Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 5
Due Thursday, February 20

This week's mastery quiz has four topics. Everyone should submit M2, S3, and S4. If you have a 4/4 on M1 you don't need to submit it again; this is your last chance on M1 before the midterm.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals

Name:

Recitation Section:

M1: Transcendental Functions

(a) Compute $\dfrac{d}{dx}x^{\ln(x)}$.

Solution: The simplest approach is to use logarithmic differentiation.

$$\begin{aligned} y &= x^{\ln(x)} \\ \ln(y) &= \ln(x)\ln(x) = \ln(x)^2 \\ \frac{y'}{y} &= 2\ln(x)\frac{1}{x} \\ y &= \frac{2\ln(x)}{x}y = \frac{2\ln(x)x^{\ln(x)}}{x}. \end{aligned}$$

Alternatively, we could compute

$$\begin{aligned} \frac{d}{dx}x^{\ln(x)} &= \frac{d}{dx}\left(e^{\ln(x)}\right)^{\ln(x)} = \frac{d}{dx}e^{\ln(x)^2} \\ &= e^{\ln(x)^2}\cdot 2\ln(x)\frac{1}{x} = \frac{2\ln(x)x^{\ln(x)}}{x}. \end{aligned}$$

(b) $\displaystyle\int 3^x(4+3^x)^3\,dx =$

Solution: Set $u = 4+3^x$ so that $du = 3^x\ln(3)\,dx$. Then

$$\begin{aligned} \int 3^x(4+3^x)^3\,dx &= \int \frac{1}{\ln(3)}u^3\,du \\ &= \frac{u^4}{4\ln(3)} + C = \frac{(4+3^x)^4}{4\ln(3)} + C \end{aligned}$$

(c) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx.$

Solution: We want $4x^2 = 9u^2$ so we have $u = 2x/3$ and $du = 2/3\,dx$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{9-4x^2}}\,dx &= \int \frac{1}{\sqrt{9-9u^2}}\frac{3}{2}\,du \\ &= \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(2x/3) + C. \end{aligned}$$

M2: Advanced Integration Techniques

(a) $\int x \arctan(x^2)\, dx =$

Solution:

$$\begin{aligned} \int x \arctan(x^2)\, dx &= \frac{x^2}{2}\arctan(x^2) - \int \frac{x^2}{2}\frac{2x}{1+x^4}\, dx \\ &= \frac{x^2}{2}\arctan(x^2) - \int \frac{x^3}{1+x^4}\, dx \\ &= \frac{x^2 \arctan(x^2)}{2} - \frac{1}{4}\ln|1+x^4| + C. \end{aligned}$$

(b) $\int \sec^4(3t)\, dt =$

Solution: We're going to take $u = \tan(3t)$ so that $du = 3\sec^2 3t\, dt$. Then

$$\begin{aligned} \int \sec^4(3t)\, dt &= \int \sec^2(3t)(1+\tan^2(3t))\, dt \\ &= \int \frac{1}{3}(1+u^2)\, du \\ &= \frac{u}{3} + \frac{u^3}{9} + C \\ &= \frac{1}{9}\big(3\tan(3t) + \tan^3(3t)\big) + C. \end{aligned}$$

(c) Compute $\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\, dx$.

Solution: We need to do a partial fractions decomposition. We have

$$\begin{aligned} \frac{4x^2-x+10}{(x-2)(x^2+4)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \\ 4x^2 - x + 10 &= A(x^2+4) + (Bx+C)(x-2). 2: \quad 24 &= A\cdot 8 \\ A &= 30: \quad 10 &= 4A + (C)(-2) = 12 - 2C \\ C &= 6 - 5 = 1 \\ 1: \quad 13 &= 3(1+4) + (B+1)(1-2) = 15 - B - 1 \\ B &= 1. \end{aligned}$$

Thus we compute

$$\begin{aligned}\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\,dx &= \int \frac{3}{x-2} + \frac{x+1}{x^2+4}\,dx \\ &= \int \frac{3}{x-2} + \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx \\ &= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{4}\int \frac{1}{(x/2)^2+1}\,dx \\ &= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan(x/2) + C.\end{aligned}$$

S3: Numeric Integration

(a) Let $f(x) = x^3 + x$. How many intervals do you need with the midpoint rule to approximate $\int_1^2 x^3 + x\,dx$ to within 1/10? Compute the integral with that many integrals. (Feel free to use a calculator to plug values into f, but show every step.)

Solution: We have

$$\begin{aligned}f''(x) &= 6x \\ f'(2) &= 12 \\ |E_M| &\leq \frac{12 \cdot 1^3}{24 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 5 \\ n &> 2\end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\int_1^2 x^3+x\,dx \approx \frac{1}{3}f(7/6)+\frac{1}{3}f(9/6)+\frac{1}{3}f(11/6) \approx \frac{1}{3}(2.75+4.875+8.00) = \frac{1}{3}15.625 \approx 5.21.$$

(Since the true answer is 5.25 this is in fact within our error bound.)

(b) Suppose we have

$$g(0) = 2.4 \qquad g(1) = 4 \qquad g(2) = 2.7 \qquad g(3) = 2.3 \qquad g(4) = 1.7$$

Approximate $\int_0^4 g(x)\,dx$ using the Trapezoid rule, and then using Simpson's rule.

Solution: For the trapezoid rule, we have

$$\begin{aligned}T_4 &= 1 \cdot \frac{2.4+4}{2} + 1 \cdot \frac{4+2.7}{2} + 1 \cdot \frac{2.7+2.3}{2} + 1 \cdot \frac{2.3+1.7}{2} \\ &= \frac{1}{2}(6.4 + 6.7 + 5 + 4.0) = \frac{1}{2} \cdot 22.1 = 11.05.\end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}\left(2.4 + 4 \cdot 4 + 2 \cdot 2.7 + 4 \cdot 2.3 + 1.7\right) \\ &= \frac{1}{3}(2.4 + 12 + 5.4 + 9.2 + 1.9) = \frac{1}{3} \cdot 34.7 \approx 11.5667. \end{aligned}$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} =$

Solution:

$$\begin{aligned} \int_1^2 \frac{dx}{x\ln(x)} &= \lim_{t\to 1^+} \int_t^2 \frac{dx}{x\ln(x)} \\ &= \lim_{t\to 1^+} \ln(|\ln(x)|)\Big|_t^2 \\ &= \lim_{t\to 1^+} \ln(|\ln(2)|) - \ln|\ln(t)| \end{aligned}$$

But $\lim_{t\to 1^+} \ln(t) = 0$, so $\lim_{t\to 1^+} \ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}}$.

Solution: We'll take $u = \sqrt{x}$ so $du = \frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}+x\sqrt{x}} = \frac{2du}{1+u^2}$. Then

$$\begin{aligned} \int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}} &= \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x} + x\sqrt{x}} \\ &= \lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{2du}{1+u^2} \\ &= \lim_{t\to\infty} 2\arctan(u)\Big|_1^{\sqrt{t}} = \lim_{t\to\infty} 2\arctan(\sqrt{t}) - 2\arctan(1) \\ &= 2(\pi/2) - 2(\pi/4) = \pi/2. \end{aligned}$$