The periodic version of this function is a "sawtooth wave." Then we have:

$$\begin{aligned}\frac{C}{2} &= \langle f(x), 1\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx \\ &= \frac{1}{2\pi}x^2|_{-\pi}^{\pi} = 0.\end{aligned}$$

$$\begin{aligned}
a_n &= 2\langle f(x), \sin(nx)\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{1}{\pi}\left(-x\cdot\frac{\cos nx}{n}|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(\frac{-\pi\cos(n\pi) - \pi\cos(-n\pi)}{n}\right) \\
&= -2\frac{\cos(n\pi)}{n} = (-1)^{n+1}\frac{2}{n}.
\end{aligned}$$

$$\begin{aligned}
b_n &= 2\langle f(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= \frac{1}{\pi}\left(x\cdot\frac{\sin nx}{n}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(\frac{\pi\sin(n\pi) + \pi\sin(-n\pi)}{n}\right) \\
&= 0.
\end{aligned}$$

Thus the sawtooth wave has Fourier series

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx).$$

Example 5.58. Define sgn : $[-\pi, \pi] \to \mathbb{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. (Made periodic, this is a "square wave").

$$\begin{aligned}
\frac{C}{2} &= \langle \operatorname{sgn}(x), 1/2\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{sgn}(x)dx = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \sin(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nx)dx + \int_{0}^{\pi} \sin(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{\cos(nx)}{n}|_{-\pi}^{0} - \frac{\cos(nx)}{n}|_{0}^{\pi}\right) \\
&= \frac{1}{n\pi}(\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0) \\
&= \frac{2}{n\pi}(\cos(0) - \cos(n\pi))
\end{aligned}$$

which equals $\frac{r}{n\pi}$ if n is odd and 0 if n is even.

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nx)dx + \int_{0}^{\pi} \cos(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{-\sin(nx)}{n}|_{-\pi}^{0} + \frac{\sin(nx)}{n}|_{0}^{\pi}\right) \\
&= \frac{1}{\pi n}(-\sin(0) + \sin(-n\pi) + \sin(n\pi) - \sin(0)) = 0.
\end{aligned}$$

Thus

$$\operatorname{sgn}(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)x).$$