Math 1232: Single-Variable Calculus 2
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Recitation 13

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Problem 1. We want to approximate $\sqrt[5]{x}$ near $a = 32$.

(a) We can try the brute force approach. Use derivatives to directly compute $T_2(x, 32)$ centered at 32.

(b) We don't like brute force; we want to use a series we already know instead. What series should we be looking at? Why is it hard to use directly?

(c) Consider the function $f(x) = (32 + x)^{1/5}$. Will this let us estimate $\sqrt[5]{x}$ near 32? Can we modify it to look like the binomial series?

(d) Write down a series approximation for $f(x)$ using the binomial series.

(e) Work out T_2. Is this the same as your answer in part (a)?

(f) Use a degree-two polynomial to estimate $\sqrt[5]{36}$.

(g) Use the same basic approach to Compute a degree-two approximation of $\sqrt[4]{78}$.

Solution:

(a) We compute:

$$
\begin{aligned}
f(x) &= \sqrt[5]{x} & f(32) &= 2 \\
f'(x) &= \frac{1}{5}x^{-4/5} & f'(32) &= \frac{1}{80} \\
f''(x) &= \frac{-4}{25}x^{-9/5} & f''(32) &= \frac{-1}{3200}
\end{aligned}
$$

and thus

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(32)}{n!} x^n$$
$$\approx 2 + \frac{x-32}{80} - \frac{(x-32)^2 2}{3200 \cdot 2}.$$

(b) This looks kinda like the binomial series, with $\alpha = 1/5$. But the binomial series tells us about inputs near 1, not near 32.

(c) We can factor out a two and get somtehing that works quite well:

$$\sqrt[5]{32+x} = 2(1 + \tfrac{x}{32})^{1/5}.$$

(d)

$$\sqrt[5]{32+x} = 2(1+\tfrac{x}{32})^{1/5} = 2\sum_{n=0}^{\infty}\binom{1/5}{n}x^n$$
$$= 2 + \frac{2}{5}(x/32) - \frac{4}{25}(x/32)^2 + \ldots$$

(Note this converges when $|x/32| < 1$ and thus when $-32 < x < 32$).

(e) We get

$$T_2 = 2 + \frac{x}{80} - \frac{x^2}{6400},$$

which is the same as our answer in part (a).

(f) We can estimate

$$\sqrt[5]{36} \approx 2 + \frac{4}{80} - \frac{16}{6400} = 2 + \frac{1}{16} - \frac{1}{800} \approx 2.06.$$

(g) We set $g(x) = (81+x)^{1/4} = 3(1+x/81)^{1/4}$. Then

$$(1+x/81)^{1/4} = \sum_{n=0}^{\infty}\binom{1/4}{81}(x/81)^n = 1 + \frac{1}{4}\frac{x}{81} + \frac{(1/4)(-3/4)}{2}\frac{x^2}{81^2} + \ldots$$
$$3(1+x/81)^{1/4} = \sum_{n=0}^{\infty} 3\binom{1/4}{81}(x/81)^n = 3 + \frac{3}{4}\frac{x}{81} + \frac{(3/4)(-3/4)}{2}\frac{x^2}{81^2} + \ldots$$
$$= 3 + \frac{3x}{324} - \frac{9x^2}{32 \cdot 81^2} + \ldots$$
$$\sqrt[4]{78} = g(-3) \approx 3 + \frac{-9}{324} - \frac{81}{209952}$$
$$= 77032592 \approx 2.97184.$$

In fact the true answer is about 2.97183.

Problem 2. Use a Taylor series to find $\lim_{x\to 0} \frac{\ln(1+x^2) - x^2 + x^4/2}{x^6}$.

Solution: We know

$$\begin{aligned}
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \\
\ln(1+x^2) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots \\
\ln(1+x^2) - x^2 + x^4/2 &= \frac{x^6}{3} - \frac{x^8}{4} + \dots \\
\frac{\ln(1+x^2) - x^2 + x^4/2}{x^6} &= \frac{x^6/3 - x^8/4 + \dots}{x^6} \\
&= \frac{1}{3} - \frac{x^2}{4} + \dots \\
\lim_{n\to\infty} \frac{\ln(1+x^2) - x^2 + x^4/2}{x^6} &= \lim_{n\to\infty} \frac{1}{3} - \frac{x^2}{4} + \dots = 1/3.
\end{aligned}$$

Problem 3. Suppose we want to find a maximum value for $\cos(x^2)$.

(a) Take a derivative and look for critical points. There should be lots of them, but what's the smallest one (the one closest to 0)?

(b) Take a second derivative and do the second derivative test. What does that tell you?

(c) Compute T_2, using the derivative definition. What do you get? What does that tell you about whether this is a max or min?

(d) Now find a formula for the Taylor series. (Hint: this should be easy.)

(e) Write out the first few terms of the Taylor series explicitly. What does this tell you about the shape of the graph?

Solution:

(a) $f'(x) = -2x\sin(x^2)$ has many zeroes, but the smallest one is at zero itself. So we can take $c = 0$.

(b) The second derivative is $f''(x) = -2\sin(x^2) - 4x^2\cos(x)$, which is also zero, so the second derivative test doesn't tell us anything.

(c) $f(0) = 1, f'(0) = 0, f''(0) = 0$, so $T_2 = 1$. This is a flat line and tells us nothing.

(d) We know

$$\cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$
$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!}$$

(e) The Taylor series is

$$1 - \frac{x^4}{2} + \frac{x^8}{24} + \cdots \approx 1 - \frac{x^4}{2}.$$

Thus the function looks roughly like $1 - x^4/2$ near $a = 0$, and so it has a maximum at zero.

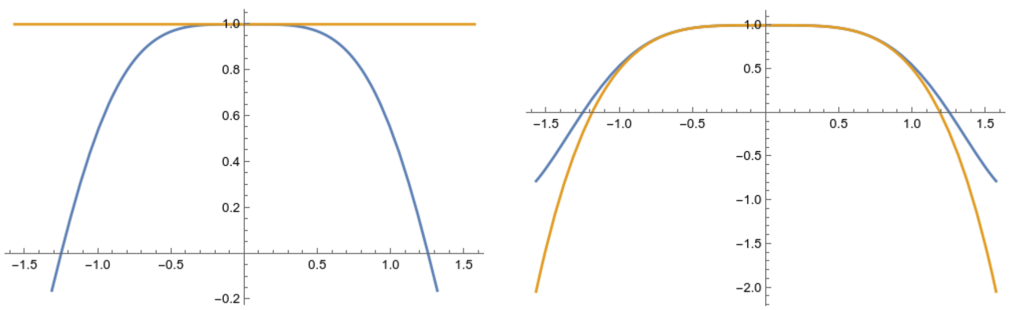

Figure 0.1: The second-order Taylor expansion on the left isn't very helpful, but the fourth-order Taylor expansion shows our function has a maximum at 0..

Problem 4. In class we worked out a Taylor series for $g(x) = \ln(x)$ centered at $a = 1$:

$$T_g(x, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

But is this actually equal to $g(x)$?

(a) Write down a formula for $R_k(x, 1)$.

(b) Compute $T_5(2, 1)$. Can you estimate the error?

(c) Compute $T_5(1.5, 1)$. Can you estimate the error?

(d) Compute $T_5(0, 1)$. Can you estimate the error here?

(e) What would you need to assume to show this goes to zero as k goes to infinity? Does that makes sense?

Solution:

(a) We know

$$R_k(x,1) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-1)^{k+1}.$$

We worked out in class that $f^{(k+1)}(x) = (-1)^k k! x^{-k-1}$, which tells us that $f^{(k+1)}(z) = \frac{(-1)^k k!}{z^{k+1}}$. Thus we have

$$R_k(x,1) = \frac{(-1)^k k!}{(k+1)! z^{k+1}} x^{k+1} = \frac{(-1)^k}{(k+1)} \frac{(x-1)^{k+1}}{z^{k+1}}$$

where z is somewhere in between 1 and x.

(b) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(2,1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60} \approx .7833.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6} \frac{(x-1)^6}{z^6}$$
$$R_5(2,1) = \frac{-1}{6} \cdot \frac{1}{z^6}.$$

We need z to between 1 and 2, so this is maximized at $\frac{-1}{6}$. So the error is at most 1/6.

(c) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(1.5,1) = .5 - \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{8} - \frac{1}{4} \cdot \frac{1}{16} + \frac{1}{5} \cdot \frac{1}{32}$$
$$= \frac{391}{960} \approx .4073.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6} \frac{(x-1)^6}{z^6}$$
$$R_5(1.5,1) = \frac{-1}{6} \cdot \frac{.5^6}{z^6} = \frac{-1}{384 z^6}$$

We need z to between 1 and 1.5, so this is maximized at $\frac{-1}{385}$. So the error is at most 1/384. This is pretty good!

(d) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(0,1) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} = \frac{-135}{60} \approx -2.28333.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6}\frac{(x-1)^6}{z^6}$$
$$R_5(0,1) = \frac{-1}{6} \cdot \frac{1}{z^6} = \frac{-1}{6z^6}.$$

This looks like our answer from part (b), but it really isn't. this time we need z between 0 and 1, so this is maximized when z is as small as possible—this error bound could be extremely big.

And that makes sense, because the power series doesn't converge here. And the function goes to infinity here. Our error "really is" infinity—we can't get a good bound.

(e) We need $x - 1$ to be small. So in particular, if we assume that x is between 1 and 2, then we have

$$|R_k(x,1)| = \left|\frac{(-1)^k}{(k+1)}\frac{(x-1)^{k+1}}{z^{k+1}}\right|$$
$$\leq \frac{1}{k+1}\frac{1}{z^{k+1}}$$

and since z has to be between 1 and 2, this is $\leq \frac{1}{k+1}$, and the error tends to zero as k tends to infinity.

If $x > 2$, we can't show this error is controlled. And that makes sense—because we know the series doesn't converge for $x > 2$, from our theory of geometric series.

Problem 5 (Special Relativity)**.** Relativity includes a number of interesting phenomena that occur when your velocity is relatively large compared to the speed of light. But we know that at low velocities, special relativity should "look like" Newtonian mechanics.

(a) Most of the relativity equations feature a variable γ ("gamma", the Greek letter "g"), given by $\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}$ where c is the speed of light. Find a formula for the Taylor series for $\gamma(v)$ centered at $v = 0$.

(b) What is the first-order approximation $T_1(v,0)$? What is the second-order approximation $T_2(v,0)$?? When do we expect these to be accurate?

(c) The formula $E = mc^2$ is famous, but it's actually incomplete; it gives energy of an object at rest. The energy of a moving object is $E(v) = mc^2\gamma(v)$.

What is the first-order approximation to this formula? What is the second-order approximation? Do you recognize that formula from elsewhere, and does it make sense to you?

(d) In special relativity, time is dilated: the faster you're moving, the more slowly you experience time. Specifically, the time is dilated by a factor of γ.

What is a first-order approximation to the amount of dilation you experience? Does that answer make sense? Why?

Solution:

(a) We know $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$. We'd like to use the binomial expansion, so we write

$$\begin{aligned}\gamma(v) = (1-(v/c)^2)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{-v^2}{c^2}\right)^n \\ &= \sum_{n=0} \binom{-1/2}{n} \frac{(-1)^n}{c^{2n}} v^{2n}.\end{aligned}$$

(b) We have $T_1(v, 0) = 1 + 0v = 1$, and

$$T_2(v, 0) = 1 + \frac{1}{2}\frac{v^2}{c^2}.$$

(c) Since $T_1(v, 0) = 1$, the first-order approximation to $E(v)$ is just mc^2, which is the famous formula.

The second order approximation is

$$E(v) \approx mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2.$$

The second term, $\frac{1}{2}mv^2$, is the classical formula for kinetic energy! So the second-order approximation to this formula just gives us classical, Newtonian physics.

(d) The first-order approximation to $\gamma(v)$ is just 1, so to the first order your time dilation is...a factor of one. Which makes sense; at normal speed we don't really notice any dilation at all.