Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 10
Due Thursday, April 3

This week's mastery quiz has two topics. Everyone should submit both M3 and S8.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution: This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful: $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty} \frac{n/n^2+1}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

(b) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2+n-3}{n^2 4^n}$

Solution: We use the ratio test. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)^2+n+1-3/(n+1)^2 4^{n+1}}{n^2+n-3/n^2 4^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n^2+2n-1)n^2 4^n}{(n+1)^2 4^{n+1}(n^2+n-3)} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^4+2n^3-n^2}{4n^4+12n^3-20n-15} \right| = \frac{1}{4} < 1. \end{aligned}$$

So by the ratio test this converges absolutely..

(c) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^3+2}$

Solution: This series has positive and negative terms, but it's not alternating. We basically have to look at absolute convergence.

We consider the series

$$\sum_{n=1}^{\infty} \left| \frac{n\sin(n)}{n^3+2} \right| = \sum_{n=1}^{\infty} \frac{n|\sin(n)|}{n^3+2}.$$

We can't really use the limit comparison test here, because the $\sin(n)$ will screw it up. But we can use the usual comparison test. We know that $0 \leq |\sin(n)| \leq 1$, so

$$\frac{n|\sin(n)|}{n^3+2} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, so this series converges by the comparison test. Thus our original series converges absolutely.

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-3)^n}{\sqrt{n}}$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}} \right| = \lim_{n\to\infty} \left| \frac{(5x-3)\sqrt{n}}{\sqrt{n+1}} \right|$$
$$= |5x-3| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3| < 1$ or $-1 < 5x-3 < 1$, or $2 < 5x < 4$ or $2/5 < x < 4/5$. We need to have x in the interval $(3/5 - 1/5, 3/5 + 1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(2-3)^n}{\sqrt{n}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by p-series test

Thus the interval of convergence is $[2/5, 4/5)$.

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}(x+2)^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{((n+1)!)^2 (x+2)^{n+1}/(3n+3)!}{(n!)^2 (x+2)^n/(3n)!} \right| = \lim_{n\to\infty} |x+2| \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \leq \frac{|x+2|}{n} = 0$$

for any x. So the radius of convergence is infinity, and this converges for all x.