2 Vector Functions

2.1 Curves in Space

In this section we want to study curves through space. By a curve we mean, essentially, any shape that is in some sense "one-dimensional". So a line, a circle, and a curving spiral through three-dimensional space are all curves.

The essence of a curve is the one-dimensionality. We capture this idea by requiring position on our curves to be described by one single real number. That is, we can describe our position on the curve with exactly one coordinate. We say a system of coordinates for an object is a "parametrization", because it describes the object with some number of parameters.

Definition 2.1. We say a function $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ is a *parametrization of a curve.*

If we're being technical, the image of $\vec{r}$, the set $\{\vec{r}(t) : t \in \mathbb{R}\}$ is a *curve* and the function $\vec{r}$ is a parametrization of that curve.

Sometimes we want to consider the *components* of the function. We will usually write $\vec{r}(t) = (x(t), y(t), z(t))$ and say that the single-variable functions $x(t), y(t), z(t)$ are the components of $\vec{r}$.

Example 2.2. Let's find a parametrization for the curve $y = x^2$.

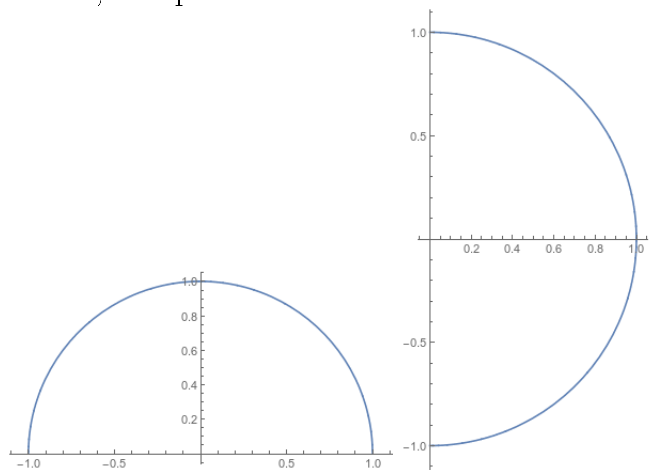
We see that we can parametrize this by the function $\vec{r}(t) = (t, t^2)$. You'll notice that this is basically the original function formula: we have $x = t$ and $y = t^2 = x^2$.

Any time we have a curve that is the graph of a single-variable function, we effectively have a parametrization for free; the entire graph is determined by the input variable, which acts as our single parameter.

Example 2.3. Let's parametrize a circle of radius 1. We know it's the graph of the equation $x^2 + y^2 = 1$, but that doesn't make it the graph of a function; for a given x value we may get two different y values, so the trick we just used won't work. In particular, we could try something like $x(t) = t, y(t) = \sqrt{1 - t^2}$ for $-1 \leq t \leq 1$. This sort of works, but only captures the top half of the circle. We could keep trying to make this idea work, but it basically won't.

Instead, we take advantage of the fact that circles are fundamentally trigonometric. We see that $\vec{r}(t) = (\cos(t), \sin(t))$ will give us every point on the circle—in fact, this is the usual unit circle definition of sin and cos. In particular, we have $\vec{r}(0) = (1, 0)$ is the rightmost point of the circle, and as t increases we move counterclockwise around the circle.

However, this isn't the only possible parametrization. For instance, we could instead take $\vec{s}(t) = (\sin(t), \cos(t))$. This will still parametrize the circle, but it starts at $\vec{s}(0) = (0, 1)$ which is the top of the circle, and proceeds clockwise.

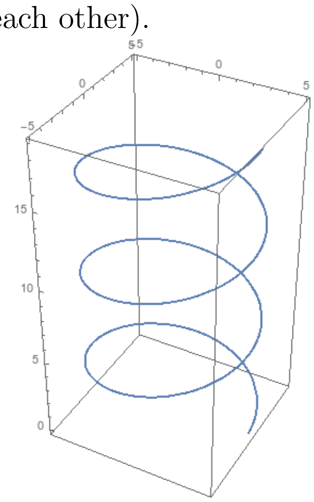


The graphs of $\vec{r}$ and $\vec{s}$ for $0 \leq t \leq \pi$

In general, choices of parametrization aren't unique. Often we can make a problem easier (or harder) by changing our choice of coordinates.

Example 2.4. Let's consider the curve given by $\vec{r}(t) = (5\cos t, 5\sin t, t)$. This gives us a circle of radius 5 if we consider only the x and y coordinates, but now the z coordinate is increasing. Thus we are spiraling around as our z coordinate increases. This gives us a shape caled a "helix".

(You might recall that DNA is described as a "double helix". This is because it is two of these helixes spiraling around each other).



The two most common shapes to parametrize are probably circles and lines. We've looked at circles already; now let's consider lines.

Example 2.5. Let's parametrize the line through $(1, 3, 5)$ in the direction of $2\vec{i} - \vec{j} + 3\vec{k}$.

This is simple and straightforward. We get $\vec{r}(t) = (1, 3, 5) + t(2, -1, 3) = (1 + 2t, 3 - t, 5 + 3t)$

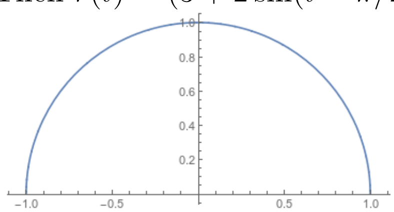
In general, a line is described by a point and a direction. Therefore, if we want to parametrize a line, we can use the equation $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ where $\vec{r}_0$ is the known point and $\vec{v}$ is the direction.

Example 2.6. Another nice property of parametrizations is that it's easy to shift them in space. Let's parametrize a circle of radius 2 centered at $(3, 2)$, going counterclockwise starting from the right-hand point.

We know that a circle of radius 1 centered at the origin is $\vec{r}(t) = (\cos(t), \sin(t))$. To get radius 3, we multiply by 3; then to shift the center, we add $(3, 2)$, leaving us with the parametrization $\vec{r}(t) = (3 + 2\cos(t), 2 + 2\sin(t))$.

If we want to start from left-hand point and go clockwise, we can do a couple things. One is to flip the circle upside down and start halfway around; this would give $\vec{r}(t) = (3 + 2\cos(t + \pi), 2 - 2\sin(t + \pi))$.

Alternatively, we could start from the parametrization $(\sin(t), \cos(t))$, which already goes clockwise. Then we would get Then $\vec{r}(t) = (3 + 2\sin(t - \pi/2), 2 + 2\cos(t - \pi/2))$.

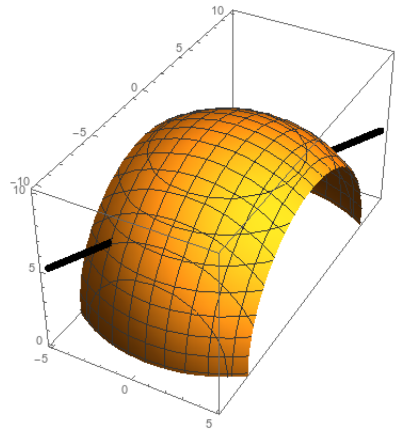


We can use parametrizations of curves to find where they intersect surfaces.

Example 2.7. Where does the curve $(t, 2t, t + 3)$ intersect the sphere of radius 9 centered at the origin?

$$\begin{aligned} x^2 + y^2 + z^2 &= 81 \\ t^2 + 4t^2 + t^2 + 6t + 9 &= 81 \\ 6t^2 + 6t &= 72 \\ t^2 + t - 12 &= 0 \\ (t + 4)(t - 3) &= 0 \end{aligned}$$

and we have $t = 3$ and $t = -4$. Thus our line intersects the sphere at $(3, 6, 6)$ and $(-4, -8, -1)$.

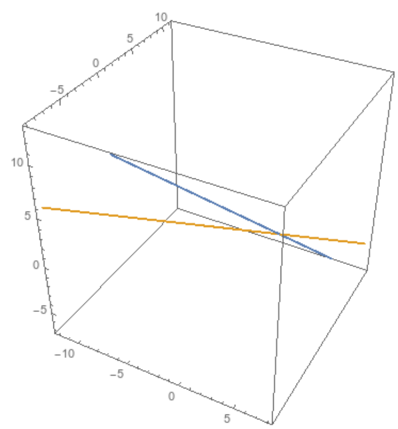


We can also see when (and if) two curves intersect.

Example 2.8. We have particles moving along the paths $\vec{r}_1(t) = (t, 1 + 2t, 3 - 2t)$ and $\vec{r}_2 = (-2 - 2t, 1 - 2t, 1 + t)$. Do the particles hit each other? Do their paths cross?

For the particles to hit each other, we need them to have the same coordinates at the same time. We see they share x coordinates when $t = -2/3$; they share y coordinates when $t = 0$ and they share z coordinates when $t = 2/3$. Thus they never collide.

If we want to see if their paths cross, we just need to test whether they ever pass through the same point. So we solve the system $t_1 = -2 - 2t_2, 1 + 2t_1 = 1 - 2t_2, 3 - 2t_1 = 1 + t_2$. The second equation gives us $t_2 = -t_1$. The other equations then are $t_1 = 2t_1 - 2$ and $3 - 2t_1 = 1 - t_1$; we can see that these both give us $t_1 = 2$ (and thus $t_2 = -2$). So the paths do cross.

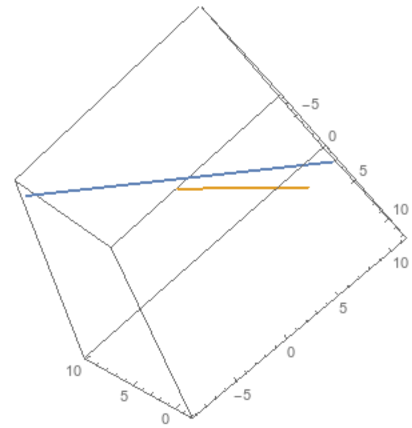


Remark 2.9. Here we have three equations in two variables, so it's very easy for the paths not to cross. But it's also quite possible for them to cross in two or more points.

Finally, we can test the relationship lines have to each other.

Example 2.10. $\vec{r}_1 = (t - 1, 1 + 2t, 5 - t)$ and $\vec{r}_2 = (2 + 2t, 4 + t, 3 + t)$.

Not parallel, because vectors/slopes are $(1, 2, -1)$ and $(2, 1, 1)$. Also don't intersect, because no solutions. So not parallel but also not intersecting. ("skew").



2.2 Motion and Calculus

So far we've discussed parametric equations as giving position as a function of time, and talking about the direction and sometimes the speed of motion. As in the single-variable case, we can make this more precise by the theory of derivatives.

Speed is change in position with respect to time. We can define this pretty easily:

Definition 2.11. The *velocity* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

This definition by itself is a bit hard to work with. However, we can make it much simpler by realizing that the x, y, and z coordinates all change independently, so we can consider them independently. (This is implicitly because derivatives are always linear, so we can write the derivative of a sum as the sum of the derivatives).

Proposition 2.12. *Let $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable function. Then*

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t)).$$

Proof.

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\
&= \lim_{h \to 0} \frac{x(t+h)\vec{i} + y(t+h)\vec{j} + z(t+h)\vec{k} - x(t)\vec{i} - y(t)\vec{j} - z(t)\vec{k}}{h} \\
&= \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}\vec{i} \frac{y(t+h) - y(t)}{h}\vec{j} + \frac{z(t+h) - z(t)}{h}\vec{k} \\
&= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.
\end{aligned}$$

□

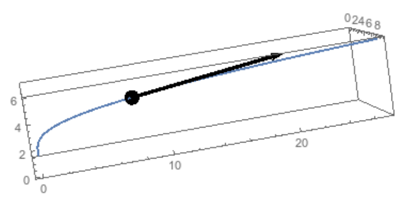
Example 2.13. Consider the circle parametrized by $(\cos(t), \sin(t))$. Then the derivative is $\vec{r}\,'(t) = (-\sin(t), \cos(t))$.

If we want to find the tangent vector at the point $(1, 0)$, we compute the derivative and plug in $t = 0$, so we get $\vec{r}\,'(0) = (0, 1)$ as your vector, and the tangent line is $(1, 0 + t)$.

Now suppose want the tangent line at $(\sqrt{2}/2, \sqrt{2}/2)$. This occurs at time $t = \pi/4$ and so we compute $\vec{r}\,'(\pi/4) = (-\sqrt{2}/2, \sqrt{2}/2)$. Thus the tangent line is $\sqrt{2}/2(1 - t, 1 + t)$.

Example 2.14. Now let's consider the curve given by $\vec{r}(t) = (t^2, t^3, 2t)$. We compute the derivative is $\vec{r}\,'(t) = 2t\vec{i} + 3t^2\vec{j} + \vec{2}k$.

If we want to find the tangent line at $t = 2$, we compute $\vec{r}\,'(2) = (4, 12, 2)$, and thus we get an equation for the line $\vec{r}(2) + t\vec{r}\,'(2) = (4, 8, 4) + t(4, 12, 2)$.



After taking the first derivative, we can also take the second (and further) derivatives. As in the single variable case, if the function gives position, and the derivative gives velocity, then the second derivative gives acceleration.

Definition 2.15. The *acceleration* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2} = \lim_{h \to 0} \frac{\vec{r}\,'(t + h) - \vec{r}\,'(t)}{h}.$$

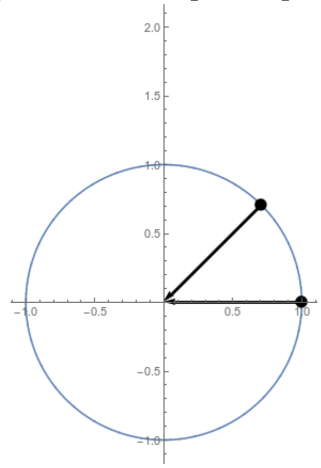
As you'd expect, we can compute the acceleration just by taking the componentwise second derivatives: we have

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = (x''(t), y''(t), z''(t)).$$

Example 2.16. Consider again the circle parametrized by $\vec{r}(t) = (\cos(t), \sin(t))$. Then we know that $\vec{r}\,'(t) = (-\sin(t), \cos(t))$, and thus the second derivative is $\vec{r}\,''(t) = (-\cos(t), -\sin(t))$.

Then we compute that $\vec{r}\,''(0) = (-1, 0)$ and $\vec{r}\,''(\pi/4) = (-\sqrt{2}/2, -\sqrt{2}/2)$.

We notice that the acceleration arrows in a circle always point inwards! This is because the motion is at a constant speed, so we can't speed up in the direction of our velocity.



Example 2.17. Suppose we have the function $\vec{r}(t) = (2, 6, 0) + (t^3 + t)(4, 3, 1)$. Then we can compute the velocity to be $\vec{r}'(t) = (3t^2 + 1)(4, 3, 1)$, and the acceleration is given by $\vec{r}''(t) = 6t(4, 3, 1)$.

Finally, we can integrate these functions. Suppose we have a function $\vec{v}(t)$ which gives the velocity of a particle as a function of time; we'd like to recover the position. Fortunately we can do this coordinate-wise, just like everything else.

Definition 2.18. Let $\vec{v} : \mathbb{R} \to \mathbb{R}^3$ be a vector-valued function, with $\vec{v}(t) = \langle x(t), y(t), z(t)\rangle$. We can define the indefinite and definite integrals of $\vec{v}$ by

$$\int \vec{v}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$
$$\int_a^b \vec{v}(t)\,dt = \left(\int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right)$$

Example 2.19. Suppose a particle moves with velocity $\vec{v}(t) = (3t + 2)\vec{i} + t^2\vec{j} - (t - 1)\vec{k}$. What is the displacement between $t = -2$ and $t = 2$?

$$\int_{-2}^{2} (3t + 2)\vec{i} + t^2\vec{j} - (t - 1)\vec{k}\,dt = (\vec{3}t^2/2 + 2t)\vec{i} + t^3/3\vec{j} - (t^2/2 - t)\vec{k}\Big|_{-2}^{2}$$
$$= \left(10\vec{i} + 8/3\vec{j} - 0\vec{k}\right) - \left(2\vec{i} - 8/3\vec{j} - 4\vec{k}\right) = 8\vec{i} + 4\vec{k}.$$

So between $t = -2$ and $t = 2$ the particle moves 8 units in the $\vec{i}$ direction and 4 in the $\vec{k}$ direction.