Math 1232: Single-Variable Calculus 2 George Washington University Spring 2025 Recitation 14

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Problem 1. Consider the curve $\vec{r}(t) = \left(\frac{t}{1+t}, \ln(1+t)\right)$.

- (a) At what time does this curve pass through the origin?
- (b) Does this curve hit the point $(2, \ln(3))$?
- (c) Does it hit the point $(1/2, \ln(2))$?
- (d) Try to sketch a graph of this curve. What do you know about it?
- (e) Find a parametric equation for the tangent line to the curve at the time t = 3. Find an implicit equation for the same line.
- (f) Set up an integral to compute the length of the curve for $0 \le 2 \le 2$?

Solution:

- (a) t = 0.
- (b) No. We can either compute that if $y = \ln(3)$ then t = 2 so x = 2/3, or that if x = 2 we have 2 + 2t = t so t = -2 and $y = \ln(-1)$ is undefined.
- (c) Yes. We can either compute that if $y = \ln(2)$ then t = 1 so x = 1/2, or that if x = 1/2then 1/2 + t/2 = t so t = 1 and thus $\ln(1 + t) = \ln(2)$.



(e) We have

$$x(3) = 3/4$$

$$y(3) = \ln(4)$$

$$x'(t) = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y'(t) = \frac{1}{1+t}$$

$$x'(3) = \frac{1}{16}$$

$$y'(3) = \frac{1}{4}$$

So we get the parametric equation

$$T(t) = (3/4, \ln(4)) + t(1/16, 1/4),$$

or we can compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/4}{1/16} = 4$$

and get the implicit equation

$$y - \ln(4) = 4(x - 3/4).$$

(f) The arc length formula is

$$L = \int_0^2 \sqrt{x'(t)^2 + y'(t)^2} dt$$
$$= \int_0^2 \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}} dt$$

Problem 2. Let $\vec{r}(t) = (\cos^3(t), \sin^3(t))$.

- (a) Find the length of the curve for $0 \le t \le 2$.
- (b) Did you get zero? Does that make any sense?
- (c) Where did that go wrong? Can you fix it?

Solution: g The obvious calculation is

$$\begin{aligned} x'(t) &= 3\cos^2(t)(-\sin(t))\\ y'(t) &= 3\sin^2(t)\cos(t)\\ L &= \int_0^{2\pi} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} \, dt\\ &= \int_0^{2\pi} 3\sin(t)\cos(t)\sqrt{\cos^2(t) + \sin^2(t)} \, dt\\ &= \int_0^{2\pi} 3\sin(t)\cos(t) \, dt = \frac{3}{2}\sin^2(t)\big|_0^{2\pi} = 0 - 0 = 0. \end{aligned}$$

But this doesn't make sense; the length shouldn't be zero!

We screwed up when we said $\sqrt{\sin^2(t)\cos^2(t)} = \sin(t)\cos(t)$. That only applies when the product is positive, and in this problem that really matters. So instead we want to compute

$$\int_0^{2\pi} 3|\sin(t)\cos(t)|\,dt.$$

The only reasonable way to do that is to split it up into pieces:

$$\int_{0}^{2\pi} 3|\sin(t)\cos(t)| dt = \int_{0}^{\pi/2} 3\sin(t)\cos(t) dt - \int_{\pi/2}^{\pi} 3\sin(t)\cos(t) dt + \int_{\pi}^{3\pi/2} 3\sin(t)\cos(t) dt - \int_{3\pi/2}^{2\pi} 3\sin(t)\cos(t) dt = \frac{3}{2}\sin^{2}(t)|_{0}^{\pi/2} - \frac{3}{2}\sin^{2}(t)|_{\pi/2}^{\pi} + \frac{3}{2}\sin^{2}(t)|_{\pi}^{3\pi/2} - \frac{3}{2}\sin^{2}(t)|_{3\pi/2}^{2\pi} = \left(\frac{3}{2} - 0\right) - \left(0 - \frac{3}{2}\right) + \left(\frac{3}{2} - 0\right) - \left(0 - \frac{3}{2}\right) = 6.$$



Problem 3. Consider the polar curve $r = e^{\theta}$.

- (a) Sketch a graph of this curve.
- (b) At what points (r, θ) does this intersect the x-axis?
- (c) What are the Cartesian coordinates of the point where $\theta = 4\pi/3$?
- (d) Can we write this curve as a parametric equation?
- (e) Find the points (r, θ) where the tangent line is horizontal.
- (f) Find the points (r, θ) where the tangent line is vertical.

Solution:

(a) Counterclockwise spiral.



(b) We would need $\theta = 0$ or $\theta = \pi$, so we have (1,0) and (e^{π}, π) . (It's not *wrong* to look at $\theta \ge 2\pi$ but we don't generally by default.)

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(c)

$$x = r\cos(\theta) = e^{4\pi/3}\cos(4\pi/3) = -\frac{1}{2}e^{4\pi/3}$$
$$y = r\sin(\theta) = e^{4\pi/3}\sin(4\pi/3) = \frac{\sqrt{3}}{2}e^{4\pi/3}.$$

(d)

$$x(\theta) = r\cos(\theta) = e^{\theta}\cos(\theta)$$
$$y(\theta) = r\sin(\theta) = e^{\theta}\sin(\theta)$$

(e)

$$\frac{dx}{d\theta} = e^{\theta} \cos(\theta) - e^{\theta} \sin(\theta)$$
$$\frac{dy}{d\theta} = e^{\theta} \sin(\theta) + e^{\theta} \cos(\theta)$$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta) + \sin(\theta)}{\cos(\theta) - \sin(\theta)}.$$

So we have a horizontal tangent line when

$$\cos(\theta) + \sin(\theta) = 0$$
$$\cos(\theta) = -\sin(\theta)$$

And thus $\theta = 3\pi/4$ or $\theta = 7\pi/4$.

(f) We have a horizontal tangent line when

$$\cos(\theta) - \sin(\theta) = 0$$
$$\cos(\theta) = \sin(\theta)$$

And thus $\theta = \pi/4$ or $\theta = 5\pi/4$.

- **Problem 4.** (a) Find the area under the curve $\vec{r}(t) = (\cos(t), e^t)$ and above the line y = 1 for $x \ge 0$.
 - (b) Find the area enclosed by one petal of $r = 3\cos(2\theta)$.

Solution:

(a) We know our area formula is $\int y \, dx$. In this case the height is $e^t - 1$ and we have $dx = -\sin(t) \, dt$. A little work shows that x = 0 when $t = \pi/2$ and y = 1 when t = 0 so our bounds are $0, \pi/2$. But $\pi/2$ is on the left, so that's our bottom bound! Then we have

$$A = \int_{\pi/2}^{0} (e^{t} - 1)(-\sin(t)) dt = \int_{\pi/2}^{0} -\sin(t)e^{t} + \sin(t) dt.$$
$$\int_{\pi/2}^{0} -\sin(t)e^{t} dt = \cos(t)e^{t}|_{\pi/2}^{0} - \int_{\pi/2}^{0} \cos(t)e^{t} dt$$
$$= 1 - \left(\sin(t)e^{t}|_{\pi/2}^{0} - \int_{\pi/2}^{0} \sin(t)e^{t} dt\right)$$
$$= 1 + e^{pi/2} + \int_{\pi/2}^{0} \sin(t)e^{t} dt$$
$$-2 \int_{\pi/2}^{0} \sin(t)e^{t} dt = 1 + e^{\pi/2}$$
$$\int_{\pi/2}^{0} -\sin(t)e^{t} dt = \frac{1 + e^{\pi/2}}{2}.$$
$$\int_{\pi/2}^{0} \sin(t) dt = -\cos(t)|_{\pi/2}^{0} = -1$$
$$A = \int_{\pi/2}^{0} (e^{t} - 1)(-\sin(t)) dt = \int_{\pi/2}^{0} -\sin(t)e^{t} + \sin(t) dt$$
$$= \frac{1 + e^{\pi/2}}{2} - 1 = \frac{e^{\pi/2} - 1}{2}.$$

(b) The function intersects the origin at $2\theta = -\pi/2, \pi/2, 3\pi/2, \ldots$ and so at $\theta = -\pi/4, \pi/4, 3\pi/4, \ldots$. So we will integrate from $-\pi/4$ to $\pi/4$.

We know that polar area is found by $a = \int_a^b \frac{1}{2} r^2 d\theta$. So we get

$$\begin{split} A &= \int_{-\pi/4}^{\pi/4} \frac{9}{2} \cos^2(2\theta) \, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{9}{2} \cdot \frac{1 + \cos(4\theta)}{2} \, d\theta \\ &= \frac{9}{4} \left(\theta + \frac{1}{4} \sin(4\theta) \mid \right) \Big|_{-\pi/4}^{\pi/4} \\ &= \frac{9}{4} \left(\pi/4 + 0 - (-\pi/4 + 0) \right) \qquad \qquad = \frac{9\pi}{8}. \end{split}$$

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