Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 7
Due Thursday, March 6

This week's mastery quiz has three topics. Everyone should submit S6. If you have a 4/4 on M2, or a 2/2 on S5, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 5: Geometric Applications
- Secondary Topic 6: Differential Equations

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx =$

Solution:

$$\begin{aligned}
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\
x^2+x-4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\
2 &= -2B \Rightarrow B = -1 \\
-4 &= 4C \Rightarrow C = -1 \\
-4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\
\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx \\
&= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C.
\end{aligned}$$

(b) $\displaystyle\int \cos^5(2x)\,dx =$

Solution:

$$\begin{aligned}
\int \cos^5(2x)\,dx &= \int \cos(2x)(1-\sin^2(2x))^2\,dx \\
&= \int \cos(2x) - 2\sin^2(2x)\cos(2x) + \sin^4(2x)\cos(2x)\,dx \\
&\frac{1}{2}\sin(2x) - \frac{1}{3}\sin^3(2x) + \frac{1}{10}\sin^5(2x) + C.
\end{aligned}$$

(If you want you can explicitly do the substitution $u = \sin(2x), du = 2\cos(2x)\,dx$, but you don't have to write it out explicitly.)

(c) $\displaystyle\int e^{-t}\cos(3t)\,dt =$

Solution:

$$\begin{aligned}
\int e^{-t}\cos(3t)\,dt &= \frac{1}{3}\sin(3t)e^{-t} - \int -e^{-t}\sin(3t)/e\,dt \\
&= \frac{1}{3}\sin(3t)e^{-t} + \frac{1}{3}\int e^{-t}\sin(3t)\,dt \\
\int e^{-t}\sin(3t)\,dt &= \frac{1}{3}e^{-t}(-\cos(3t)) - \int (-e^{-t})\frac{-1}{3}\cos(3t)\,dt \\
\int e^{-t}\cos(3t)\,dt &= \frac{1}{3}\sin(3t)e^{-t} - \frac{1}{9}\cos(3t)e^{-t} - \frac{1}{9}\int e^{-t}\cos(3t)\,dt \\
\frac{10}{9}\int e^{-t}\cos(3t)\,dt &= \frac{1}{3}\sin(3t)e^{-t} - \frac{1}{9}\cos(3t)e^{-t} + C \\
\int e^{-t}\cos(3t)\,dt &= \frac{3}{10}\sin(3t)e^{-t} - \frac{1}{10}\cos(3t)e^{-t} + C.
\end{aligned}$$

S5: Geometric Applications

(a) Compute the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ as x varies from 1 to 2.

Solution: We have $y' = \frac{x^3}{2} + \frac{-1}{2x^3}$, and thus

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + (x^3/2 - x^{-3}/2)}\,dx = \int_1^2 \sqrt{1 + x^6/4 - 1/2 + x^{-6}/4}\,dx \\
&= \int_1^2 \sqrt{x^6/4 + 1/2 + x^{-6}/4}\,dx = \int_1^2 \sqrt{(x^3/2 + x^{-3}/2)^2}\,dx \\
&= \int_1^2 x^3/2 + x^{-3}/2\,dx = \frac{x^4}{8} - \frac{1}{4x^2}\Big|_1^2 \\
&= \frac{16}{8} - \frac{1}{16} - \left(\frac{1}{8} - \frac{1}{4}\right) = \frac{33}{16}.
\end{aligned}$$

(b) Compute the area of the surface obtained by taking the curve $y = \sqrt{15 - x}$ as x goes from 3 to 5 and rotating it around the x-axis.

Solution: We have $y' = \frac{-1}{2\sqrt{15-x}}$. So we get

$$\begin{aligned}
A &= \int_3^5 2\pi y\sqrt{1+y'^2}\,dx \\
&= \int_3^5 2\pi\sqrt{15-x}\sqrt{1+\frac{1}{4(15-x)}}\,dx \\
&= 2\pi\int_3^5 \sqrt{15-x+\frac{1}{4}}\,dx \\
&= \pi\int_3^5 \sqrt{61-4x}\,dx \\
&= \pi\frac{2}{3\cdot(-4)}(61-4x)^{3/2}\Big|_3^5 = \frac{-\pi}{6}\left(41^{3/2}-49^{3/2}\right) \\
&= \frac{\pi}{6}(343-41\sqrt{41}) =\approx 42.13.
\end{aligned}$$

S6: Differential Equations

(a) Find a general solution to the equation $xy' = y^2+1$.

Solution:

$$\begin{aligned}
\frac{dy}{y^2+1} &= \frac{dx}{x} \\
\arctan(y) &= \ln(x)+C \\
y &= \tan(\ln(x)+C).
\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $-2x+4y^3\sqrt{x^2+3}\cdot y' = 0$ if $y(1)=1$

Solution:

$$\begin{aligned}
4y^3y'\sqrt{x^2+3} &= 2x \\
4y^3\,dy &= \frac{2x}{\sqrt{x^2+3}}\,dx \\
y^4 &= 2\sqrt{x^2+3}+C \\
y &= \sqrt[4]{2\sqrt{x^2+3}+C}.
\end{aligned}$$

Then we have

$$\begin{aligned}
1 &= \sqrt[4]{2\sqrt{4}+C} = \sqrt[4]{4+C} \\
1 &= 4+C \\
C &= -3 \\
y &= \sqrt[4]{2\sqrt{x^2+3}-3}.
\end{aligned}$$