Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 12
Due Thursday, April 17

This week's mastery quiz has three topics. Everyone should submit S10. If you have a 4/4 on M4, or a 2/2 on S9, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series
- Secondary Topic 10: Parametrization

Name:

Recitation Section:

M4: Taylor Series

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $e^{3x} - e^x$, and then write down the degree-three polynomial explicitly.

Solution: We can take this from the known series for e^x. So we have

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{3x} &= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n \\ e^{3x} - e^x &= \sum_{n=0}^{\infty} \frac{3^n - 1}{n!} x^n \\ T_3(x,0) &= 0 + \frac{2}{1}x + \frac{8}{2}x^2 + \frac{26}{6}x^3 \\ &= 2x + 4x^2 + \frac{13}{3}x^3. \end{aligned}$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $x^2 \ln(1 - 2x^3)$, and then write down the degree-eleven polynomial explicitly.

Solution: We can take this from the series for $\ln(1 + x)$. So we have

$$\begin{aligned} \ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \\ \ln(1-2x^3) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-2x^3)^n}{n} \\ x^2 \ln(1-2x^3) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n 2^n x^{3n+2}}{n} \\ &= \sum_{n=1}^{\infty} \frac{-2^n}{n} x^{3n+2} T_{11}(x,0) \qquad = -2x^5 - 2x^8 - \frac{8}{3}x^{11}. \end{aligned}$$

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $(1 - 2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{-3}{n} (-2x)^n = \sum_{n=0}^{\infty} \binom{-3}{n} (-2)^n x^n \\ T_4(x,0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4 \\ &= 1 + 6x + 24x^2 + 80x^3 + 240x^4 \end{aligned}$$

S9: Applications of Taylor Series

(a) Use a Taylor series to compute $\lim_{x\to 0} \dfrac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} &= \lim_{x\to 0} \frac{(x + x^4 + x^7/2 + x^{10}/3! + \dots) - x + x^4}{x^7} \\ &= \lim_{x\to 0} \frac{x^7/2 + x^{10}/3! + \dots}{x^7} \\ &= \lim_{x\to 0} \frac{1}{2} - \frac{x^3}{3!} + \dots = \frac{1}{2}.\end{aligned}$$

(b) If $f(x) = \sum_{n=0}^{\infty} \dfrac{n+1}{n!+1} x^n$, compute $\int_3^5 f(x)$.

Solution:

$$\begin{aligned}\int f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!+1} x^{n+1} + C \\ \int_3^5 f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!+1} (5^{n+1} - 3^{n+1}).\end{aligned}$$

(c) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$\begin{aligned}(1.1)^{3.1} &\approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2} x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3} x^3 \\ &= 1 + 3.1x + 3.255x^2 + 1.1935x^3\end{aligned}$$

$$(1.1)^{3.1} \approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435.$$

S10: Parametrization

(a) Find a **clockwise** parametrization for an ellipse 10 units wide and 6 units tall (thus with major radius 5 and minor radius 3), centered at the point $(5, 2)$.

Solution: We can parametrize a circle clockwise with $\vec{r}(t) = (\sin(t), \cos(t))$. Thus our ellipse can be

$$\vec{r}(t) = \big(5\sin(t) + 5, 3\cos(t) + 2\big).$$

Alternate solution:

$$\vec{r}(t) = \big(5\cos(-t) + 5, 3\sin(-t) + 2\big)$$

(b) Find a (non-parametric) equation of the line tangent to the curve parametrized by $x = \cos^3(t), y = \sin^3(t)$ at the point $(1/8, -3\sqrt{3}/8)$.

Solution: We have $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$. This point happens at $t = -\pi/3$, so we have

$$\begin{aligned} x'(-\pi/3) &= 3\sqrt{3}/8 \\ y'(-\pi/3) &= 9/8 \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{9/8}{3\sqrt{3}/8} = \sqrt{3} \\ y + \frac{3\sqrt{3}}{8} &= \sqrt{3}(x - 1/8). \end{aligned}$$

(c) Find the area inside the cardioid $r = 1 + \cos(\theta)$.

Solution: We have the polar area formula

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(1+\cos(\theta))^2 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} + \cos(\theta) + \frac{1}{2}\cos^2(\theta) \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta) \, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{2}\left((2\pi + 0 + \pi + 0) - (0 + 0 + 0 + 0)\right) = \frac{3\pi}{2}. \end{aligned}$$