Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 11
Due Thursday, April 10

This week's mastery quiz has three topics. Everyone should submit M4. If you have a 4/4 on M3 or a 2/2 on S8, you don't need to submit them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$

Solution: We can work this out with the integral test. We have

$$\begin{aligned}\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\ &= \lim_{t\to\infty}\frac{-1}{2}e^{-x^2}\Big|_1^t \\ &= \lim_{t\to\infty}\frac{1}{2e}-\frac{1}{2e^{t^2}} = \frac{1}{2e} < \infty.\end{aligned}$$

Since this integral converges, the series must also converge by the integral test.

(b) Analyze the convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{2n+3}$

Solution: This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty}\frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{n}{2n+3} = 1/2.$$

Since the series $\sum\frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum\frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

(c) Analyze the convergence of the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^3+n}$

Solution: We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)^3+n+1}{(-2)^n/n^3+n}\right| &= \lim_{n\to\infty}\frac{2(n^3+n)}{(n+1)^3+n+1} \\ &= \lim_{n\to\infty} 2 > 1.\end{aligned}$$

This limit is greater than 1, so by the ratio test this diverges.

Alternatively, we could note that

$$\lim_{n\to\infty} \frac{(-2)^n}{n^3+n} = \pm\infty,$$

so by the divergence test this diverges. But it's a little tricky to cleanly argue that this goes to infinity; we can't really use L'Hospital's rule without getting the negative sign out of there somehow.

M4: Taylor Series

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered at 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}
\frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty}(-4x)^n \\
\frac{2x^2}{1+4x} &= 2x^2\sum_{n=0}^{\infty}(-4)^n x^n \\
&= \sum_{n=0}^{\infty} 2\cdot(-4)^n x^{n+2} \\
\text{(or)} \quad &= \sum_{n=2}^{\infty} 2^{2n-3}(-1)^n x^n.
\end{aligned}$$

The radius of convergence is $1/4$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n}\right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $x^3 e^{(x^5/4)}$, and then write down the first four non-zero terms of this series.

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$e^{x^5/4} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^5/4)^n = \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n}$$
$$x^3 e^{x^5/4} = \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n+3}$$

The first four non-zero terms are

$$x^3 + \frac{1}{4}x^8 + \frac{1}{32}x^{13} + \frac{1}{6 \cdot 64}x^{18}.$$

(Note: this is *not* T_3 or T_4. It's T_{18}!)

(c) If $f(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}(x+2)^n$, compute $\frac{d}{dx} f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{3^n}{(n-1)!}(x+2)^{n-1} (\text{or much much better}) \quad = \sum_{n=1}^{\infty} \frac{3^n}{(n-1)!}(x+2)^{n-1}$$
$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}(x+2)^{n+1} + C$$
$$(\text{or}) = \sum_{n=1}^{\infty} \frac{3^{n-1}}{n!}(x+2)^n + C.$$

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{n}{5^n}(x-3)^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}/5^{n+1}}{(n)(x-3)^n/5^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x-3}{5} \right|$$
$$= |x-3|/5 \lim_{n\to\infty} \frac{n+1}{n} = |x-3|/5.$$

So we need $|x-3|/5 < 1$ or $-5 < x-3 < 5$, or $-2 < x < 8$ or $3-5 < x < 3+5$. So the radius is 5.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{n}{5^n} 5^n = \sum_{n=0}^{\infty} n$$

diverges by divergence or p-series test

$$\sum_{n=0}^{\infty} \frac{n}{5^n} (-5)^n = \sum_{n=0}^{\infty} (-1)^n n$$

diverges by divergence test

Thus the interval is $(-2, 8)$.

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} (n(x-3))^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}(x-3)^{n+1}}{n^n (x-3)^n} \right| = \lim_{n\to\infty} |x-3| \left(\frac{n+1}{n} \right)^n (n+1) \quad \geq |x-3| \lim_{n\to\infty} (n+1) = \infty$$

unless $x = 3$. So the radius of convergence is 0, and the series converges if and only if $x = 3$.