Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 4
Due Thursday, February 13

This week's mastery quiz has three topics. Everyone should submit M2. If you have a 4/4 on M1 or a 2/2 on S2, you don't need to submit that topic.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 2: L'Hospital's Rule

Name:

Recitation Section:

M1: Transcendental Functions

(a) Compute $\frac{d}{dx}\cos(x)^{\sin(x)}$

Solution:

$$\begin{aligned} y &= \cos(x)^{\sin(x)} \\ ln|y| &= \sin(x)\ln|\cos(x)| \\ y'/y &= \cos(x)\ln|cos(x)| + \sin(x)\frac{-\sin(x)}{\cos(x)} \\ y' &= \cos(x)^{-\sin(x)}\left(\cos(x)\ln|\cos(x)| - \frac{\sin^2(x)}{\cos(x)}\right) \end{aligned}$$

(b) Compute $\int \frac{\sin(2t)+\cos(2t)}{\sin(2t)-\cos(2t)}\,dt$.

Solution: We can take $u = \sin(2t) - \cos(2t)$. Then $du = (2\cos(2t) + 2\sin(2t))dt$, and we get

$$\begin{aligned} \int \frac{\sin(2t)+\cos(2t)}{\sin(2t)-\cos(2t)}\,dt &= \int \frac{\sin(2t)+\cos(2t)}{u}\frac{du}{2\cos(2t)+2\sin(2t)} \\ &= \int \frac{1}{2}u\,du = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|\sin(2t)-\cos(2t)| + C. \end{aligned}$$

(Note this is a definite integral)

$\int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx =$

Solution: We can take $u = e^{2x}$ so $du = 2e^{2x}\,dx$ and

$$\begin{aligned} \int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx &= \int_1^{e^4} \frac{1}{2}\frac{1}{u^2+1}\,du = \frac{1}{2}\arctan(u)\Big|_1^{e^4} \\ &= \frac{1}{2}\arctan(e^4) - \frac{1}{2}\arctan(1) = \frac{1}{2}\arctan(e^4) - \frac{\pi}{8}. \end{aligned}$$

M2: Advanced Integration Techniques

(a) $\int \frac{\sqrt{1-x^2}}{x^2}\,dx =$

Solution: We're going to set $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Then

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int \frac{\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)}\cos(\theta)\,d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\,d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C.
\end{aligned}$$

But we know that $\sin(\theta) = x$, so $x = \arcsin(\theta)$ and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sqrt{1-x^2}}{x}$. Then we get

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = -\frac{\sqrt{1-x^2}}{x} - \arcsin(x) + C.$$

(b) Compute $\int \frac{3x}{(x+4)(x-2)}\,dx =$

Solution:

$$\begin{aligned}
\frac{3x}{(x+4)(x-2)} &= \frac{A}{x+4} + \frac{B}{x-2} \\
3x &= A(x-2) + B(x+4) \\
6 &= 6B \Rightarrow B = 1 \\
-12 &= -6A \Rightarrow A = 2 \\
\frac{3x}{(x+4)(x-2)} &= \frac{2}{x+4} + \frac{1}{x-2} \\
\int \frac{3x}{(x+4)(x-2)}\,dx &= \int \frac{2}{x+4} + \frac{1}{x-2}\,dx \\
&= 2\ln|x+4| + \ln|x-2| + C.
\end{aligned}$$

(c) Compute $\displaystyle\int \sin(2x)e^{5x}\,dx$.

Solution:

$$\begin{aligned}
\int \sin(2x)e^{5x}\,dx &= \sin(2x)\frac{e^{5x}}{5} - \int 2\cos(2x)\frac{e^{5x}}{5}\,dx \\
&= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{5}\left(\cos(2x)\frac{e^{5x}}{5} - \int -2\sin(2x)\frac{e^{5x}}{5}\,dx\right) \\
&= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} + \frac{4}{25}\int \sin(2x)e^{5x}\,dx \\
\frac{29}{25}\int \sin(2x)e^{5x}\,dx &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} + C \\
\int \sin(2x)e^{5x}\,dx &= \frac{5}{29}\sin(2x)e^{5x} - \frac{2}{29}\cos(2x)e^{5x} + C.
\end{aligned}$$

S2: L'Hospital's Rule

(a) $\displaystyle\lim_{x\to 0}\left(\frac{e^x+1}{2}\right)^{1/x} =$

Solution:

$$\begin{aligned}
\ln y &= \frac{1}{x}\ln\left(\frac{e^x+1}{2}\right) \\
\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln\left(\frac{e^x+1}{2}\right)^{\nearrow 0}}{x_{\searrow 0}} \\
&\stackrel{\text{L'H}}{=} \lim_{x\to 0}\frac{2}{e^x+1}\cdot\frac{e^x}{2} = \lim_{x\to 0}\frac{e^x}{e^x+1} = 1/2 \\
\lim_{x\to 0} y &= e^{1/2}.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to+\infty}\frac{\arctan(x)}{\arctan(x)+1} =$

Solution: $\lim_{x\to+\infty}\arctan(x) = \pi/2$, so this limit is $\frac{\pi/2}{\pi/2+1} \approx .611$.

Note: you cannot use L'Hospital's rule here! If you tried, you would get

$$\lim_{x\to+\infty}\frac{1/(x^2+1)}{1/(x^2+1)} = \lim_{x\to+\infty} 1 = 1$$

but that is not in fact the limit.

(c) $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{\arcsin(2x-2)} =$

Solution: The top and bottom both approach 0, so we can use L'Hospital's Rule:

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{\arcsin(x-1)_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 1} \frac{1/x}{\frac{2}{\sqrt{1-(2x-2)^2}}}$$

$$= \lim_{x\to 1} \frac{\sqrt{1-(x-1)^2}}{2x} = \frac{1}{2}.$$