Math 1232: Single-Variable Calculus 2
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Recitation 1

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Problem 1. (a) Is the function $f(x) = |x|$ one-to-one? Prove it is, or find a counterexample.

(b) Is the function $g(x) = 5x^3 + 3$ one-to-one? Prove it is, or find a counterexample.

(c) Find an inverses for any of these functions that were one-to-one.

Solution:

(a) No, because $f(-1) = 1 = f(1)$.

(b) Yes. Suppose $g(x) = g(y)$. Then we have

$$\begin{aligned} 5x^3 + 3 &= 5y^3 + 3 \\ 5x^3 &= 5y^3 \\ x^3 &= y^3 \\ \sqrt[3]{x^3} &= \sqrt[3]{y^3} \\ x &= y. \end{aligned}$$

(If you want to be clever, you can get to the line $x^3 = y^3$ and then remember the function x^3 is one-to-one

(c) Now we want to solve the equation $y = g(x)$.

$$\begin{aligned} y &= 5x^3 + 3 \\ y - 3 &= 5x^3 \\ \frac{y-3}{5} &= x^3 \\ \sqrt[3]{\frac{y-3}{5}} &= x \end{aligned}$$

So the inverse is

$$g^{-1}(y) = \sqrt[3]{\frac{y-3}{5}}.$$

Problem 2. Consider the function $f(x) = x^4$.

(a) Is this one-to-one?

(b) Can you find a smaller, restricted domain on which it's one-to-one?

(c) Find an inverse on your restricted domain.

(d) Can you find a completely different restricted domain? Find an inverse on that domain.

Solution:

(a) $f(-2) = 16 = f(2)$, so this function is not one-to-one.

(b) f is one-to-one on $[0, +\infty)$. (There are lots of choices but this is the most obvious.)

(c) On this domain, the inverse is $\sqrt[4]{x}$.

(d) If you want to be trolly, you could say something like $[1, +\infty)$, or $[0, 5)$, or any number of other choices.

But to be *completely* different we want to flip things around. we'll say the domain is $(-\infty, 0]$. On this domain, the inverse is $-\sqrt[4]{x}$.

This is why there are two fourth roots of any positive number. (Over the complex numbers there are in fact four, but we don't have do deal with that yet.)

Problem 3. Consider $f(x) = \cos(x)$.

(a) Is this function one-to-one? Why or why not?

(b) What domains can you restrict it to to get a one-to-one function?

(c) What value "should" you pick to solve $\cos(x) = 0$? What about $\cos(x) = 1$? $\cos(x) = -1$?

(d) What domain should you pick to create an inverse?

Solution:

(a) No. $\cos(0) = 1 = \cos(2\pi)$.

(b) Looking at a graph, we need to go from a peak to a trough or a trough to a peak. So we need something that looks like $[n\pi, (n+1)\pi]$ for some integer n.

(c) This is pretty subjective. We definitely want $\cos(0) = 1$. It makes most sense to me to take $\cos(\pi/2) = 0$ and $\cos(\pi) = -1$, but you could maybe argue for $\cos(-\pi/2)$ and $\cos(-\pi)$ instead.

(d) Consequently we want to define cosine on $[0, \pi]$ to get a one-to-one function.

Problem 4. Let $f(x) = x^5 + x$.

(a) Is this function one-to-one? You won't be able to prove it directly from the definition, but you can use calculus to make a clear argument.

(b) Can you find an inverse for this function?

(c) Can you find $f^{-1}(2)$? $f^{-1}(34)$? $f^{-1}(-2)$?

(d) Can you find $(f^{-1})'(2)$?

(e) Can you find $(f^{-1})'(34)$? $(f^{-1})'(-2)$?

Solution:

(a) Yes! We see that $f'(x) = 5x^4 + 1 \geq 1$, so the function is always increasing. That means it can't repeat, and so must be one-to-one.

(b) No! I mean this in a fairly robust way. If we go ask Wolfram Alpha to find the inverse to this function, it gives the answer

$$x\,{}_2F_2\left(\frac{3 \pm 1}{10}, \frac{7 \pm 1}{10}; \frac{5}{4}, \frac{5 \pm 1}{8}; -\frac{3125x^4}{256}\right).$$

I don't know what that means, either, but it's not helpful. (It is apparently a "hypergeometric pfq".)

If we ask Mathematica to solve the equation $y = x^5 + x$ we get the even more wonderful answer that x is the solution to the polynomial $x^5 + x - y = 0$. None of this is helpful.

There definitely is an inverse. But you can't find it and neither can I.

(c) We don't have a formula, but we can still find these answers by guess-and-check. Plugging in small numbers gives

$$f(0) = 0 \qquad f(1) = 2 \qquad f(2) = 34$$
$$f(-1) = -2 \qquad f(-2) = -34.$$

Thus $f^{-1}(2) = 1$, and $f^{-1}(34) = 2$, and $f^{-1}(-2) = -1$.

(d) The inverse function theorem tells us that

$$\begin{aligned}(f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} \\ &= \frac{1}{f'(1)} = \frac{1}{5(1)^4 + 1} = \frac{1}{6}.\end{aligned}$$

(e) Again,

$$\begin{aligned}(f^{-1})'(34) &= \frac{1}{f'(f^{-1}(34))} \\ &= \frac{1}{f'(2)} = \frac{1}{5(2)^4 + 1} = \frac{1}{81} \\ (f^{-1})'(-2) &= \frac{1}{f'(f^{-1}(-2))} \\ &= \frac{1}{f'(-1)} = \frac{1}{5(-1)^4 + 1} = \frac{1}{6}.\end{aligned}$$

Problem 5. Let $g(x) = \sqrt[3]{x^3 + x + 6}$.

(a) Can you compute an inverse for g?

(b) Can you find $\left(g^{-1}\right)'(2)$?

Solution:

(a) The function is invertible, since it's increasing. You even, in theory, could find the inverse. But realistically you're not going to; the formula is:

$$g^{-1}(y) = \frac{\sqrt[3]{\frac{2}{3}}}{\sqrt[3]{-9y^3 + \sqrt{3}\sqrt{27y^6 - 324y^3 + 976} + 54}} - \frac{\sqrt[3]{-9y^3 + \sqrt{3}\sqrt{27y^6 - 324y^3 + 976} + 54}}{\sqrt[3]{2} \cdot 3^{2/3}},$$

and I wouldn't expect anyone to successfully find or work with that.

(b) This is much easier. By the Inverse Function Theorem we know that

$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}$$
$$g'(x) = \frac{1}{3}(x^3 + x + 6)^{-2/3}(3x^2 + 1).$$

We just need to find $g^{-1}(2)$, which we can, essentially, solve by guessing and checking: and it turns out that $g(1) = 2$, so $g^{-1}(2) = 1$. So we have

$$g'(1) = \frac{1}{3}(1^3 + 1 + 6)^{-2/3}(3(1)^2 + 1) = \frac{1}{3}8^{-2/3}(4) = \frac{1}{3}$$
$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(1)} = 3.$$

Problem 6. (a) Consider the functions $f(x) = x^3 - x^2 + x$ and $g(x) = x^3 - x^2 - x$. Which one is invertible and why?

(b) Consider the functions $f(x) = 3^x + x$ and $g(x) = 3^x - x$. Can you figure out which one is invertible?

Solution:

(a) We might try guess-and-check; in that case we might see that $g(1) = -1 = g(-1)$, and thus g isn't one-to-one or invertible.

For a more systematic approach, we can compute derivatives. We see that

$$f'(x) = 3x^2 - 2x + 1$$
$$g'(x) = 3x^2 - 2x - 1$$

$g'(0) = -1 < 0$ but $g'(2) = 7 > 0$ (and in fact $g'(1) = 0$). This tells us that g goes down and then back up, and so it will fail the horizontal line test.

In contrast, with a little work we can see that $f'(x) \geq 0$ for all values of x. For instance we know that $x^2 - 2x + 1 = (x-1)^2 \geq 0$ and thus $f'(x) = 2x^2 + (x-1)^2 \geq 0$. So f is always increasing, and that means that it is one-to-one.

(b) This would be easy if we knew how to compute the derivative of 3^x, but we don't yet. (Soon!)

But we *do* know that 3^x is an increasing function, and so $3^x + x$ is adding two increasing functions together and so still increasing. So this must be one-to-one.

If we look at $g(x)$, we can try plugging some numbers in. $g(0) = 1, g(1) = 2, g(2) = 7$, which all seems increasing. But $g(-1) = 1/3 + 1 = 4/3$, and so g decreases then increases again; and in particular we know that for some value of x between 0 and 1, $g(x) = 4/3$. So g is not one-to-one.