Math 1232: Single-Variable Calculus 2
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Recitation 13

Jay Daigle

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Problem 1. We want to approximate $\sqrt[5]{x}$ near $a = 32$.

(a) We can try the brute force approach. Use derivatives to directly compute $T_2(x, 32)$ centered at 32.

(b) We don't like brute force; we want to use a series we already know instead. What series should we be looking at? Why is it hard to use directly?

(c) Consider the function $f(x) = (32 + x)^{1/5}$. Will this let us estimate $\sqrt[5]{x}$ near 32? Can we modify it to look like the binomial series?

(d) Write down a series approximation for $f(x)$ using the binomial series.

(e) Work out T_2. Is this the same as your answer in part (a)?

(f) Use a degree-two polynomial to estimate $\sqrt[5]{36}$.

(g) Use the same basic approach to Compute a degree-two approximation of $\sqrt[4]{78}$.

Problem 2. Use a Taylor series to find $\lim_{x \to 0} \frac{\ln(1 + x^2) - x^2 + x^4/2}{x^6}$.

Problem 3. Suppose we want to find a maximum value for $\cos(x^2)$.

(a) Take a derivative and look for critical points. There should be lots of them, but what's the smallest one (the one closest to 0)?

(b) Take a second derivative and do the second derivative test. What does that tell you?

(c) Compute T_2, using the derivative definition. What do you get? What does that tell you about whether this is a max or min?

(d) Now find a formula for the Taylor series. (Hint: this should be easy.)

(e) Write out the first few terms of the Taylor series explicitly. What does this tell you about the shape of the graph?

Problem 4. In class we worked out a Taylor series for $g(x) = \ln(x)$ centered at $a = 1$:

$$T_g(x, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

But is this actually equal to $g(x)$?

(a) Write down a formula for $R_k(x, 1)$.

(b) Compute $T_5(2, 1)$. Can you estimate the error?

(c) Compute $T_5(1.5, 1)$. Can you estimate the error?

(d) Compute $T_5(0, 1)$. Can you estimate the error here?

(e) What would you need to assume to show this goes to zero as k goes to infinity? Does that makes sense?

Problem 5 (Special Relativity)**.** Relativity includes a number of interesting phenomena that occur when your velocity is relatively large compared to the speed of light. But we know that at low velocities, special relativity should "look like" Newtonian mechanics.

(a) Most of the relativity equations feature a variable γ ("gamma", the Greek letter "g"), given by $\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}$ where c is the speed of light. Find a formula for the Taylor series for $\gamma(v)$ centered at $v = 0$.

(b) What is the first-order approximation $T_1(v, 0)$? What is the second-order approximation $T_2(v, 0)$?? When do we expect these to be accurate?

(c) The formula $E = mc^2$ is famous, but it's actually incomplete; it gives energy of an object at rest. The energy of a moving object is $E(v) = mc^2\gamma(v)$.

What is the first-order approximation to this formula? What is the second-order approximation? Do you recognize that formula from elsewhere, and does it make sense to you?

(d) In special relativity, time is dilated: the faster you're moving, the more slowly you experience time. Specifically, the time is dilated by a factor of γ.

What is a first-order approximation to the amount of dilation you experience? Does that answer make sense? Why?