Math 2233 Practice Midterm 1 Solutions

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(a) These are the instructions you will see on the real test, next week. I include them here so you know what to expect.

(b) You will have **90** minutes for this test.

(c) You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.

(d) You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine, but will probably hurt you.

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Problem 1 (M1). (a) Find the area of the triangle with vertices $(4,1,1),(3,2,2),(2,3,4)$.

Solution: We have the vectors $(-1,1,1)$ and $(-2,2,3)$, and the cross product is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ -2 & 2 & 3 \end{vmatrix} = \Big(3\vec{i} - 2\vec{j} - 2\vec{k}\Big) - \Big(-2\vec{k} + 2\vec{i} - 3\vec{j}\Big) = \vec{i} + vecj$$

So the triangle has area

$$A = \frac{1}{2}\|\vec{i} + \vec{j}\| = \frac{\sqrt{2}}{2}.$$

(b) Find the cosine of the angle between the vectors $\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$.

Solution: We know that

$$\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\cdot\|\vec{u}\|} = \frac{3-4-1}{\sqrt{14}\sqrt{6}} = \frac{-2}{2\sqrt{21}} = \frac{-1}{\sqrt{21}}.$$

(c) Let $\vec{v} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{u} = -2\vec{i} - \vec{j} + 2\vec{k}$. Compute the orthogonal decomposition of $\vec{v}$ with respect to $\vec{u}$. That is, write $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\perp}$.

Solution:

$$\begin{aligned} \vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{-6-1-2}{4+1+4}\vec{u} \\ &= \frac{-9}{9}\vec{u} = 2\vec{i} + \vec{j} - 2\vec{k} \\ \vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = \vec{i} + \vec{k}. \end{aligned}$$

Problem 2 (M2). (a) Find an equation for the tangent plane to the graph of the function $f(x,y) = e^{xy} + x/y$ at the point $(0,2)$.

Solution: We have $\nabla f(x,y) = (e^{xy}y + 1/y)\vec{i} + (e^{xy}x - x/y^2)\vec{j}$, so $\nabla f(0,2) = 5/2\vec{i} + 0\vec{j}$. Further we have $f(0,2) = 1 + 0$. Thus we get the equation

$$z = 1 + \frac{5}{2}(x - 0).$$

(b) Let $g(x,y,z) = x^2y + y^2z$. Use a linear approximation at the point $(1,2,3)$ to estimate $g(.9, 2.1, 3.2)$.

Solution:

$$\begin{aligned} \nabla g(x,y,z) &= 2xy\vec{i} + (x^2 + 2yz)\vec{j} + y^2\vec{k} \\ \nabla g(1,2,3) &= 4\vec{i} + 13\vec{j} + 4\vec{k} \\ g(x,y,z) &\approx 4(x-1) + 13(y-2) + 4(z-3) + 14 \\ g(.9,2.1,3.2) &\approx 4(-.1) + 13(.1) + 4(.2) + 14 = -.4 + 1.3 + .8 + 14 = 15.7. \end{aligned}$$

(c) Let $h(x,y) = 2xy - x^2y - 2$, and $\vec{u} = \frac{-3}{5}\vec{i} + \frac{4}{5}\vec{j}$. Compute $h_{\vec{u}}(2,1)$.

Solution: $\nabla h(x,y) = (2y - 2xy)\vec{i} + (2x - x^2)\vec{j}$, so

$$\nabla f(2,1) = -2\vec{i}$$
$$h_{\vec{u}}(2,1) = 6/5.$$

(d) Compute $\nabla\left(x^2z + \sqrt{xy}\right)$. At the point $(1,2,1)$, which direction should we move to increase the value of this function as quickly as possible?

Solution:

$$(2xz + \frac{1}{2}\sqrt{y/x})\vec{i} + \frac{1}{2}\sqrt{x/y}\vec{j} + x^2\vec{k}$$

At the point $(1,2,1)$ this has the value $(2+\sqrt{2}/2)\vec{i} + \frac{1}{2\sqrt{2}}\vec{j} + \vec{k}$, so to increase the value of the function as quickly as possible, we should move in the direction of the vector $(2+\sqrt{2}/2)\vec{i} + \frac{1}{2\sqrt{2}}\vec{j} + \vec{k}$, or $(4\sqrt{2}+2)\vec{i} + \vec{j} + 2\sqrt{2}\vec{k}$.

Problem 3 (S1)**.** Give an equation for a plane through the points $(1,1,1), (1,3,5), (3,1,-3)$.

Solution: There are two approaches here.

First, we can observe that the first two points share a x coordinate and the first and third share an y coordinate. Thus we can compute the x slope is -2 and the y slope is 2. Then our equation is

$$z = -2(x-1) + 2(y-1) + 1 = -2x + 2y + 1.$$

Alternatively, we get the vectors $2\vec{j} + 4\vec{k}$ and $2\vec{i} - 4\vec{k}$. Then we compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 4 \\ 2 & 0 & -4 \end{vmatrix} = -8\vec{i} + 8\vec{j} - 4\vec{k} = \vec{n}$$

and thus the equation for the plane is

$$0 = -8(x-1) + 8(y-1) - 4(z-1).$$

These are, non-obviously, the same plane.

Problem 4 (S2)**.** (a) Find a parametric equation for a particle moving in a straight line from $(1,7,-4)$ to $(4,4,2)$

Solution:

$$\vec{r}(t) = (1,7,-4) + t(3,-3,6) = (1+3t, 7-3t, -4+6t).$$

(b) Suppose another particle follows the path $\vec{r}_2(t) = (4t, t+3, t^2+t)$. Does this particle's path intersect the path of the particle from part (a)?

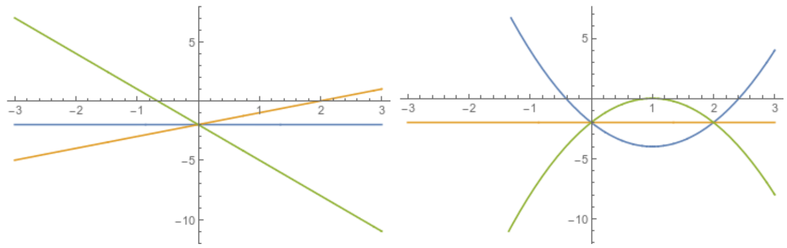
Solution: We would need

$$1 + 3t_1 = 4t_2$$
$$7 - 3t_1 = t_2 + 3$$
$$6t_1 - 4 = t_2^2 + t_2$$

The second equation gives that $t_2 = 4 - 3t_1$. Plugging that into the first equation gives $1 + 3t_1 = 16 - 12t_1$ and thus $t_1 = 1$, so $t_2 = 1$ as well. Then we see that $t_1 = t_2 = 1$ satisfies the third equation as well, so the particles' paths do intersect—and in fact the particles themselves collide, since it happens at the same time.

Problem 5 (S3)**.** Let $f(x,y) = 2xy - x^2y - 2$

(a) Sketch and clearly label cross-sections of f for $x = -1, 0, 1$ and $y = -2, 0, 2$.



Solution:

$$
\begin{array}{lll}
f(-1,y) = -2y - y - 2 = -3y - 2 & f(0,y) = -2 & f(1,y) = 2y - y - 2 = y - 2 \\
f(x,-2) = -4x + 2x^2 - 2 & f(x,0) = -2 & f(x,2) = 4x - 2x^2 - 2
\end{array}
$$

(b) Sketch and clearly label contours of f for $c = -4, -2, 0$.

Solution: Less obviously, we can write this as

$$
\begin{aligned}
2xy - x^2y - 2 &= c \\
(2x - x^2)y &= 2 + c \\
y &= \frac{2+c}{2x - x^2}
\end{aligned}
$$

as long as $2x - x^2 \neq 0$. And thus we need to graph the curves

$$
\begin{aligned}
y &= \frac{-2}{2x - x^2} \\
xy(2-x) &= 0 \\
y &= \frac{2}{2x - x^2}
\end{aligned}
$$

which gives the three curves