Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 8
Due Thursday, March 20

This week's mastery quiz has two topics. Everyone should submit S7. If you have a 2/2 on S6, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Secondary Topic 6: Differential Equations
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

S6: Differential Equations

(a) Find a general solution to the equation $y' = x^2 + 1 + x^2y + y$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= (x^2+1)(1+y) \\
\frac{dy}{y} &= x^2 + 1\,dx \\
\ln|1+y| &= {}^{x^3}\!/3 + x + C \\
1 + y &= e^{x^3/3+x+C} \\
y &= e^{x^3/3+x+C} - 1 = Ke^{x^3/3+x} - 1.
\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $y'/x = \cos^2(y)$ if $y(0) = \pi/4$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= x\cos^2(y) \\
\sec^2(y)\,dy &= x\,dx \\
\tan(y) &= x^2/2 + C \\
y &= \arctan(x^2/2 + C).
\end{aligned}$$

Then we have

$$\begin{aligned}
\pi/4 &= \arctan(C) \\
\tan(\pi/4) &= C \\
C &= 1 \\
y &= \arctan(x^2/2 + 1).
\end{aligned}$$

S7: Sequences and Series

(a) Let $b_n = \frac{(n)!}{(n+2)!}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$.

Solution: $b_1 = {}^1\!/6, b_2 = {}^2\!/24 = {}^1\!/12, b_3 = {}^6\!/120 = {}^1\!/20$, and $b_4 = {}^{24}\!/720 = {}^1\!/30$.

We compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n^2+3n+2} = 0.$$

(b) Compute $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$. Rigorously justify your computation of this limit.

Solution: We can do a partial fractions decomposition: we have

$$\begin{aligned} 2 &= A(n+1) + B(n+3) \\ 2 &= 2B && \Rightarrow B = 1 \\ 2 &= -2A && \Rightarrow A = -1. \end{aligned}$$

Then our partial sums are

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{2}{n^2+4n+3} \\ &= \sum_{n=1}^{k} \frac{1}{n+1} - \frac{1}{n+3} \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\ &\quad + \cdots + \left(\frac{1}{k+1} - \frac{1}{k+3}\right) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

(c) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{2n}}$.

Solution: This is a geometric series with $a = \frac{4}{9}$ and $r = \frac{2}{9}$. Since $r < 1$ this series converges, and the sum of the series is

$$\frac{4/9}{1-2/9} = \frac{4/9}{7/9} = \frac{4}{7}.$$