Math 1007: Mathematics and Politics

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1.1 Two-Candidate Elections

1.1.1 Introductory Scenario

A student organization needs to elect a new president, but unfortunately the bylaws do not specify how this is to be done. One candidate, called candidate A, is the incumbent president and a senior. Running against candidate A is a sophomore, candidate B. The organization has 57 members, all of whom attend a meeting called to elect a new president.

Candidate A presides over the meeting, and opens the floor to suggestions. Someone responds "We should just take a vote, and whoever gets the most votes should win." Everyone agrees this is a good idea. Assuming everyone votes, what is the minimum number of votes a candidate needs to win? What might go wrong if some members abstain? What do the words "majority" and "plurality" mean?

Membership records show that in this organization there are 8 seniors and 22 juniors. It is a given that all the seniors will vote for candidate A, while all the freshmen and sophomores will vote for candidate B. However, the votes of the juniors are evenly divided between the two. If the vote were taken right away, who would win?

Just before the vote, one of the seniors speaks. "In the past," he recalls, "seniors each got three votes, and juniors each got two." Without thinking about it, everyone agrees to adopt this new "weighted" voting system. If the vote were taken at this point, who would win? Where in the real world does weighted voting seem appropriate?

Just before the ballots are distributed, a certain junior who supports candidate A says "It is unfair to discriminate against freshmen and sophomores. 'One person one vote' is a solid American tradition." He points out, however, that in the past "we always required 2/3 of the members to vote against a sitting president to unseat him." Under this new system, are there enough votes to unseat candidate A? Where in the real world does it seem appropriate to give one alternative an advantage over another in an election?

The vote is finally ready to be taken, but just before that happens, a representative from the Student Association arrives to oversee the election. "This year," he says, "all student organizations must use the 'parity method' in which a candidate can be elected only if he or she receives an even number of votes." Who would win the election under this new method? What features of this method would you mention if your goal was to highlight the absurdity of this Student Association mandate? In particular, what would occur if candidate A, in frustration with campus politics, decided to vote for the opposing candidate B?

1.1.2 Defining our terms

Big part of math is coming up with precise definitions so we're consistently talking about the same thing. This is really different from a lot of fields, where definitions can often be a lot fuzzier. Is the United States "a capitalist society"? Is it "a democracy"? Sort of but also it doesn't fit those idealized descriptions but also nothing really does. Categories in general are fuzzy.

In order to "do math" we need to write precise definitions so we don't have fuzzy categories; a thing either does, or does not, meet the definition. And then we can talk about what's true of *every* rule that falls into that category, because there are no edge cases that kinda sorta fall into the category but might be missing some important pieces.

But we also take the definition seriously. If we come up with a definition and it includes things we didn't think about originally, either we include those things in the category even if they "obviously shouldn't" count, or we have to start over and come up with a different definition. We'll do some of both in this course!

This is unlike the way we talk in the sciences, or the social sciences, or humanities, or every-day life. It is similar to the way we do *legal reasoning*, though. A criminal offense will have a specific list of elements, and you commit the crime if and only if you check off all those elements. A regulation will apply to everything meeting certain listed criteria, and sometimes that leads to weird results. Infamously, in California, a bee is a fish (for the purpose of certain environmental laws), because the law code defines "fish" to include "invertebrates" and bees are, in fact, invertebrates.

> Each of these statutes provides that covered species include "native species or subspecies of a bird, mammal, fish, amphibian, reptile, or plant[.]" This portion of the code, however, does not elaborate on what qualifies as a bird, mammal, fish, and so forth. Based only on the qualified species listed above, bees and other land-dwelling invertebrates would not receive protection under the law. The court looked elsewhere in the Fish and Game Code for definitions to help clarify whether bees may qualify for protection under CESA. Importantly, the section 45 of the code defines "fish" as "a wild fish, mollusk, crustacean, invertebrate, amphibian, or part, spawn, or ovum of any of those animals." (Emphasis added). According to the court, the term "invertebrate" under the definition of fish includes both aquatic and terrestrial invertebrates, such as bees. [**?**]

Mathematical reasoning is similar to legal (and philosophical) reasoning, in that we are necessarily concerned with the implications of the precise rules we have set up—and if we don't like those implications, we have to change the rules themselves.

Some of the definitions we're about to lay out will seem obvious. Others will seem quite strange! But it's important to have them all laid out clearly so that we are following the same rules and discussing the same ideas and procedures for the rest of the course. When we want to give a formal definition, we will lay it out in a block like this:

Definition 1.1. A *function* is a rule that assigns exactly one output to every valid input. We call the set of valid inputs the *domain* of the function, and the set of possible outputs the *codomain* or sometimes the *range*. A function must be deterministic, in that given the same input it will always yield the same output.

We will try to use "normal" English in our definitions as much as possible. This will sometimes lead to awkward phrasing or high levels of wordiness as we try to give precise and unambiguous definitions without resorting to too much technical, mathematical jargon. For instance, instead of definition 1.1 we could have written:

Definition 1.2 (Unnecessarily technical definition)**.** Let A,B be sets. We define a function $f : A \to B$ to be a set of ordered pairs $\{(a,b) : a \in A, b \in B\}$ such that for each a in A there is exactly one pair whose first element is a. We call A the domain and B the codomain of f.

That technically conveys the same information, but is much harder to read if you're not used to it; we'll avoid that sort of thing as much as we can.

For right now we want to talk about two-candidate elections. Everything we say will also apply to votes on whether or not to pass a bill, or whether to get pizza or Chinese food; the only thing that matters is that we have exactly two choices, A and B. But for brevity we'll usually refer to them as candidates.

Obviously many elections have more than two options. (We could get Thai!) but it's convenient to start out looking at the two-candidate case. This allows us to deal with some of the issues that come up in general elections without having to think of all of them at once. This is a common technique in mathematical reasoning: whatever we want to study, we often start by working on a much simpler version where some of the difficulty disappears.

Definition 1.3. A *social choice function* for two candidates is a function whose domain is the set of all possible preferences that voters could have, and whose codomain is a set with three options: A wins; B wins; or A and B tie.

It's tempting to require social choice functions to output a definite winner, and remove the possibility of ties. But as we'll see, that can wind up being impractical. We'll return to that idea later.

1.1.3 Some methods of voting

There is a fairly obvious social choice function to start with:

Definition 1.4. The *simple majority method* for a two-candidate election is the social choice function that selects as the winner the candidate who gets more than half of all the votes cast. If each candidate gets exactly half of the votes, then the result is a tie.

So if 6 people vote for A and 4 people vote for B, then A will win. If 5 vote for A and 5 vote for B, then it will be a tie.

Weighted voting methods But note we made an implicit assumption in that last paragraph: we only gave *counts* of votes. What needs to happen for that to work? It needs to not matter who votes, just what the total is. That's true for the simple majority method, but it's not true for every possible social choice function.

Definition 1.5. A *weighted voting method* works as follows. There are n voters, and each voter is assigned a positive number of votes; we will say that voter number i has w_i votes, or a *weight* of w_i.

Set $t = w_1 + w_2 + \cdots + w_n$ be the total number of votes. A candidate who gets more than $t/2$ votes is the winner. If no candidate gets more than $t/2$ votes, the result is a tie.

Example 1.6. Suppose we have ten voters, with the following vote preferences:

A	B	A	B	A	A	A	A	B	B

We call this a *profile* of the electorate. If we don't care about who gives which vote, we can summarize this in a *tabulated profile*:

6	4
A	B

Thus we see that A gets 6 votes and B gets 4 votes. In the simple majority method, A will win.

But now suppose we have a weighted voting method where the first four voters get 5 votes each, the next three get 1 vote each, and the last three get 3 votes each. Then we have $t = 5+5+5+5+1+1+1+3+3+3 = 32$. Adding up the votes we see that A gets $5+5+1+1+1+3 = 16$ votes and B gets $5+5+3+3 = 16$ votes. That election is a tie! But note we could never have figured that out from the tabulated profile.

Poll Question 1.1.1. When does weighted voting make sense?

These weighted voting methods give a preference toward some voters over others. In the limit we can make this preference absolute:

Definition 1.7. In the *dictatorship method*, one of the voters is the *dictator*. Whoever the dictator prefers is the winner.

Example 1.8. Suppose we have a weighted voting method with ten voters, but the first voter gets twelve votes and each other voter gets one vote. Then the preference of the first voter will always win the election.

Obviously the dictatorship method is undemocratic. It is, however, mathematically a social choice function. And it does actually get used in some elections.

Example 1.9. The company Meta has two classes of stock: Class A shares get one vote per share, and Class B shares get 10 votes per share. Founder Mark Zuckerbert holds the overwhelming majority of the Class B stock, which means that while he only owns about 13% of the company, he has more than 50% of the voting power. Facebook shareholder elections are effectively run by the dictatorship method.

Supermajority Methods There's a different way we can tweak voting systems: we can have a bias toward ties.

Definition 1.10. Let p be a number such that $1/2 < p \leq 1$. The *supermajority method* with parameter p selects as the winner the candidate who gets a fraction p or more of all the votes. If there are t votes in total, a candidate must get at least $p \cdot t$ votes to win. If no candidate gets $p \cdot t$ votes, the result is a tie.

Example 1.11. A common value for p is 2/3. If we look at the same profile above, we have ten voters and thus need $2/3 \cdot 10 = 20/3 \approx 6.66$ votes to win. A gets six votes, and B gets four votes, so the result is a tie.

The number q of votes a candidate needs to win is sometimes called the *quota* and supermajority methods are sometimes called *quota methods*. It's tempting to write that $q = pt$ but that isn't quite true, since no candidate can actually get $20/3$ votes. Instead, the quota is $q = \lceil pt \rceil$, which we read as "the ceiling of the number pt" and means the smallest integer that is greater than or equal to pt.

Poll Question 1.1.2. When would this social choice function make sense?

If $p = 1/2$, this is just the simple majority method. If $p = 1$, this is the *unanimity* method or the *consensus* method, where everyone needs to agree.

Poll Question 1.1.3. Why don't we allow $p < 1/2$?

Status quo methods A weighted voting method favors some voters; a supermajority method favors ties. But sometimes we want to favor some specific result.

Definition 1.12. Start with some social choice function (such as the simple majority method or the supermajority method), which we will call the "base method". A *status quo method* designates one of the two candidates as the status quo, and the other as the challenger. If either candidate wins under the base method, then that candidate also wins under the status quo method. If there is a tie under the base method, then the status quo method names the status quo candidate as a winner.

Example 1.13. Suppose as in example 1.11 we have a supermajority method with $p = 2/3$, and we designate candidate B is the status quo. We have the same voter profile, so A gets 6 votes and B gets 4 votes. The supermajority method would declare this a tie, so candidate B wins.

Poll Question 1.1.4. When does this method get used regularly? When is it a good idea?

Like with the weighted voting method, this can also be taken to an extreme:

Definition 1.14. In the *monarchy method*, one of the candidates is a *monarch*. That candidate wins regardless of how anybody votes.

Mathematically this is a *constant function*: all inputs have the same output.

Note that a monarch is different from a dictator. A dictator doesn't necessarily win the election; they just select who wins the election. A monarch doesn't get any say; they win the election whether they want to or not.

Poll Question 1.1.5. When does a monarchy method make sense?

Bloc Voting

Definition 1.15. A *block voting method* partitions the electorate into n blocs, so that every voter is in exactly one bloc. Then each bloc i is assigned a positive number w_i of votes. Each bloc conducts an election using the simple majority method, possibly with some method for resolving ties. Then the bloc casts all of its w_i votes in the main election for the candidate that won that simple majority election. The winner is the candidate who receives the most votes in the main election, with a tie if both candidates receive the same number of votes.

Example 1.16. Suppose we have the following profile, split into five blocs which each gets one vote:

A	B	A	B	A	B	A	B	B	A	A	A	B	A	B

A wins the first bloc and the fourth bloc, for 2 votes. B wins the second, third, and fifth blocs, for 3 votes. So B wins the election.

If we ignore the blocs, the tabulated voting profile here is:

8	7
A	B

Is that enough information to recover the winner of the bloc election? Who would win this election by the simple majority method?

Poll Question 1.1.6. When would this voting method make sense? Where does it get used in real life?

Testing the limits There are many, many social choice functions, and we can't possibly describe them all. But when we're analyzing social choice functions, it can be really helpful to have a list of options to consider. Especially when we're trying to prove something "always" or "never" happens, it can be useful to consider some very silly social choice functions to check our intuitions.

Two such silly methods are the dictatorship method of definition 1.7 and the monarchy method of definition 1.14. But here are two more to consider:

Definition 1.17. In the *all-ties method*, the election is a tie, no matter how the electorate votes.

Like the monarchy method, this is a constant function.

Definition 1.18. In the *parity method*, if just one candidate gets an even number of votes, then that candidate wins. If both candidates get an odd number of votes, or both candidates get an even number of votes, then the result is a tie.

Example 1.19. Suppose we have the tabulated profile

7	4
A	B

Then B has an even number of votes and A has an odd number of votes, so B wins. If one voter who currently prefers A switches their vote to B, we would get the tabulated profile

6	5
A	B

Now A has an even number of votes so A wins.

1.1.4 Voting method criteria

We've seen a bunch of social choice methods. We haven't said which ones are better than others, and on some level this isn't a mathematical question. We can't "prove" that weighted voting is better or worse than bloc voting, or anything like that. It depends on what you want, and what you mean by "better"!

What math can do is help you figure that out—what are your goals, and how well each method does at achieving your goals. So first we have to come up with some things we care about, and write them down as precisely as we've specified the social choice functions.

Poll Question 1.1.7. What features do you think a good voting system will have?

Followup: what kind of election were you thinking about there?

One way of thinking about our attempt to develop criteria is that there are some social choice methods that we know we don't like. For each of those, we can figure out what we don't like about it, and generate some criterion that expresses that judgment.

Definition 1.20. A method satisfies the *anonymity condition* (or *is anonymous*) if it treats all voters equally.

Another way of expressing this is that an anonymous method will always give the same result if the voters exchange ballots among themselves.

It's clear that the dictatorship method, and weighted voting methods are not anonymous. The simple majority and supermajority methods are anonymous. The monarchy, parity, and all-ties methods are also anonymous.

When we have a result we have established, we will state it as a *proposition*. Major results will be called *theorems*, and minor technical results that are useful mainly to prove other results are *lemmas*.

Proposition 1.21. *A method is anonymous if and only if its outcomes depend only on the tabulated profile.*

Proof. The phrase "if and only if" means that we have to prove two separate things. We need to show that if the method is anonymous, then it depends only on the tabulated profile; and we need to show that if a method depends only on the tabulated profile, then it is anonymous.

First let's suppose we have an anonymous method. We want to show that if two different voter profiles give the same tabulated profile, they will have the same result. Since the two profiles give the same tabulated profile, the same number of voters prefer candidates A and B respectively in each profile. That means we can start with the first profile and exchange ballots around until we have reached the second profile. Since the method is anonymous, this can't change the result; so the outcome depends only on the tabulated profile.

Conversely, suppose we have a social choice method whose outcome depends only on the tabulated profile. If we have two profiles that differ only by exchanging ballots between voters, they will produce the same tabulated profile and thus produce the same result. Therefore, the method must be anonymous.

□

Definition 1.22. A method satisfies the *neutrality criterion* or *is neutral* if it treats both candidates equally.

Status quo methods and monarchy are not neutral. Majority and supermajority methods are neutral, as are weighted voting methods and dictatorships. The parity and all-ties methods are neutral.

So far simple majority and supermajority satisfy both our criteria, as do parity and all-ties. But parity, in particular, is obviously absurd. We should come up with a criterion that explains why parity is a bad voting method. And there is in fact a criterion, which feels so obvious that you might not think to state it if you're not confronted with something as silly as parity. (But we'll see that it can be trickier than it sounds in multi-candidate elections!)

Definition 1.23. A method satisfies the *monotonicity criterion* or *is monotone* if a candidate is never hurt by getting more votes.

That is: suppose the votes are cast and the method selects one candidate as the winner. Suppose the method is applied again after one or more voters change their votes from the losing candidate to the winning candidate. The candidate who was the winner before the change must remain the winner after the change.

Monotonicity is necessary if we want to avoid strategic voting: in a monotone social choice method, it's always reasonable to vote for the candidate you actually prefer. In a two-candidate election any reasonable method will be monotone, but not every method will be.

Proposition 1.24. *The parity method violates monotonicity.*

Proof. This proposition only requires us to present a *counterexample.* To say a method *is* monotone is to say that a certain thing always happens. To show it is not monotone, we don't need to show that the thing never happens; we only need to show that it doesn't always happen. (Note that "it doesn't always rain" is very different from "it never rains"!)

Consider the profile

A	B	A	B	A	A	A	A	B

In this profile, A gets 6 votes and B gets 3 votes, so A wins.

Now suppose the second voter changes their mind and starts to prefer A. We get the following profile:

A	**A**	A	B	A	A	A	A	B

Now A gets 7 votes and B gets 2. By the parity method, B wins this election.

Thus a voter changing their vote from B to A causes B to win, and the parity method violates monotonicity.

□

The all-ties method also seems bad. What's wrong with it? Well, it always produces ties. A voting method should, ideally, give us a result.

Definition 1.25. A method satisfies the *decisiveness criterion* or is *decisive* if it always chooses a winner, that is, never produces a tie.

Poll Question 1.1.8. Which of the methods we've discussed so far are decisive?

This criterion is ideal, but it's actually too strong to be useful. Sometimes a tie is the only really reasonable output, like if both candidates get exactly the same number of votes. (We'll make this claim precise in section 1.1.5). So we want a slightly weaker criterion that isn't too strong but still rules out nonsense like the all-ties method.

Definition 1.26. A method satisfies the *near decisiveness criterion* or is *nearly decisive* if ties can only occur when both candidates receive the same number of votes.

These two criteria aren't exclusive. Any decisive method is also nearly decisive. Note that this isn't normally how we use English; but it is definitely the way that we wrote these definitions. However, many methods are nearly decisive, but not decisive.

1.1.5 May's Theorem

Proposition 1.27. *The simple majority method is nearly decisive.*

Proof. Suppose the number t of voters is odd. No candidate can receive exactly half the votes, since $t/2$ is not an integer. So one must receive more than half; they win a majority and win the election.

Now suppose t is even. If both candidates get $t/2$ votes, they receive the same number of votes, and tie. If not, one gets more than $t/2$ and thus has a majority and wins. So ties only occur when each candidate gets exactly $t/2$ votes. □

So far we haven't found any method that checks all our boxes. The simple majority method checks most of them, but it isn't decisive. However, Kenneth May in 1952 proved that if we want to check off all our criteria, we're asking for too much.

Theorem 1.28 (May's Theorem)**.** *In an election with two candidates, the only voting method that is anonymous, neutral, monotone, and nearly decisive is the simple majority method.*

Proof. Suppose we have an anonymous, neutral, monotone, nearly decisive social choice function for two candidates. Since it is anonymous, we only need to consider tabulated profiles by proposition 1.21. Then let a be the number of voters who support candidate A, and b the number of voters who support candidate B; set $t = a + b$ to be the total number of voters. We want to show that the method we are imagining must be the simple majority method.

First consider the case where t is even. If $a = b = t/2$ then each candidate gets the same number of votes, so by neutrality the result must be a tie, which is the result that the simple majority method would give.

Suppose candidate A has a majority, that is, $a > t/2$. We want to show that for *any* method that is neutral, monotone, and nearly decisive, A must win. By near decisiveness, the result cannot be a tie, so either A wins or B wins.

But B can't win this election. We see that B gets $b = t - a$ votes, and since $a > t/2$ then $b < t/2$. If B wins with $b < t/2$ votes, then by monotonicity, B must win with $t/2$ votes. But we saw that if B gets $t/2$ votes then the election is a tie.

So if A gets a majority, we've seen the election can't be a tie and B can't win; thus A wins. By neutrality, the same is true of B; so in either case, a candidate with a simple majority of votes must win.

Now consider the case where t is odd. In this case, neither candidate can get $t/2$ votes, since it's not an integer; by near decisiveness, some candidate must win.

Suppose A gets $a > t/2$ votes. By near decisiveness, this can't be a tie. We claim that B cannot be the winner. We see that B gets $b = t - a < t/2$ votes, and necessarily $b < a$. If B wins with $b < a$ votes, then by monotonicity B would also have to win with a votes. But by neutrality, that means that A would also win with a votes.

So if A gets $a > t/2$ votes, then the election isn't a tie and B doesn't win; so A must win the election. By neutrality, if B gets more than $t/2$ votes, then B must win as well. So a candidate with a simple majority of votes must always win. □

The important thing about this theorem is it doesn't just prove that we haven't come up with a better method than simple majority. We've shown that any method that satisfies these criteria has to be the simple majority method; there's no possible way to be "more clever" and come up with a better option. In particular, we have the following *impossibility result*:

Corollary 1.29. *It is impossible for a voting system with two candidates to be anonymous, neutral, monotone, and decisive.*

Proof. If a method is decisive, it must be nearly decisive. If a method is anonymous, neutral, monotone, and nearly decisive, it must be the simple majority method by Theorem 1.28. So any method fitting these criteria must be the simple majority method; but the simple majority method is not decisive, so no method fits all these criteria. □

A similar argument to the proof of May's theorem can give us the somewhat more general result below:

Theorem 1.30. *In an election with two candidates, a voting method that is anonymous, neutral, and monotone must be the simple majority method, a supermajority method, or the all-ties method.*

We're not going to prove this right now but I encourage you to think about why that should be true, and how you'd prove it.

1.2 Social Choice Functions

1.2.1 Ballots and Simple Functions

We want to consider elections with more than 2 candidates. We call the set of candidates the *slate* and the set of voters the *electorate*. We'll generally call the candidates $A, B, C, \ldots$. (Again, instead of "candidates" these could represent bills or policies or pizza toppings, but we'll call them candidates to keep the terminology simple.)

For most of this section we'll assume each voter submits a *preference ballot*, in which they list the candidates in decreasing order of preference. So if there are four candidates $\{A, B, C, D\}$ a voter might submit a ballot that says they prefer $B > D > C > A$, which we might represent

B
D
C
A

We assume that voters are *rational*, in a specific way: we assume that each voter's preferences are *transitive*, meaning that if a voter prefers B to D and prefers D to C they must also prefer B to C. (It's not clear that people work this way in real life! But it's a convenient simplifying assumption for us to make now.)

We can represent the set of all ballots with a *profile*, similar to the profiles of section 1.1.3. But now these profiles need to have more rows, so we might get a profile like

A	B	A	C	A	B	C
B	C	B	A	B	C	A
C	A	C	B	C	A	B

Figure 1.1

In this profile, there are seven voters. We can see the first and third prefer A to B and B to C, while the second and sixth prefer B to C and C to A. In many cases we can summarize this in a *tabulated profile*

3	2	2
A	B	C
B	C	A
C	A	B

Definition 1.31. A *social choice function* for a slate of candidates is a function that takes in a voter profile, and outputs a non-empty subset of the slate.

We think of this subset as the list of "winners" of the election. We must have at least one winner, but we can have multiple winners (which you might interpret as a tie).

There is a tremendously large set of possible social choice functions; even with just three candidates and four voters, the number of possible social choice functions is a thousand-digit number. But most of those functions aren't very interesting.

We're going to start by listing off a few obvious social choice functions that are basically reasonable and easy to interpret as plausible methods of social choice. Then we'll introduce a few more sophisticated approaches. In section 1.2.2 we'll look at some of the properties we might want a voting system to have, and in section 1.3 we'll take a tour of a number of voting methods that try to achieve those criteria.

Perhaps the most obvious thing to do is build on the work we did in the two-candidate case. Our favorite method there was the simple majority method. We can't quite ask for a majority in a multi-candidate election, because it's very possible that no candidate gets a majority. (See also homework 3 on this idea.) Instead, we ask for a *plurality*.

Definition 1.32. A candidate who gets more votes than any other candidate is said to have a *plurality* of the votes.

In the *plurality method* we select as the winner the candidate who is ranked first by the largest number of voters. In the case that there is a tie for the most first-choice votes, we select all the candidates who tie for the most first-choice votes.

Let's apply this to the voting profile in figure 1.1. Candidate A gets 3 first-place votes, while B and C each get 2. So A is the plurality candidate and wins the election.

The plurality method is familiar and simple. It also doesn't use very much information; it only needs to know each voter's top preference (sometimes called a "vote-for-one" ballot.) In some ways, that's an advantage, since it makes voting and ballots much simpler. But it doesn't use much information about voter preferences, so it seems like we could probably produce better social choice methods somehow.

For instance, consider the following tabulated profile 1.2:

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

Figure 1.2

In this profile, A gets 5 first-choice votes, while B, C, D each get 4 and E gets 3. So A is the plurality winner. However, we see that all the voters who don't rank A first rank them *last*; A seems widely disliked, and probably a bad choice overall. In particular, B seems widely liked, with 4 first-place votes and *thirteen* second-place votes, out of twenty. Maybe we can develop a method that takes that other information into account?

Definition 1.33. The *Borda count method* works as follows. If there are n candidates, give each candidate $n-1$ points for each voter who ranks them first; $n-2$ points for each voter who ranks them second; $n-3$ points for each candidate who ranks them third; and so on, until they get 1 point for each voter who ranks them second-to-last (and 0 points for each voter who ranks them last.)

Add up all the points; the candidate who gets the most points wins. If more than one candidate ties for the most points, all of them win.

In profile 1.2, there are 5 candidates, so we give 4 points for a first-place vote. We can calculate that:

- A gets $5 \cdot 4 + 4 \cdot 0 + 4 \cdot 0 + 4 \cdot 0 + 3 \cdot 0 = 20$ points;
- B gets $5 \cdot 3 + 4 \cdot 4 + 4 \cdot 3 + 4 \cdot 3 + 3 \cdot 2 = 61$ points;
- C gets $5 \cdot 2 + 4 \cdot 3 + 4 \cdot 4 + 4 \cdot 1 + 3 \cdot 1 = 45$ points;
- D gets $5 \cdot 0 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 4 + 3 \cdot 3 = 37$ points;
- E gets $5 \cdot 1 + 4 \cdot 2 + 4 \cdot 1 + 4 \cdot 2 + 3 \cdot 4 = 37$ points.

Thus B wins by the Borda count method.

The Borda method is pretty compelling, and uses all the information from the voter profile. But it does raise some issues. For instance, consider the following profile:

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

Figure 1.3

Here the Borda counts are $A : 49$; $B : 54$; $C : 31$; $D : 28$; $E : 38$. So B is the Borda count winner. B is also the plurality winner, in fact. But we might notice that of the 20 voters, 15 prefer A to B; in a head-to-head matchup between the two, A would win. We could try to build a method that takes those head-to-head matchups into account. There are a few ways to do this, and we'll see several of them in this course. We'll start with the one that is currently being used in New York City for primary elections:

Definition 1.34. *Hare's method* operates as follows. Unless all candidates have the same number of first-place votes, identify the candidate (or candidates) who have the fewest first-place votes, and eliminate them from consideration. Create a new profile where that candidate is removed, and each voter moves up their lower-ranked candidates by one place.

Repeat the process, either until only one candidate remains, or until all remaining candidates have the same number of first-place votes. The remaining candidates are the winners.

Let's apply this to profile 1.3. We see that E has the fewest first-place votes, and so can be removed. That generates the profile

5	4	4	4	3
B	C	A	D	A
C	A	B	A	B
D	B	D	B	D
A	D	C	C	C

Now D and C are tied for the fewest first-place votes, with 4 each. We eliminate them both and get the profile

5	4	4	4	3
B	A	A	A	A
A	B	B	B	B

B has 5 first-place votes and A has 15, so B is removed and A is left as the winner.

But now let's go back and apply Hare's method to profile 1.2. Again, E has the fewest first-place votes, and is eliminated. That gives us the following profile:

5	4	4	4	3
A	B	C	D	D
B	C	B	B	B
C	D	D	C	C
D	A	A	A	A

Now A has 5 first-place votes, D has 7, while B and C each have 4. We eliminate B and C simultaneously again, giving the profile

5	4	4	4	3
A	D	D	D	D
D	B	B	B	B

So this time D wins. And this happens even though a majority prefer B to D!

Definition 1.35. *Coombs's method* operates as follows. Unless all candidates have the same number of last-place votes, identify the candidate (or candidates) with the most last-place votes, and eliminate them. As in Hare's method, when a candidate is eliminated, remove them from the profile and let each voter move the candidates they ranked below the eliminated candidate up a spot. Repeat the process, eliminating candidates, until either one candidate remains, or all remaining candidates have the same number of last-place votes. The candidate or candidates who remain at the end are the winners.

We can apply this method to profiles 1.2 and 1.3.

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

$\to^A$

5	4	4	4	3
B	B	C	D	E
C	C	B	B	D
E	E	D	E	B
D	D	E	C	C

=

9	4	4	3
B	C	D	E
C	B	B	D
E	D	E	B
D	E	C	C

$\to^D$

9	4	4	3
B	C	B	E
C	B	E	B
E	E	C	C

$\to^E$

9	4	4	3
B	C	B	B
C	B	C	C

=

16	4
B	C
C	B

$\to^C$

20
B

and so B wins profile 1.2 by Hare's method, just as with the Borda count, but unlike with Hare's method.

For profile 1.3 we get

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

$\to^C$

5	4	4	4	3
B	A	A	D	E
E	B	B	A	A
D	E	E	B	B
A	D	D	E	D

=

5	8	4	3
B	A	D	E
E	B	A	A
D	E	B	B
A	D	E	D

$$\to^{D} \begin{array}{|c|c|c|c|}\hline 5 & 8 & 4 & 3 \\ \hline\hline \text{B} & \text{A} & \text{A} & \text{E} \\ \hline \text{E} & \text{B} & \text{B} & \text{A} \\ \hline \text{A} & \text{E} & \text{E} & \text{B} \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 5 & 12 & 3 \\ \hline\hline \text{B} & \text{A} & \text{E} \\ \hline \text{E} & \text{B} & \text{A} \\ \hline \text{A} & \text{E} & \text{B} \\ \hline \end{array} \to^{E} \begin{array}{|c|c|c|}\hline 5 & 12 & 3 \\ \hline\hline \text{B} & \text{A} & \text{A} \\ \hline \text{A} & \text{B} & \text{B} \\ \hline \end{array} \to^{B} \begin{array}{|c|}\hline 20 \\ \hline\hline \text{A} \\ \hline \end{array} \text{ and so } A \text{ wins.}$$

Hare and Coombs have similar structures, but somewhat different incentives. Hare cares about the number of first-place votes, and so rewards candidates with a lot of strong support, even if they have a lot of strong detractors or haters. Coombs cares about the number of *last*-place votes, so it rewards candidates no one hates, even if no one really loves them either; it rewards middle-of-the-road moderation and widespread acceptability.

Definition 1.36. *Copeland's method* is the social choice function in which each candidate earns one point for every candidate they beat in a head-to-head matchup (using a simple majority method). A candidate earns half a point for every candidate they tie. The candidate with the most points at the end becomes the winner. If there's a tie for the most points, all those candidates are winners.

Again looking at profile 1.2:

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

We need to look at each pairwise competition. For instance, we see that 5 voters prefer A to B, but 15 prefer B to A, so B wins that matchup 15 to 5, and B gets one point there. We get the following table:

	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
Votes for first candidate	5	5	5	5	16	13	17	13	13	8
Votes for second candidate	15	15	15	15	4	7	3	7	7	12
Winner	B	C	D	E	B	B	B	C	C	E

We see that B gets 4 points, C gets 3 points, E gets 2 points, and D gets 1 point. So B wins by Copeland's method.

A better way to organize this is to make a *matrix*. We can make a table

	A	B	C	D	E	Total
A	-	0	0	0	0	0
B	1	-	1	1	1	4
C	1	0	-	1	1	3
D	1	0	0	-	0	1
E	1	0	0	1	-	2

In each cell, we put a 1 if the candidate whose row it is wins, a 0 if the candidate whose column it is wins, and a 1/2 if the match is a tie. (We put a - on the diagonal, since it doesn't make sense to ask of A beats A.) Then each candidate gets the sum of their row as a score.

Now let's apply this to profile 1.3:

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

We get

	A	B	C	D	E	Total
A	-	1	1	1	1	4
B	0	-	1	1	1	3
C	0	0	-	0	0	0
D	0	0	1	-	0	1
E	0	0	1	1	-	2

We can also adapt some of the other, arguably sillier, methods from section 1.1.3.

Definition 1.37. In the *dictatorship method*, one voter is the *dictator*. Their first-choice candidate is the unique winner.

In the *monarchy method*, one candidate is the monarch. That candidate is the unique winner regardless of how anyone votes.

In the *all-ties method*, every candidate is selected as a winner.

1.2.2 Voting System Criteria

Definition 1.38. *Unanimity criterion* or *unanimous* if whenever all voters place the same candidate at the top of their preference orders, that candidate is the unique winner.

Desirable, too strong.

Definition 1.39. A method is *decisive* if it always selects a unique winner.

Also too strong!

Definition 1.40. A method satisfies the *majority criterion* if, whenever a candidate receives a majority of the first-place votes, that candidate must be the unique winner.

Definition 1.41. A method is *anonymous* if the outcome is unchanged whenever two voters exchange their ballots.

Lemma 1.42. *A social choice function is anonymous if and only if it depends only on the tabulated profile.*

Proof. See proposition 1.21. □

Definition 1.43. A method is *neutral* if it treats all candidates the same, in the following sense: suppose we have some profile that names A to be a winner. Now suppose there is another candidate B, and all voters exactly swap their preferences for A and B. In the new profile, B should be a winner.

Also should be the case that voters liking you more should never hurt you.

Definition 1.44. A method is *monotone* if: suppose there is a profile in which candidate A wins, but some voter puts another candidate B immediately ahead of A. If that voter moves A up one place to be ahead of B, then the method must declare A to be a winner in the new profile as well.

It follows that if A moves up any number of places for any number of voters, they still must win. This is extremely desirable because it rules out "tactical" voting: if you like A more than B you should always rank A over B. Ranking A over B can never cause A to lose.

Example 1.45.

A	C	B
B	A	C
C	D	D
D	B	A

Who *shouldn't* win?

Definition 1.46. A method is *Pareto* or satisfies the *Pareto criterion* if whenever every voter prefers a candidate A to another candidate B, then the method does not select B as a winner.

Named after Italian economist Vilfredo Pareto. A choice is "Pareto optimal" if there's no other choice that's better for everyone. This guarantees that a winner is Pareto optimal. (Note there can be many Pareto optimal outcomes.)

Does not mean that if everyone prefers A to B then A must be a winner! Maybe everyone prefers C to both.

7	6
D	C
C	D
A	A
B	B

Definition 1.47. A candidate is a *Condorcet candidate* if they beat every other candidate in a head-to-head. They are an *anti-Condorcet candidate* if they lose to every other candidate in a head-to-head.

Not always a Condorcet candidate. Famously

A	B	C
B	C	A
C	A	B

A beats B, B beats C, but C beats A.

People sometimes call Copeland's method the Condorcet method, but that's not quite right. And people sometimes say that the Condorcet method picks the Condorcet winner, but that's not actually a method—doesn't always give an answer!

Definition 1.48. A method satisfies the *Condorcet criterion* if whenever there's a Condorcet candidate, they're the unique winner.

A method satisfies the *anti-Condorcet criterion* if whenever there's an anti-Condorcet candidate, they don't win.

Definition 1.49. A method is *independent* if:

Suppose there are two profiles where no voter changes their mind about whether candidate A is preferred to candidate B: if a voter ranks A above B in the first profile, they must also rank A above B in the second profile. Suppose that in the first profile, the method makes A a winner but not B. Then the method must not choose B as a winner for the second profile.

The idea here is that to decide if A beats B, it shouldn't matter what voters think about another candidate C. So if we consider the two profiles:

A	C	B	C	C	A	B
B	A	C	B	B	B	A
C	B	A	A	A	C	C

A	C	B	C	C	A	B
B	A	C	B	B	B	C
C	B	A	A	A	C	A

The only difference between the two is that in profile 1, the last voter prefers A to C, while in profile 2 they prefer C to A; preferences between A and B are unchanged. If A wins and B loses in the first profile, that shouldn't change in the second!

Unfortunately, that's much harder than it sounds.

Poll Question 1.2.1. What methods have we looked at violate this?

Proposition 1.50. *Any social choice function that satisfies anonymity and neutrality must violate decisiveness.*

Proof. Since it's anonymous, we only have to look at tabulated profiles. Suppose we have $2n$ voters.

n	n
A	B
B	A

n	n
B	A
A	B

By neutrality, if A wins in the first B must win in the second, and vice versa. So both have to win.

□

Proposition 1.51 (Taylor)**.** *No social choice function involving at least three candidates satisfies both the independence criterion and the Condorcet criterion.*

Proof. Suppose we have an independent Condorcet method. Suppose there are three candidates and three voters, and we have the profile

A	C	B
B	A	C
C	B	A

We see that A can't be a winner. Because if we look at

A	C	C
B	A	B
C	B	A

C is the Condorcet candidate, and so must be the unique winner, and so A is not a winner. But since this doesn't change the relative positions of A and C, by independence A can't win in the first profile either.

We can make the exact same argument to show that B and C also can't be named winners in the original profile. But every profile must have a winner. So no method can be both Condorcet and independent.

□

Proposition 1.52. *If a method is Condorcet then it satisfies the majority criterion.*

Proof. Suppose A has a majority of first-place votes. Then they will win any head-to-head matchup, and thus A is the Condorcet candidate. So any method that satisfies the Condorcet criterion will cause A to win, also satisfying the majority criterion. □

In this sense we can say the Condorcet criterion is stronger than the majority criterion.

1.3 Evaluating Social Choice Functions

To summarize, we have discussed the following social choice functions:

- Plurality
- dictatorship
- monarchy
- all ties
- Borda count
- Hare's method
- Coombs's Method
- Copeland's Method
- Sequential agenda
- Positional methods

We also have several criteria we looked at:

- unanimous
- decisive
- majoritarian
- anonymous
- neutral
- monotone
- Pareto
- condorcet
- anti-condorcet
- independent

We want to think about how each of these criteria apply to each of the social choice functions we've studied.

1.3.1 Plurality method

Proposition 1.53. *The plurality method is majoritarian, monotone, and Pareto, but not Condorcet, anti-Condorcet, or independent.*

Proof. The majority is always a plurality, so the plurality method is majoritarian.

It's monotone because raising a candidate on some preference lists can't reduce their number of first-place votes, or increase any other candidate's. So if a candidate wins before the raising they will still win after.

It's Pareto because if A is ahead of B on every preference list, then B will have no first-place votes at all and thus will not win.

But consider the profile In this profile, B wins the plurality vote. But A beats B in a

2	3	2
A	B	C
C	A	A
B	C	B

Figure 1.4

head-to-head (four to three), A beats C head-to-head (five to four), and C beats B head to head (four to three). So A is the Condorcet candidate, and does not win; B is the anti-Condorcet candidate, and yet wins. So plurality fails both Condorcet and anti-Condorcet.

Now consider the following two profiles, "before" and "after".

A	A	A	A	B	B	B		A	A	C	C	B	B	B
B	B	C	C	A	A	A	→	B	B	A	A	A	A	A
C	C	B	B	C	C	C		C	C	B	B	C	C	C

In the first profile, A wins, with four first-place votes. In the second profile, two voters have changed their minds and now rank C above A; this causes B to win.

(This is exactly the sort of "strategic voting" we see in a lot of primary elections, but also in some general elections. Imagine that A is the Democrat, B is the Republican, and C is a far-left candidate. People will say that voting for C is "throwing away your vote" and just giving the election to B.)

□

1.3.2 Antiplurality method

Proposition 1.54. *The antiplurality method is monotone, but not majoritarian, Condorcet, anti-Condorcet, Pareto, or independent.*

Proof. It's monotone because raising a candidate in a preference list can never increase their number of last-place votes, or decrease the last-place votes of any other candidate. If A wins before the change, they will also win after.

Consider the profile In this profile, B has a majority first-place votes, but also a plurality

C	C	B	B	B
A	A	C	A	A
B	B	A	C	C

Figure 1.5

of last-place votes, and so loses under anti-plurality and so the method is not majoritarian.

Similarly, we see that B is the Condorcet candidate, and A is the anti-Condorcet candidate. But A wins and B loses. That means this method is neither Condorcet nor anti-Condorcet.

We can also construct a profile where A is rated higher than B by all voters, but B still wins. That's a little bit weird, because it means A can't have any last-place votes; but B

can tie with A as a winner by not having any last-place votes either. So this method isn't Pareto.

A	A	A
B	B	B
C	C	C

Finally, we want to think about independence. Consider the following two profiles:

C	A	A	B	B
A	B	B	A	A
B	C	C	C	C

$\rightarrow$

C	A	A	B	B
A	B	B	C	C
B	C	C	A	A

Figure 1.6

In the first profile, A wins the anti-plurality election, and B loses (along with C). In the second profile, we have swapped some voters between C and A, but keep the same relative positions of A and B; now B wins the anti-plurality election, and A And C both lose.

□

1.3.3 Borda count

Proposition 1.55. *The Borda count method is monotone, anti-Condorcet, and Pareto, but not majoritarian, Condorcet, or independent.*

Proof. Monotone because raising A on some preference lists will never decrease A's score, or increase anyone else's. So if A wins before the change, they will also win after.

It's Pareto because if A is ranked above B by every voter, then A will definitely get a higher score than B, and so B will not win.

The anti-Condorcet argument is kind of tricky. (Note this is the first time we're showing something *is* anti-Condorcet, so this will be a new type of argument.) Suppose that A is the anti-Condorcet candidate; we need to show that it's not possible for them to win. That means it's not enough to just give an example; we need to give a totally universal argument.

Instead, we need to make a tricky argument with counting. Let's suppose there are n candidates and m voters. Then every voter gives out $\frac{n(n-1)}{2}$ points, and the total number of points is $\frac{mn(n-1)}{2}$. That means the *average* number of points each candidate can get is $\frac{m(n-1)}{2}$ (the total divided by the number of candidates).

The most total points any candidate can get is $m(n-1)$, since there are m voters and a first-place vote gives $n-1$ points. We want to show that the anti-Condorcet candidate A gets less than half the maximum number of points.

Another way of thinking about the Borda count is that A gets one point for each pair (X,v) of (candidate, voter) such that the voter v ranks candidate X below A. That is, A gets one point for each time a voter ranks a candidate below them. But since A is the anti-Condorcet candidate, they are ranked below each candidate more often than they are ranked above that candidate—that's what it means to lose each round head-to-head. So A gets less than half the maximum number of possible points: less than $\frac{m(n-1)}{2}$. This is precisely the average number of points a candidate gets.

Since A gets less than the average, someone must get more than average. Thus someone gets more points than A, and A cannot win.

Now we want to show that the Borda count isn't majoritarian, Condorcet, or independent. For these we can simply show examples again.

Consider the profile

A	A	A	B	B
B	B	B	C	C
C	C	C	A	A

In this case, A is the Condorcet candidate, with a majority of the first-place votes. But A only gets 6 Borda points, while B gets 7, so B wins the Borda count election.

The profiles in figure 1.6 also show that the Borda count isn't independent. In the first profile A gets 7 points, B gets 6, and C gets 2, so A wins. In the second profile, we've only changed the relative positions of A and C; but now A gets 5 points, B still gets 6, and C gets 4. Now B wins.

□

1.3.4 Hare's method

One of the most popular methods is Hare's method. But it is actually surprisingly weak by our criteria, and in particular it's the first method we're going to talk about that, surprisingly, isn't monotone.

Proposition 1.56. *Hare's method is majoritarian and Pareto, but not monotone, Condorcet, anti-Condorcet, or independent.*

Proof. A candidate at the top of a majority of preference lists will always in in Hare's method, because they will never have less first-place votes than anyone else and so will never be eliminated.

Conversely, if a character is not at the top of any preference lists, they will always be eliminated in the first round. If everyone prefers A to B then B will not appear at the top of any preference lists and so will be immediately resolved.

The following pair of before-and-after preference lists shows that Hare's method violates monotonicity:

6	5	4	2
A	C	B	B
B	A	C	A
C	B	A	C

$\rightarrow$

6	5	4	2
A	C	B	A
B	A	C	B
C	B	A	C

Figure 1.7

In the "before" profile, Hare's method will first eliminate C, and then in a head-to-head A will defeat B with eleven votes to six. In the "after" profile, two voters have switched their votes to prefer A to B. this means that in the first round, B is eliminated, and then in the second round C defeats A nine to eight.

To see that Hare violates Condorcet and anti-Condorcet, we can look back to figure 1.4:

2	3	2
A	B	C
C	A	A
B	C	B

In the first round, A and C are eliminated, leaving B as the unique winner. But A is the Condorcet winner and B is the Condorcet loser.

Finally, to consider independence, consider the following tabulated profiles:

In the "before" figure, C is eliminated in the first round, then B is eliminated in the second round, leaving A as the unique winner. In the "after" profile, two voters have switched their preferences on A and C, but this leaves the relative rankings of A and B unchanged. But now, A loses in the first round, and in the second round C is eliminated leaving B as the unique winner. This violates independence.

2	2	1
B	A	A
A	C	B
C	B	C

$\rightarrow$

2	2	1
B	C	A
A	A	B
C	B	C

□

1.3.5 Coombs's method

Proposition 1.57. *Coombs's method is Pareto but not majoritarian, monotone, Condorcet, anti-Condorcet, or independence.*

Proof. If candidate A is ahead of candidate B on every preference list, then A will never have any last-place votes until B is eliminated. Therefore A will survive each round, with no last place votes, until B is eliminated; the election cannot come to an end until that happens. So B must lose, and Coombs's method is Pareto.

We saw in figure 1.5 that a candidate can have a majority and still lose an antiplurality election:

C	C	B	B	B
A	A	C	A	A
B	B	A	C	C

Here B is the majority winner, but is eliminated in the first round of Coombs's method and thus does not win. This method violates the majority criterion, and thus also the Condorcet criterion.

It *seems* like Coombs's method should satisfy the anti-Condorcet criterion, since even if they survive to the final round, they will lose the final head-to-head matchup (since they lose any head-to-head matchup). But this is not true! If we look at this same profile 1.5 again, we can see that A is the anti-Condorcet candidate losing to both B and C. But A wins the election, precisely because there is no "final" head-to-head: B and C are both eliminated at the same time, leaving A the unique winner.

The remaining two properties we leave as an exercise for the student. □

Exercise 1.58. *You should prove that Coombs's method isn't monotone or independent. Each of these arguments requires generating an example pair of profiles. You may wish to* *use the proof of proposition 1.56 as a hint or guideline as you construct these.*

1.3.6 Copeland's method

The two elimination methods were very similar; the positional methods are very similar. Copeland's method is structurally quite different and therefore produces a very different set of results. In particular, this system is constructed almost specifically to satisfy the Condorcet property.

Proposition 1.59. *Copeland's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.*

Remark 1.60. We'll see later than while you can different tradeoffs than this, you can't do *better* than this, in a specific way.

Proof. In the Copeland system, a Condorcet candidate gets a perfect, maximum score, since they beat each other candidate head-to-head. Each other candidate loses at least one head-to-head and so does not get a perfect score. So the Condorcet candidate always wins. This also means the system satisfies the majority property, since a majority candidate will be a Condorcet candidate.

An anti-Condorcet candidate loses each head-to-head and thus gets a score of zero. Since some candidate will have a score greater than zero, the anti-Condorcet candidate will always lose.

We now need to prove this method is monotone and Pareto. Again, this will require a bit of a sophisticated argument; we can't just generate an example, but need to provide an argument that will work for any possible profile, no matter how strange.

To prove Pareto: suppose A is above B on every preference list. In this case, B may win some head-to-head matchups, but any matchup B wins, A will also win. This means that A gets a point every time B gets a point, and A gets at least half a point every time B gets half a point. Further, A will defeat B, so A will have at least one more point than B does. So B cannot win.

To prove monotonicity: moving candidate A up on a preference list can never hurt A in any head-to-head matchup, so it cannot reduce A's score. A change that only involves moving A up can't affect any other head-to-head matchup, so it won't ever increase any other candidate's score. Therefore, if A was a winner before the change, they will still win after the change.

But even this argument may suggest why Copeland's method isn't independent. An improvement in A's place can't improve the scores of other candidates; but changing the relative positions of other candidates can change *their* scores. So consider the following two profiles:

B	A	A
C	B	C
A	C	B

$\rightarrow$

B	A	C
C	B	A
A	C	B

Figure 1.8

In the first profile, A gets 2 points, and B gets one, so A is the unique winner. In the second profile, we have swapped the positions of A and C for one voter. Now C beats A, getting one point; A beats B, getting one point; and B beats C, getting one point. All three candidates are declared winners.

□

1.3.7 Black's method

Here is a new method that also tries to prioritize Condorcet winners.

Definition 1.61. *Black's method* is the social choice function that chooses the Condorcet candidate as the unique winner if there is a Condorcet candidate, and chooses the Borda count winner if there is not.

We can also think of this as taking the Borda count, and "fixing" the "glitch" where it doesn't satisfy the Condorcet property.

Proposition 1.62. *Black's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.*

Proof. By definition, if there's a Condorcet candidate, Black's method selects them as the unique winner. So Black's method is Condorcet, and therefore majoritarian.

We know that an anti-Condorcet candidate can't be the Condorcet candidate, and we've also see that they can't be a Borda count winner in the proof of proposition 1.55. So an anti-Condorcet candidate can't win in Black's method.

To see Black's method is monotone, we need to split into two cases. Suppose the Black winner was a Condorcet winner. Then moving them up in some rankings will not change that; they will still be the Condorcet winner and will still win. If the Black winner was not a Condorcet candidate, they must have been the Borda count winner. Moving them up in the rankings will still leave them as the Borda count winner. And no *other* candidate can become a Condorcet candidate when the original winner moves up in some rankings. So either the original winner becomes the Condorcet candidate and thus wins, or no one is a Condorcet candidate and the original winner still wins by Borda count. Thus Black's method is monotone.

To see the method is Pareto, imagine that A is favored over B by all voters. Then B cannot be the Condorcet candidate because they lose to A head-to-head, and they can't be the Borda count winner because A will have more points than B does. Thus B cannot win.

Finally, we know that Black's method violates independence, because it is Condorcet. We get a concrete example from figure 1.8: in the before profile, A is the Condorcet candidate and wins, but in the after profile, we get a three-way tie in which each candidate has three Borda points.

□

Proposition 1.63. *The dictatorship method is monotone, Pareto, and independent, but not Condorcet, anti-Condorcet, or majoritarian.*

Proof. If A is the winner, they are at the top of the dictator's preference list. No improvement on other preference lists can change that, so A remains the dictator's top choice, and thus still wins. So the method is monotone.

If A is higher on ever voter's list than B, then in particular A is above B on the dictator's list and so B does not win. So the dictatorship method is Pareto.

If A is at the top of the dictator's preference list in one profile, and then a second profile has A and B in the same relative position, it can't have B at the top of the dictator's preference list. So B can't win in the second profile. Thus the dictatorship method is independent.

But now consider the following profile, where the third voter is dictator.

A	A	B
B	B	A

Figure 1.9

A is the Condorcet and the majority candidate, while B is the anti-Condorcet candidate. But B wins.

□

Proposition 1.64. *The all-ties method and the monarchy method are monotone and independent, but not Condorcet, anti-Condorcet, majority, or Pareto.*

Proof. These methods are both what we call constant functions, meaning that the output to the method ignores the input completely and doesn't care how anyone votes. These methods are all monotone and independent, because no candidate can win on one profile and lose on another. But they also violate Condorcet, anti-Condorcet, majority, and Pareto, because some profile can have a candidate with a Condorcet candidate, an anti-Condorcet candidate, a majority candidate, or a candidate preferred universally to another candidate—and that has no effect on who wins. □

	anon	neu	unan	dec	maj	Con	AC	mono	Par	ind
Plurality	Y	Y	Y	N	Y	N	N	Y	Y	N
Antiplur	Y	Y	Y	N	N	N	N	Y	N	N
Borda	Y	Y	Y	N	N	N	Y	Y	Y	N
Hare	Y	Y	Y	N	Y	N	N	N	Y	N
Coombs	Y	Y	Y	N	N	N	N	N	Y	N
Copeland	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Black	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Dictator	N	Y	Y	Y	N	N	N	Y	Y	Y
All-ties	Y	Y	N	N	N	N	N	Y	N	Y
Monarchy	Y	N	N	Y	N	N	N	Y	N	Y
Bloc	N	Y	Y	N	N	N	N	Y	Y	N
Agenda	Y	N	Y	N	Y	Y	Y	Y	N	N

Figure 1.10: : A summary of the properties of various social choice methods

1.4 Arrow's Theorem

We've seen that it's impossible to produce a "perfect" social choice function. Fundamentally, the problem is cycles like this, first seen in the proof of proposition 1.51:

A	C	B
B	A	C
C	B	A

In this profile, a majority of voters prefer A to B, a majority prefer B to C, and a majority prefer C to A. Even though each individual voter has rational, transitive preferences, the electorate as a whole does not. This is known as the *Condorcet paradox*.

Your textbook presents the following extension that's even worse: suppose we have the preference order

D	C	B
E	D	C
F	E	D
G	F	E
A	G	F
B	A	G
C	B	A

Imagine these represent a series of different policies we could have on some issue, and that we start with policy D. Someone proposes policy C, and that wins by majority vote over D. Then someone else proposes an amendment to B, and that defeats C by majority vote. Then finally, someone else proposes an amendment to shift to A, and that defeats B. In each case the vote was a clear majority vote. But we have moved from D to A, even though every single voter strongly prefers D to A.

We say in proposition 1.51 that no social choice function can be both independent and Condorcet, due to these loops. But it's worse than that. The insight behind these Condorcet paradox loops lead the economist Kenneth Arrow to formulate the following theorem:

Theorem 1.65 (Arrow). *If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.*

This is often phrased in the following way:

Theorem 1.66 (Arrow's theorem, alternate version). *It is impossible for a social choice function with at least three candidates to be Pareto, independent, and non-dictatorial.*

Lemma 1.67 (decisiveness lemma). *A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.*

Proof. □

Corollary 1.68. *It is impossible for a method to satisfy Pareto, independence, anonymity, and neutrality.*

Proof of theorem 1.65. □