# Math 2233 Practice Final Solutions

Instructor: Jay Daigle

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made for yourself ahead of time. You may not use a calculator.
- The exam has 7 problems, one on each mastery topic. The exam has 9 pages total.
- Each part of each major topic is worth 10 points. The question on topic S5 is worth 10 points.

  This practice test has too many questions so you can get in a broad spectrum of practice. I expect one question per topic for M1 through M4, and two questions on M5 and M6, on the real final.
- The real final will have optional questions on S1 through S4. Answering one correctly can earn you up to two bonus points on the test. More importantly, answering one correctly can raise your overall mastery score.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all.

  When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

## Name:

## Recitation Section:

**Problem 1** (M1)**.** The final will have *one* problem like this.

(a) Find the area of the parallelogram with vertices $(1,3,2),(1,5,3),(2,4,5),(2,6,6)$.

**Solution:** The area of the parallelogram is the magnitude of the cross product of the vectors spanning it. Those vectors are $\vec{u}=(0,2,1)$ and $\vec{v}=(1,1,3)$ so we compute

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 2 & 1\\ 1 & 1 & 3\end{vmatrix} = (6-1)\vec{i}+(1-0)\vec{j}+(0-2)\vec{k} = 5\vec{i}+\vec{j}-2\vec{k}\\ \|\vec{u}\times\vec{v}\| &= \sqrt{25+1+4}=\sqrt{30}.\end{aligned}$$

(b) Find the orthogonal decomposition of $2\vec{i}+5\vec{j}-4\vec{k}$ with respect to $5\vec{i}-\vec{j}+2\vec{k}$.

**Solution:**

$$\begin{aligned}\vec{v}_{parallel} &= \frac{(2,5,-4)\cdot(5,-1,2)}{(5,-1,2)\cdot(5,-1,2)}(5,-1,2)\\ &= \frac{10-5-8}{25+1+4}(5,-1,2) = \frac{-1}{10}(5,-1,2) = \left(\frac{-1}{2},\frac{1}{10},\frac{-1}{5}\right)\\ \vec{v}_{\perp} = \vec{v}-\vec{v}_{parallel} &= (2,5,-4)-\left(\frac{-1}{2},\frac{1}{10},\frac{-1}{5}\right) = \left(\frac{5}{2},\frac{49}{10},\frac{-19}{5}\right).\end{aligned}$$

**Problem 2** (M2)**.** The final will have *one* problem like this.

(a) Find a linear approximation of $f(x,y)=\sin(x)\sqrt{1-y^2}$ near the point $(0,0)$. Use it to estimate $f(.1,.1)$.

**Solution:**

$$\begin{aligned}\nabla f(x,y) &= (\cos(x)\sqrt{1-y^2}, \sin(x)y/\sqrt{1-y^2}\\ \nabla f(0,0) &= (1,0)\\ f(x,y) &\approx 0+1(x-0)+0(y-0)=x\\ f(.1,.1) &\approx .1.\end{aligned}$$

(b) Find an equation for the plane tangent to $g(x,y)=4xy^2+3xy$ at the point $(3,2)$.

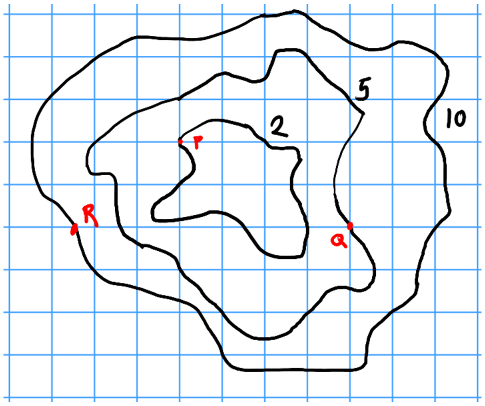
**Solution:** We compute that

$$\begin{aligned}g_x(x,y) &= 4y^2+3y & g_x(3,2) &= 16+6=22\\ g_y(x,y) &= 8xy+3x & g_y(3,2) &= 48+9=57\\ & & g(3,2) &= 48+18=66\end{aligned}$$

and so the plane will have equation

$$z=22(x-3)+57(y-2)+66.$$

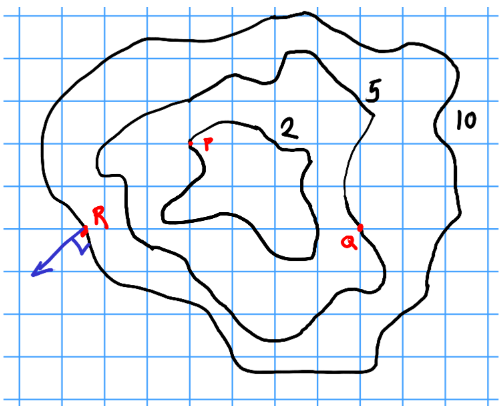
(c) Consider the contour plot below. Estimate $\frac{\partial f}{\partial x}(Q)$ and $\frac{\partial f}{\partial y}(P)$. Sketch the gradient vector at $R$.




**Solution:** At $P$, we see that moving to the right two units will increase the output by five, and moving to the left by one unit will decrease it by 3. So the partial derivative is somewhere between 5/2 and 3/1; we can estimate it as the average 11/4 but anything in that range is a good estimate.

At $P$, we see that moving in the $y$ direction moves tangent to the contour. So this $\frac{\partial f}{\partial x}(P) = 0$.

At $R$ we know the gradient should be perpendicular to the contour. And since it points in the direction of greatest increase, it should be pointing out of the shape, since the contours are lower towards the inside.




**Problem 3** (M3)**.** The final will have *one* problem like this.

(a) Find and classify all the critical points of $g(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.

**Solution:**

$$g_x(x, y) = 2x - 3y + 5$$
$$g_y(x, y) = -3x + 12y - 2$$
$$0 = -9y + 15 + 24y - 4 = 15y + 11$$

so we see that $y = -11/15$ and $x = -18/5$. This is the only critical point. The second derivatives are

$$g_{xx}(x, y) = 2 > 0$$
$$g_{xy}(x, y) = -3$$
$$g_{yy}(x, y) = 12$$
$$D = g_{xx}g_{yy} - g_{xy}^2 = 24 - 9 = 15 > 0$$

so this point is a local minimum.

(b) Find the minimum value of $f(x, y) = 4xy$ on the unit circle.

**Solution:** Our constraint equation is $x^2 + y^2 = 1$. So we have:

$$\begin{aligned}
4y &= \lambda 2x \\
4x &= \lambda 2y \\
\lambda &= 2y/x \\
4x &= 4y^2/x \\
4x^2 &= 4y^2 \\
x^2 &= y^2 \\
x &= \pm y
\end{aligned}$$

Plugging either of these into our constraint equation gives $2x^2 = 1$ and thus $x = \pm\sqrt{1/2}$. Thus we have four critical points: $(\sqrt{1/2}, \sqrt{1/2}), (\sqrt{1/2}, -\sqrt{1/2}), (-\sqrt{1/2}, \sqrt{1/2}), (-\sqrt{1/2}, -\sqrt{1/2})$. Plugging these in gives $2, -2, -2, 2$ respectively. So the absolute minimum value is $-2$.

**Problem 4** (M4)**.** The final will have *one* problem like this.

Let $g(x, y, z) = z^2(x^2 + y^2)$ and let $W$ be a cone with its point at the origin and base given by the circle $z = 2, x^2 + y^2 = 2$.

(a) Set up integrals to compute $\int_W g\, dV$ in cartesian, cylindrical, and spherical coordinates.

**Solution:**

$$\int_0^2 \int_{-z/\sqrt{2}}^{z/\sqrt{2}} \int_{-\sqrt{z^2/2-x^2}}^{\sqrt{z^2/2-x^2}} z^2(x^2+y^2)\, dy\, dx\, dz \quad \text{or} \quad \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{2(x^2+y^2)}}^{2} z^2(x^2+y^2)\, dz\, dy\, dx$$

$$\int_0^2 \int_0^{z/\sqrt{2}} \int_0^{2\pi} z^2 r^2 \cdot r\, d\theta\, dr\, dz \quad \text{or} \quad \int_0^{\sqrt{2}} \int_{\sqrt{2}r}^{2} \int_0^{2\pi} z^2 r^2 \cdot r\, d\theta\, dz\, dr$$

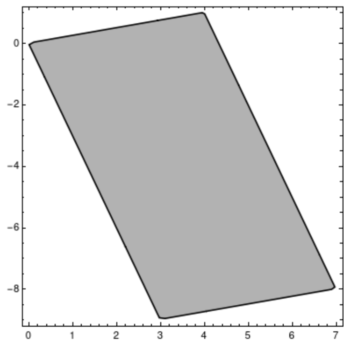
$$\begin{aligned}
&\int_0^{\arctan(1/\sqrt{2})} \int_0^{2\pi} \int_0^{2/\cos\phi} (\rho\cos\phi)^2(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta\rho^2\sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\arctan(1/\sqrt{2})} \int_0^{2\pi} \int_0^{2/\cos\phi} (\rho\cos\phi)^2(\rho^2\sin^2\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi
\end{aligned}$$

(b) Choose one of the integrals from part (a) and evaluate it.

**Solution:** The cylindrical coordinates are probably the easiest to work with. We compute

$$\begin{aligned}
\int_0^2 \int_0^{z/\sqrt{2}} \int_0^{2\pi} r^3 z^2\, d\theta\, dr\, dz &= 2\pi \int_0^2 \int_0^{z/\sqrt{2}} r^3 z^2\, dr\, dz \\
&= 2\pi \int_0^2 \frac{r^4}{4} z^2 |_0^{z/\sqrt{2}}\, dz = 2\pi \int_0^2 z^6/16\, dz \\
&= 2\pi z^7/112|_0^2 = 256\pi/112 = 16\pi/7.
\end{aligned}$$

(c) Compute $\iint_R x + y\, dA$ over the parallelogram with vertices $(0,0), (4,1), (7,-8), (3,-9)$.




**Solution:** We want to reparametrize this with $x = 4s + 3t, y = s - 9t$. [You could also use $x = 4s + t, y = s - 3t$, which would work out about the same.] Then we get bounds of $s, t \in [0, 1] \times [0, 1]$, and we're integrating the function $5s - 6t$.

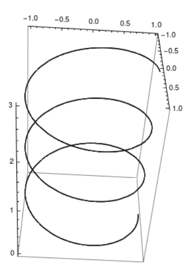
To compute the Jacobian get

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 1 & -9 \end{vmatrix} = -36 - 3 = -39$$

so $\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 39$. Then the integral is

$$\begin{aligned} \iint_R x + y \, dA &= \int_0^1 \int_0^1 5s - 6t \cdot 39 \, dt \, ds \\ &= 39 \int_0^1 5st - 3t^2|_0^1 \, ds = 39 \int_0^1 5s - 3 \, ds \\ &= 39 \left(5s^2/2 - 3s\right)\Big|_0^1 = 39(5/2 - 3) = -39/2. \end{aligned}$$

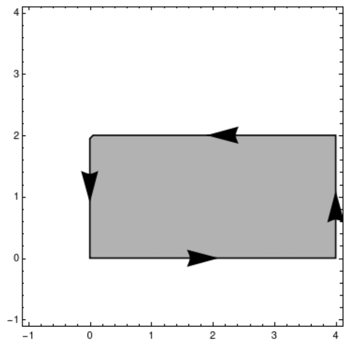
**Problem 5** (M5). (a) Set up an integral to compute the work done by the force field $\vec{F}(x^2y, yz^3, x+y+z)$ on a particle that moves from $(1, 0, 0)$ to $(1, 0, 3)$ by spiraling clockwise around the $z$-axis three times with radius 1.




**Solution:** We can parametrize with $\vec{r}(t) = (\cos(2\pi t), -\sin(2\pi t), t)$ for $t \in [0, 3]$. (This makes sure we both move clockwise and start at $(1, 0, 0)$; the $2\pi$ is to make a change of 1 in $t$ cause a complete rotation.) Then the integral is

$$\begin{aligned} &\int_0^3 \left(-\cos^2(2\pi t)\sin(2\pi t), -\sin(2\pi t)t^3, (\cos(2\pi t) - \sin(2\pi t) + t)\right) \cdot (-2\pi \sin(2\pi t), -2\pi \cos(2\pi t), 1) \, dt \\ &\qquad = \int_0^3 2\pi \sin^2(2\pi t)\cos^2(2\pi t) + 2\pi \sin(2\pi t)\cos(2\pi t)t^3 + \cos(2\pi t) - \sin(2\pi t) + t \, dt. \end{aligned}$$

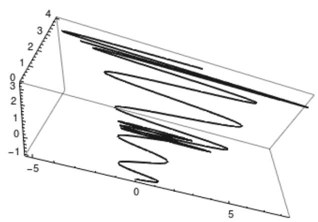
(b) Find the circulation of $\vec{F}(x, y) = -3y\vec{i} + 2x\vec{j}$ counterclockwise around the rectangle $0 \leq x \leq 4, 0 \leq y \leq 2$.




**Solution:** We compute $\|\nabla \times \vec{F}(x,y)\| = |2-(-3)| = 5$. The curve is oriented so the interior is on the left-hand side, so by Green's Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \|\nabla \times \vec{F}\| \, dA = \int_0^2 \int_0^4 5 \, dx \, dy = 40.$$

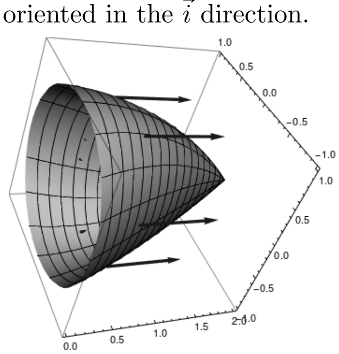
(c) Find the integral of the vector field $\vec{F}(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ over the path $\vec{r}(t) = (t + \sin(10\pi t)e^t, t^2 - \cos(2\pi t), 2^t)$ as $t$ varies from 0 to 2.



**Solution:** We observe that $\vec{F}(x,y,z) = \nabla xyz$. Thus by the fundamental theorem of line integrals we can just plug in the two endpoints. We have

$$\begin{aligned} \vec{r}(0) &= (0,-1,1) \\ \vec{r}(2) &= (2,3,4) \\ \int_C \vec{F} \cdot \vec{r} &= 24 - 0 = 24. \end{aligned}$$

**Problem 6** (M6)**.** (a) Let $\vec{F}(x,y,z) = \sqrt{x^5 + x}\vec{i} + (x^2yz - z)\vec{j} + (x\sqrt{z^3+y} + y)\vec{k}$. Compute the flux of the vector field $\nabla \times F$ through a net whose rim is the unit circle $y^2 + z^2 = 1$ in the $x = 0$ plane, oriented in the $\vec{i}$ direction.



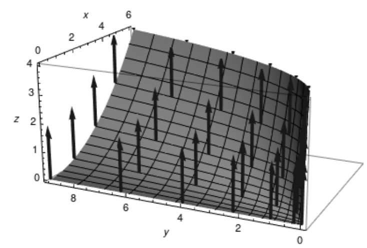
**Solution:** Instead of trying to parametrize the net, we use Stokes's theorem to just compute the circulation of $\vec{F}$ along the boundary. This means we don't even need to take the curl!

If the net is oriented in the $\vec{i}$ direction, that's the same as the circle being oriented counterclockwise when viewed from the positive $x$-axis. So we can parametrize the circle with $\vec{r}(t) = (0, \cos(t), \sin(t))$. Then by Stokes's theorem, we have

$$\begin{aligned}\int_S \nabla \times \vec{F} \cdot d\vec{A} &= \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (0, -\sin(t), \cos(t)) \cdot (0, -\sin(t), \cos(t))\, dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.\end{aligned}$$

(b) Find the flux of the vector field $\vec{F}(x, y, z) = (x, xy, z)$ through the surface parametrized by $\vec{r}(s, t) = (st, s^2, t^2)$ oriented upwards, for $0 \leq s \leq 3, 0 \leq t \leq 2$.

*Note: the arrows in the diagram are the orientation of the surface, not a representation of $F$.*



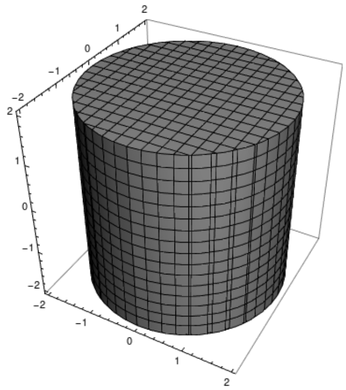

**Solution:** We need the normal vector. We have

$$\begin{aligned}\vec{r}_s &= (t, 2s, 0) \\ \vec{r}_t &= (s, 0, 2t) \\ \vec{r}_s \times \vec{r}_t &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2s & 0 \\ s & 0 & 2t \end{vmatrix} = (4st - 0)\vec{i} + (0 - 2t^2)\vec{j} + (0 - 2s^2)\vec{k}\end{aligned}$$

is oriented downwards, so instead we take $-4st\vec{i} + 2t^2\vec{j} + 2s^2\vec{k}$. Then the integral is

$$\begin{aligned}&\int_0^3 \int_0^2 (st, s^3t, t^2) \cdot (-4st, 2t^2, 2s^2)\, dt\, ds \\ &= \int_0^3 \int_0^2 -4s^2t^2 + 2s^3t^3 + 2s^2t^2\, dt\, ds \\ &= \int_0^3 \int_0^2 2s^3t^3 - 2s^2t^2\, dt\, ds \\ &= \int_0^3 \frac{1}{2}s^3t^4 - \frac{2}{3}s^2t^3 \Big|_0^2 dx \\ &= \int_0^3 8s^3 - \frac{16}{3}s^2\, ds = 2s^4 - \frac{16}{9}s^3 \Big|_0^3 \\ &= 162 - 48 = 114.\end{aligned}$$

(c) Compute $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F}(x, y, z) = xy^2\vec{i} + x^2y\vec{j} + (x^2y^2 + z)\vec{k}$ and $S$ is the surface (including both ends!) of a closed cylinder with radius 2 centered on the $z$-axis, from $z = -2$ to $z = 2$.




**Solution:** We want to use the divergence theorem here. We compute $\nabla \cdot \vec{F} = y^2 + x^2 + 1$, so we can integrate $x^2 + y^2 + 1$ over the cylinder. We use cylindrical coordinates, and get

$$\begin{aligned}\int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{2}(r^2+1)\cdot r\,dr\,d\,\theta\,dz &= \int_{-2}^{2}\int_{0}^{2\pi}\frac{1}{4}r^4+\frac{1}{2}r^2|_0^2\,d\theta\,dz \\ &= \int_{-2}^{2}\int_{0}^{2\pi}6\,d\theta\,dz = 48\pi.\end{aligned}$$

**Problem 7** (S5)**.** Let

$$\vec{F}(x,y,z) = (0,x,y) \qquad \vec{G}(x,y,z) = (2x,z,y) \qquad \vec{H}(x,y,z) = (3y,2x,z).$$

(a) For each field, either find a scalar potential function or prove that none exists.

**Solution:** We have

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & y \end{vmatrix} = (1-0)\vec{i} + (0-0)\vec{j} + (1-0)\vec{k} \neq \vec{0}$$

$$\nabla \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & z & y \end{vmatrix} = (1-1)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = \vec{0}$$

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 2x & z \end{vmatrix} = (0-0)\vec{i} + (0-0)\vec{j} + (2-3)\vec{k} \neq \vec{0}$$

so the only field that could be conservative is $\vec{G}$. To find a potential function, we would need

$$\frac{\partial g}{\partial x} = 2x$$
$$\frac{\partial g}{\partial y} = z$$
$$\frac{\partial g}{\partial z} = y$$

The first equation tells us $g(x,y,z) = x^2 + h(y,z)$. The second tells us that $g(x,y,z) = yz + i(x,z)$ and the third tells us that $g(x,y,z) = yz + j(x,y)$. Putting this all together, we can take $g(x,y,z) = x^2 + yz$.

(b) For each field, either find a vector potential function or prove that none exists.

**Solution:** $\nabla \cdot \vec{F} = 0$ so $\vec{F}$ is irrotational. We set up a system

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= 0 \\ \frac{\partial F_1}{\partial z} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= y. \end{aligned}$$

The first equation tells us that $F_2 = g(x, y)$, and the second equation tells us that $F_1 = xz + h(x, y)$. Then the third equation tells us that $g_x(x, y) - h_y(x, y) = y$; one reasonable solution for this is $g(x, y) = xy$. Thus $\vec{F}$ has a vector potential of $(xz, xy, 0)$.

$\nabla \cdot \vec{G} = 2$, so $\vec{G}$ is not a curl field. $\nabla \cdot \vec{H} = 1$, so $\vec{H}$ is not a curl field.