Math 1232 Spring 2025
Single-Variable Calculus 2
Mastery Quiz 9
Due Thursday, March 27

This week's mastery quiz has two topics. Everyone should submit M3. If you have a 2/2 on S7, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$

Solution: You can't really use the limit comparison test here, because the obvious comparsion $\sum 1/n^2$ leads to you needing to compute $\lim_{n\to\infty} \sin(1/n)$, which doesn't exist.

But you can use the usual comparison test. We know that

$$\sin(1/n) \leq 1$$
$$\frac{\sin(1/n)}{n^2} \leq \frac{1}{n^2}$$

Since $\sum 1/n^2$ converges, by the comparison test our series also converges.

(In fact it's true that $\sin(1/n) < 1/n$, so $\sum \frac{\sin(1/n)}{n}$ also converges, but that seemed a little hard to see.)

(b) Analyze the convergence of the series $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{x\to+\infty} \frac{x^{\nearrow\infty}}{\ln(x)_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to+\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

(c) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$

Solution: We have

$$\int_1^{\infty} \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty} \int_1^t \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty} \ln(x^4+x+3)\Big|_1^t$$
$$= \lim_{t\to\infty} \ln(t^4+t+3) - \ln(5) = \infty.$$

This diverges, so by the integral test the series $\sum_{i=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$ diverges.

We can also try to use the Comparison Test here, but it's a little tricky, because $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, and that doesn't help because $\sum_{n=1}^{\infty} \frac{4}{n} = \infty$. If we want to do comparison, we can try to argue that while $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, it's also true that $\frac{4n^3+1}{n^4+n+3} > \frac{1}{n}$. But that's not super obvious and would take some work.

Or we could use the Limit Comparison Test, and argue that

$$\lim_{n\to\infty} \frac{\frac{4n^3+1}{n^4+n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n^4+n}{n^4+n+3} = 4$$

is a finite non-zero limit. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-series test, or because it's the harmonic series), we know by the Limit Comparison Test that $\sum_{n=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$ diverges.

S7: Sequences and Series

(a) Let $b_n = \frac{n!}{2^n}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$$b_1 = \frac{1}{2} \qquad b_2 = \frac{2}{4}$$
$$b_3 = \frac{6}{8} \qquad b_4 = \frac{24}{16}.$$

We see that

$$\frac{n!}{2^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{2(2)(2)\dots(2)(2)} \geq \frac{n}{2}.$$

Since $\lim_{n\to\infty} \frac{n}{2} = \infty$, we know that $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$.

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+4}{n+3}\right) =$

Solution: This is the same as

$$\begin{aligned}\sum_{n=1}^{\infty} \ln(n+4) - \ln(n+3) &= \lim_{k\to\infty} \sum_{n=1}^{k} \ln(n+4) - \ln(n+3) \\ &= \lim_{k\to\infty} \big(\ln(5) - \ln(4)\big) + \big(\ln(6) - \ln(5)\big) + \cdots + \big(\ln(k+4) - \ln(k+3)\big) \\ &= \lim_{k\to\infty} \ln(k+4) - \ln(4) = \infty.\end{aligned}$$

Thus this sum diverges to infinity.

(c) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^{2n}}$.

Solution: This is a geometric series with $a = 4/9$ and $r = 1/9$, so we have

$$\sum_{n=1}^{\infty} \frac{4}{3^{2n}} == \frac{4/9}{1-1/9} = 1/2.$$