Math 1232: Single-Variable Calculus 2
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Recitation 9

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Problem 1. Let $(a_n) = \left(-6, 4, \frac{-8}{3}, \frac{16}{9}, \frac{-32}{27}, \ldots\right)$.

(a) Find a closed-form formula for a_n.

(b) Is there a real function f so that $f(n) = a_n$?

(c) What is $\lim_{n\to\infty} a_n$? Why?

Solution:

(a) $a_n = 6 \cdot \left(\frac{-2}{3}\right)^n$.

(b) There isn't really a natural one, because you can't just take $\left(\frac{-2}{3}\right)^x$ for x irrational. (Or for x rational with even denominator; you can't take the square root.)

It is *possible* to find a function that interpolates this, though. It's just adding a bunch of noise. A good example would be

$$f(x) = 6 \cdot \left(\frac{2}{3}\right)^n \cos(n\pi).$$

(c) The limit is zero. There are a few ways to argue this, but they pretty much all spring back to the squeeze theorem.

My approach would be to observe that

$$-6 \cdot \frac{2^n}{3^n} \leq a_n \leq 6 \cdot \frac{2^n}{3^n}.$$

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{x\to+\infty} (2/3)^x = 0$$

because $0 < 2/3 < 1$. So we know

$$\lim_{n\to\infty} -6 \cdot \frac{2^n}{3^n} = 0 \lim_{n\to\infty} 6 \cdot \frac{2^n}{3^n} \qquad = 0$$

so by the Squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

Problem 2 (Factorials). (a) What is 4!? What is $\frac{4!}{3!}$?

(b) What is $\frac{5!}{4!}$? What is $\frac{5!}{3!}$?

(c) Can you figure out what $\frac{202!}{200!}$ is?

Solution:

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. $\frac{4!}{3!} = \frac{24}{6} = 4$.

(b) We know $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Then $\frac{5!}{4!} = \frac{120}{24} = 5$. But there's a better way: we have

$$\frac{5!}{4!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 5.$$

Thus we have

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20.$$

(c) $\frac{202!}{200!} = 202 \cdot 201 = 40602$.

Problem 3. (a) Compute $\lim\limits_{n\to\infty} \dfrac{n}{n!}$. Justify your answer.

(b) Compute $\lim_{n\to\infty} \frac{e^n}{n!}$.

(c) Now compute $\lim_{n\to\infty} \frac{n^k}{n!}$, where $k > 0$ is a fixed integer.

Solution:

(a)

$$\lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{n}{n \cdot (n-1)!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0.$$

If we want to justify that last limit, we can observe that $\frac{1}{(n-1)!} < \frac{1}{n}$ as long as $n > 3$, and use the squeeze theorem.

(b) For $k > 2$ we know that $e/k < 1$, so

$$\frac{e^n}{n!} = \frac{e \cdot e \cdot e \cdot \dots \cdot e \cdot e \cdot e}{n(n-1)(n-2)\dots(3)(2)(1)}$$
$$\leq \frac{e}{n} \cdot \frac{e^2}{2} \leq \frac{e^3}{n} \to 0.$$

Since $0 \leq \frac{e^n}{n!} \leq \frac{e^3}{n}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{e^3}{n}$, by the squeeze theorem we know $\lim_{n\to\infty} \frac{e^n}{n!} = 0$.

(c) This one is tricky. For large k and small n this can be pretty big. But if $n > 2k$ we have

$$\frac{n^k}{n!} = \frac{n \cdot n \cdot \dots \cdot n}{n(n-1)(n-2)\dots(3)(2)(1)}$$
$$= \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \frac{n}{n-3} \dots \frac{n}{n-k+1} \cdot \frac{1}{(n-k)!}$$
$$\leq 2^k \frac{1}{(n-k)!} \leq \frac{2^k}{(n-k)}.$$

But remembering k is a constant, we know that $\lim_{n\to\infty} \frac{1}{n-k} = 0$, so $\lim_{n\to\infty} \frac{2^k}{n-k} = 0$. By the squeeze theorem, $\lim_{n\to\infty} \frac{n^k}{n!} = 0$.

Problem 4. Write out the first five terms of:

(a) $\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{3k}$

(b) $\sum_{k=1}^{\infty} \frac{k+1}{k!}$

(c) $\sum_{k=3}^{\infty} \frac{k+3}{k^2-k-2}$

Solution:

(a) $\frac{4}{3} - \frac{8}{6} + \frac{16}{9} - \frac{32}{12} + \frac{64}{15}$.

(b) $\frac{2}{1} + \frac{3}{2} + \frac{4}{6} + \frac{5}{24} + \frac{6}{120}$.

(c) $\frac{6}{4} + \frac{7}{10} + \frac{8}{18} + \frac{9}{28} + \frac{10}{40}$.

Problem 5. Write in series/summation notation:

(a) $1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$

(b) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \dots$

(c) $2 + 7 + 14 + 23 + 34 + \dots$

Solution:

(a) $\sum_{k=1}^{\infty} \frac{k}{2k-1}$.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$.

(c) $\sum_{k=1}^{\infty} k^2 + 2k - 1$.

Problem 6. (a) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \frac{1}{k^2+3k+2}$.

(b) Compute $\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2}$.

(c) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \frac{2}{k^2+2k}$.

(d) Compute $\sum_{k=1}^{\infty} \frac{2}{k^2+2k}$.

(e) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \ln\left(\frac{k+1}{k+3}\right)$.

(f) Compute $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+3}\right)$.

Solution:

(a)

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k^2+3k+2} &= \sum_{k=1}^{n} \frac{1}{k+1} - \frac{1}{k+2} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{2} - \frac{1}{n+2}.\end{aligned}$$

(b)

$$\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2} = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} = \frac{1}{2}.$$

(c)

$$\begin{aligned}\sum_{k=1}^{n} \frac{2}{k^2+2k} &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2} \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.\end{aligned}$$

(d)

$$\sum_{k=1}^{\infty} \frac{2}{k^2+2k} = \lim_{n\to\infty} 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2}.$$

(e)

$$\begin{aligned}\sum_{k=1}^{n} \ln\left(\frac{k+1}{k+3}\right) &= \sum_{k=1}^{n} \ln(k+1) - \ln(k+3) \\ &= \Big(\ln(2) - \ln(4)\Big) + \Big(\ln(3) - \ln(5)\Big) + \Big(\ln(4) - \ln(6)\Big) \\ &\quad + \cdots + \Big(\ln(n) - \ln(n+2)\Big) + \Big(\ln(n+1) - \ln(n+3)\Big) \\ &= \ln(2) + \ln(3) - \ln(n+2) - \ln(n+3).\end{aligned}$$

(f)

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+3}\right) = \lim_{n\to\infty} \ln(2)+\ln(3)-\ln(n+2)-\ln(n+3) = \ln(6)-\ln(n^2+5n+6) = -\infty.$$

Problem 7 (Geometric Series)**.** Compute:

(a) $\sum_{k=1}^{\infty} \frac{2^k}{3^k}$

(b) $\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}}$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \dots$

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \dots$

Solution:

(a) $\sum_{k=1}^{\infty} \frac{2^k}{3^k} = \frac{2/3}{1-2/3} = 2.$

(b) $\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}} = \frac{625/64}{1+5/8} = \frac{625/64}{13/8} = \frac{625}{104}.$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \cdots = \sum_{k=1}^{\infty} \frac{5}{2^k} = \frac{5/2}{1-1/2} = 5.$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \cdots = \sum_{k=1}^{\infty} \frac{-2}{3} \frac{4^{k-1}}{3^{k-1}}$ and since the ratio $r = \frac{4}{3} > 1$ this series diverges.

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{3^k} = \frac{1/3}{1+1/3} = \frac{1/3}{4/3} = \frac{1}{4}.$