Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 6
Due Thursday, February 27

This week's mastery quiz has four topics. Everyone should submit S5. If you have a 4/4 on M2, or a 2/2 on S3 or S4, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Geometric Applications

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\displaystyle\int \frac{\sqrt{4x^2-1}}{x}\,dx =$

Solution: We set $2x = \sec(\theta)$, so $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{4x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta-1}}{\frac{1}{2}\sec\theta}\frac{1}{2}\sec(\theta)\tan\theta\,d\theta\\ &= \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta)-1\,d\theta\\ &= \tan(\theta)-\theta+C\end{aligned}$$

Then we know $\sec\theta = 2x$ so we can make a triangle with hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan(\theta)=\sqrt{4x^2-1}$. Then we can say either $\theta = \operatorname{arcsec}(2x)$ or $\theta = \arctan(\sqrt{4x^2-1})$, and we have

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \sqrt{4x^2-1}-\arctan(\sqrt{4x^2-1})+C = \sqrt{4x^2-1}-\operatorname{arcsec}(2x)+C.$$

(b) $\displaystyle\int_0^3 x^2e^{2x}\,dx =$

Solution:

$$\begin{aligned}\int_0^3 x^2e^{2x}\,dx &= \frac{1}{2}x^2e^{2x}\Big|_0^3 - \int_0^3 xe^{2x}\,dx\\ &= \frac{9}{2}e^6 - \int_0^3 xe^{2x}\,dx\\ \int_0^3 xe^{2x}\,dx &= \frac{1}{2}xe^{2x}\Big|_0^3 - \int_0^3 \frac{1}{2}e^{2x}\,dx\\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\Big|_0^3\\ &= \frac{3}{2}e^6 - \frac{1}{4}e^6 + \frac{1}{4}\\ \int_0^3 x^2e^{2x}\,dx &= \frac{9}{2}e^6 - \frac{3}{2}e^6 + \frac{1}{4}e^6 - \frac{1}{4} = \frac{13}{4}e^6 - \frac{1}{4}.\end{aligned}$$

(c) $\displaystyle\int \frac{x^2+x+3}{x^2+2}\,dx =$

Solution: Polynomial long division gives

$$\frac{x^2+x+3}{x^2+2} = 1 + \frac{x+1}{x^2+2} = 1 + \frac{x}{x^2+2} + \frac{1}{x^2+2}$$

and therefore

$$\begin{aligned}\int \frac{x^2+x+3}{x^2+2}\,dx &= \int 1 + \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx \\ &= x + \frac{1}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan(x/\sqrt{2}) + C.\end{aligned}$$

S3: Numeric Integration

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_5^9 (x+4)^{3/2}\,dx$ to within 1/10? Use the trapezoid rule with that many intervals to approximate the integral.

(Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$\begin{aligned} f'(x)\frac{3}{2}(x+4)^{1/2} f''(x) &= \frac{3}{4}(x+4)^{-1/2} = \frac{3}{4\sqrt{x+4}} \\ f''(5) &= \frac{1}{4} \\ |E_M| &\leq \frac{{}^1\!/_4 \cdot 4^3}{12 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 40/3 \approx 13.3 \\ n &\geq 4 \end{aligned}$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$\begin{aligned}\int_5^9 (x+4)^{3/2}\,dx &\approx \frac{\sqrt{9}^3+\sqrt{10}^3}{2} + \frac{\sqrt{10}^3+\sqrt{11}^3}{2} + \frac{\sqrt{11}^3+\sqrt{12}^3}{2} + \frac{\sqrt{12}^3+\sqrt{13}^3}{2} \\ &\approx \frac{1}{2}9^{3/2} + 10^{3/2} + 11^{3/2} + 12^{3/2} + \frac{1}{2}13^{3/2}.\end{aligned}$$

We can stop there, but numerically this is roughly 146.61. The true answer is approximately 146.54 so this is within the expected error bound.

(b) Suppose we have

$$g(3)=2 \quad g(5)=5 \quad g(7)=3 \quad g(9)=7 \quad g(11)=8 \quad g(13)=9 \quad g(15)=1$$

Approximate $\int_3^9 g(x)\,dx$ using the midpoint rule and Simpson's rule.

Solution: For the midpoint rule, we have

$$\begin{aligned} M_3 &= 4g(5) + 4g(9) + 4g(13) \\ &= 20 + 28 + 36 = 84. \end{aligned}$$

For Simpson's rule, weh ave

$$\begin{aligned} S_6 &= \frac{2}{3}\left(g(3) + 4g(5) + 2g(7) + 4g(9) + 3g(11) + 4g(13) + g(15)\right) \\ &= \frac{2}{3}\left(2 + 4\cdot 5 + 2\cdot 3 + 4\cdot 7 + 2\cdot 8 + 4\cdot 9 + 1\right) \\ &= \frac{2}{3}(2 + 20 + 6 + 28 + 16 + 36 + 1) = \frac{2}{3}\cdot 109 = \frac{218}{3} \approx 72.67. \end{aligned}$$

S4: Improper Integrals

(a) Compute $\int_0^{+\infty} \frac{1}{x^2}\,dx$.

Solution: This is improper in two ways: there's a singularity at 0, and it goes to $+\infty$. Thus we have to compute

$$\begin{aligned} \int_0^{+\infty} \frac{1}{x^2}\,dx &= \int_0^1 \frac{1}{x^2}\,dx + \int_1^{+\infty} \frac{1}{x^2}\,dx \\ &= \lim_{s\to 0^+} \int_s^1 \frac{1}{x^2}\,dx + \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx \\ &= \lim_{s\to 0^+} \left.\frac{-1}{x}\right|_s^1 + \lim_{t\to+\infty} \left.\frac{-1}{x}\right|_1^t \\ &= \lim_{x\to 0^+} \left(-1 + \frac{1}{s}\right) + \lim_{t\to+\infty} \left(\frac{-1}{t} + 1\right). \end{aligned}$$

The second limit is 1, but the first limit is $+\infty$, so the whole limit is $+\infty$ and thus the integral does not converge.

(b) Compute $\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx$.

Solution: We know that $\frac{1}{\sqrt[3]{x^2}}$ is undefined at zero. So we need to split this in half:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx &= \int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx \\
&= \lim_{t\to 0^-} \int_{-1}^{t} x^{-2/3}\,dx + \lim_{s\to 0^+} \int_{s}^{1} x^{-2/3}\,dx \\
&= \lim_{t\to 0^-} 3x^{1/3}\Big|_{-1}^{t} + \lim_{x\to 0^+} 3x^{1/3}\Big|_{s}^{1} \\
&= 3 \lim_{t\to 0^-} \left((\sqrt[3]{t} - \sqrt[3]{-1}) + (\sqrt[3]{1} - \sqrt[3]{s})\right) \\
&= 3(0+1+1-0) = 6.
\end{aligned}$$

S5: Geometric Applications

(a) Compute the area of the surface obtained by taking the curve $y = x^3$ as x goes from 0 to 1 and rotating it around the x-axis.

Solution: We have $y' = 3x^2$, and so

$$\begin{aligned}
L &= \int_0^1 2\pi x^3 \sqrt{1+(3x^2)^2}\,dx = \int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx \\
&= \frac{\pi}{27}(1+9x^4)^{3/2}\Big|_0^1 = \frac{10^{3/2}\pi}{27} - \frac{\pi}{27}.
\end{aligned}$$

(b) Compute the arc length of the curve $(y-2)^3 = x^2$ between $y = 2$ and $y = 6$ for $x \geq 0$.

Solution: We have $x = (y-2)^{3/2}$, so $\frac{dx}{dy} = \frac{3}{2}(y-2)^{1/2}$ and

$$\begin{aligned}
L &= \int_2^6 \sqrt{1+\frac{9}{4}(y-2)}\,dy \\
&= \frac{8}{27}\left(1+\frac{9}{4}(y-2)\right)^{3/2}\Bigg|_2^6 \\
&= \frac{8}{27}\left(10^{3/2}-1\right).
\end{aligned}$$