Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 12
Due Thursday, April 17

This week's mastery quiz has three topics. Everyone should submit M4 and S9. If you have a 4/4 on M3, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n + 1}$

Solution: We use the Ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}3^{n+1}/5^{n+1}+1}{(-1)^n 3^n/5^n+1} \right| = \lim_{n\to\infty} \frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)}$$
$$= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1}$$
$$= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.$$

This limit is less than 1, so by the ratio test this converges absolutely.

(b) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n + 1}$

Solution: You might try the ratio test here, but it won't actually help:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}/(n+1)2^{n+1}+1}{(-2)^n/n2^n+1} \right| = \lim_{n\to\infty} \frac{2(n2^n+1)}{(n+1)2^{n+1}+1} = \lim_{n\to\infty} \frac{n+1/2^{n+1}}{n+1+1/2^{n+1}} = 1.$$

Instead, we observe that this is an alternating series with the terms tending to zero, since

$$\lim_{n\to\infty} \frac{(-2)^n}{n2^n+1} = \lim_{n\to\infty} \frac{(-1)^n}{n+1/2^n} = 0.$$

Thus it converges. However, if we look at the absolute value, we can compare it to the series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{2^n/n2^n+1}{1/n} = \lim_{n\to\infty} \frac{n2^n}{n2^n+1} = 1$$

and since $\sum \frac{1}{n}$ diverges, by the limit comparison test our absolute-value series also diverges. Thus the original series converges conditionally.

(c) $\sum_{n=4}^{\infty} \frac{(-1)^n}{(n^2)/5 + 3n}$

Solution: This clearly converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2/5-3n} = 0$, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n}{n^2/5+3n} \right| = \sum_{n=4}^{\infty} \frac{1}{n^2/5+3n},$$

so we can use the Limit Comparison Test to $\frac{1}{n^2}$. We compute

$$\lim_{n\to\infty} \frac{\frac{1}{n^2/5+3n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2/5+3n} = 1/5.$$

This is a nonzero real number, so since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges, by the Limit Comparison Test, $\sum_{n=4}^{\infty} \frac{1}{n^2/5+3n}$ converges. Thus our original series converges absolutely. (And thus we don't actually need to check for whether the alternating series test applies.)

M4: Taylor Series

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $(1+3x)^{2/3}$, and then write down the degree-three polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{2/3}{n} (3x)^n = \sum_{n=0}^{\infty} \binom{2/3}{n} (3)^n x^n \\ T_3(x,0) &= 1 + \frac{2/3}{1} \cdot 3x + \frac{(2/3)(-1/3)}{2} \cdot 3^2 x^2 + \frac{(2/3)(-1/3)(-4/3)}{6} \cdot 3^3 x^3 \\ &= 1 + 2x - x^2 + \frac{4}{3} x^3. \end{aligned}$$

(b) In class we computed a Taylor series for $\sin(x)$ centered at zero. Use the degree-seven Taylor polynomial to approximate $\sin(3) \approx T_7(3,0)$. (You don't need to numerically simplify this.)

Using the Taylor series remainder, find an upper bound for the error in this approximation.

Solution: We know that

$$\begin{aligned} \sin(x) &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ T_7(x,0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ T_7(x,3) &= 3 - \frac{27}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} = 3 - \frac{37}{6} + \frac{243}{120} - \frac{2187}{5040} \\ &= 3 - \frac{9}{2} + \frac{81}{40} - \frac{243}{560} = \frac{51}{560} \approx 0.091. \end{aligned}$$

We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$|R_n(x)| = \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}$$
$$|R_7(x)| \leq \frac{x^{7+1}}{(7+1)!}$$
$$|R_7(3)| \leq \frac{3^8}{8!} = \frac{729}{4480} \approx 0.16.$$

It would also be okay to observe that the eighth term is zero, so we could actually compute

$$|R_n(x)| = \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}$$
$$|R_8(x)| \leq \frac{x^{8+1}}{(8+1)!}$$
$$|R_8(3)| \leq \frac{3^9}{9!} = \frac{243}{4480} \approx 0.054.$$

(c) Write a power series expression for $\frac{x}{2+x^2}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^n$$
$$\frac{1}{2+x^2} = \frac{1}{2}\frac{1}{1-(-x^2/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-x^2/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n}$$
$$\frac{x}{2+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}.$$

The radius of convergence is $\sqrt{2}$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < x^2/2 < 1$ or $-2 < x^2 < 2$ or $-\sqrt{2} < x < \sqrt{2}$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}/2^{n+2}}{x^{2n+1}/2^{n+1}}\right| = \lim_{n\to\infty}\frac{|x|^2}{2}$$

and thus it converges when $x^2/2 < 1$.

S9: Applications of Taylor Series

(a) Use a Taylor series to compute $\lim_{x\to 0}\frac{\cos(x^2) - 1 + x^4/2}{x^8} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos(x^2)-1+x^4/2}{x^8} &= \lim_{x\to 0}\frac{(1-x^4/2+x^8/4!-x^{12}/6!+\dots)-1+x^4/2}{x^8}\\
&= \lim_{x\to 0}\frac{x^8/4!-x^{12}/6!+\dots}{x^8}\\
&= \lim_{x\to 0}\frac{1}{4!}-\frac{x^4}{6!}+\cdots=\frac{1}{24}.
\end{aligned}$$

(b) Using series, compute $\int_0^\pi 2x\cos(x^5)\,dx$.

Solution:

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\\
\cos(x^5) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{10n}\\
2x\cos(x^5) &= \sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n)!}x^{10n+1}\\
\int 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n)!(10n+2)}x^{10n+2}+C\\
\int_0^\pi 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n)!(10n+2)}\pi^{10n+2}
\end{aligned}$$

(c) Use a degree-five Taylor polynomial to estimate sin(.3).

Solution: We have

$$\begin{aligned}
\sin(x) &\approx x-x^3/6+x^5/120\\
\sin(.3) &\approx .3-.3^3/6+.3^5/120\approx .29552.
\end{aligned}$$