3 Partial Derivatives

In this section, we want to shift our attention to *multivariable functions*: functions which take in multiple inputs, but have a single number output. (We can write $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^3 \to \mathbb{R}$.) These are more complex than the vector-valued functions we studied in section 2

Now that we have a basic understanding of multivariable functions, we want to apply calculus to them. Our goal in this section is to define and understand the derivative, which measures the rate at which a function is changing.

3.1 Graphing multivariable functions

To describe and understand single-variable functions, we would draw a graph, with one dimension representing the input and one dimension representing the output. We would like to do the same thing for multivariable functions, but the situation is a bit more difficult because it's much harder to draw three-dimensional pictures. (And all but impossible to draw four- or six-dimensional pictures).

3.1.1 Graphing functions of two variables as surfaces

Recall that when we graphed a single-variable function f, we took all the points (x, y) such that $y = f(x)$. Similarly, we can define:

Definition 3.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Then the *graph* of f is the set $\{(x, y, z) : z = f(x, y)\}$ of all points with $z = f(x, y)$.

The graph of a two-variable function will generally look like a curved two-dimensional surface in three-dimensional space.

A graph of a two-variable function will still have to pass the vertical line test: a vertical line given by $x = a, y = b$ will intersect the surface in at most one point. This is because a given (x, y) input has only one output.

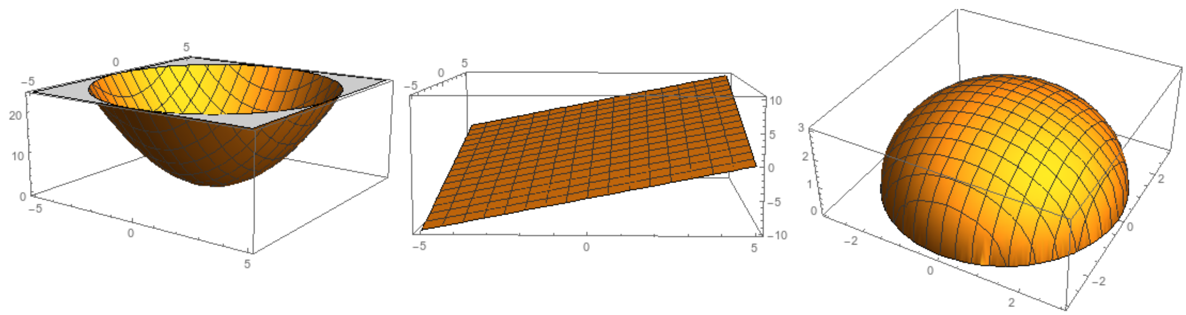


Figure 3.1: Graphs of the functions $x^2 + y^2$, $x + y$, and $\sqrt{9 - x^2 - y^2}$

3.1.2 Transformations of two-variable functions

If you already know the graph of one function, you can often figure out what the graphs of related functions must look like.

- The graph of $f(x, y) + c$ is the graph of $f(x, y)$ shifted up (along the z-axis) by c units.
- The graph of $f(x - a, y - b)$ is the same as the graph of $f(x, y)$ but shifted a units along the x axis and b units along the y axis. You can think of this as moving the center of the graph from $(0, 0)$ to (a, b).
- The graph of $f(-x, y)$ is the graph of $f(x, y)$ reflected across the yz plane, inverting the x axis.
- The graph of $f(x, -y)$ is the graph of $f(x, y)$ reflected across the xz plane, inverting the y axis.
- The graph of $-f(x, y)$ is the graph of $f(x, y)$ reflected across the xy plane, inverting the z axis and drawing the graph "upside down".

Example 3.2. Let's consider the function $f(x, y) = x^2 + y^2$ that we saw in figure 3.1. Then we can look at the following ways of shifting the function in figure 3.2:

Similarly, we can take the function $g(x, y) = x^3 + 5y$ and look at the following graphs in figure 3.3:

3.1.3 Graphing two-variable functions with cross-sections

We still don't have a good way to figure out what the graph of a two-variable function looks like if we don't already know. But the last section gives us an idea: look at each variable individually.

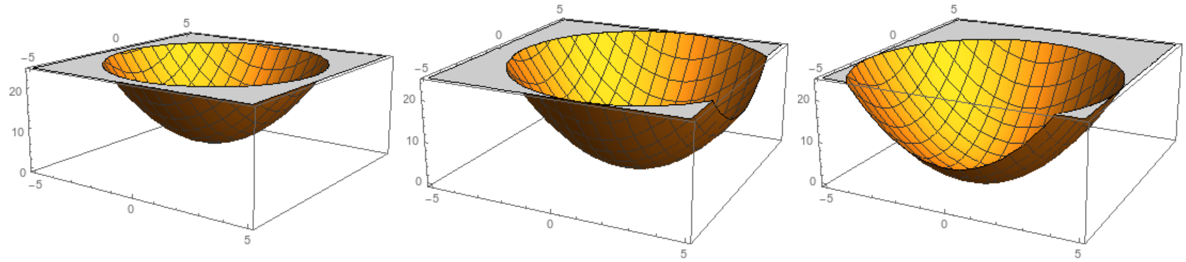


Figure 3.2: The graphs of $f(x, y) + 5$, $f(x-1, y)$, and $f(x, y+2)$

Definition 3.3. If $f(x, y)$ is a function of two variables, then we can get a function of one-variable by fixing $x = c$ and considering the function $f(c, y)$. This function is called a *cross-section* of f with x fixed. The graph of this cross-section is the curve given by intersecting the plane $x = c$ with the graph of $f(x, y)$.

Similarly, the function of one variable given by $f(x, c)$ is a cross-section of f with y fixed. The graph of this function is the curve given by intersecting the plane $y = c$ with the graph of $f(x, y)$.

Each cross-section is a single-variable function, and thus straightforward to graph. By graphing a number of cross sections we can get a good idea what the graph of the whole function looks like.

Example 3.4. Let $f(x, y) = x^2 - y^2$. First we'll take cross-sections holding y constant. We can plot these below in figure 3.4:

Thus we see that the cross-sections holding y constant are parabolas, which start lower and lower the further away we get from the $y = 0$ plane.

We can also take cross-sections holding x constant. We get the similar graphs in figure 3.5:

These show us that holding x constant, we get upside-down parabolas, with the peak being higher and higher the farther we are from the plane $x = 0$.

Putting this together, we can assemble a picture of the real function:

Example 3.5. Let $g(x, y) = x^3 + \sin(y)$. We can again take cross sections, holding x and y constant in turn:

From the left, we see that holding x constant, we have a gentle sine wave along the y axis. From the right, we see that holding y constant, x is increasing in a cubic. Putting this information together, we can get a graph for the whole surface:

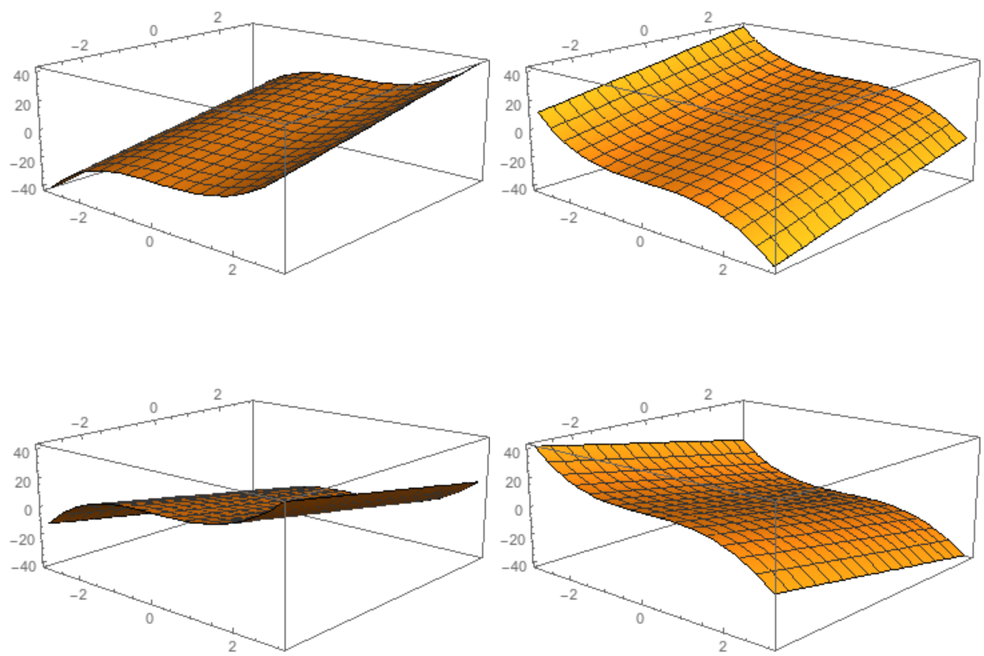


Figure 3.3: The graphs of $g(x, y)$, $g(-x, y)$, $g(x, -y)$, and $-g(x, y)$

3.1.4 Graphing two-variable functions with level sets

Sometimes we want to approach the same question from a different direction (literally!). Instead of holding x constant or y constant, we will hold z constant.

Definition 3.6. If $f(x, y)$ is a function of two variables, then the *level set* of f at level c is the set of all points (x, y) such that $f(x, y) = c$.

A *contour diagram* for f is a graph with several level sets for f at different levels.

Importantly, the level set is not a function, and doesn't need to pass any vertical line tests or anything similar.

Contour diagrams show up commonly in topographical maps.

Example 3.7. The contour plots in figure 3.9 look very similar, but the contour heights make them very different. We can see the corresponding graphs in figure 3.10.

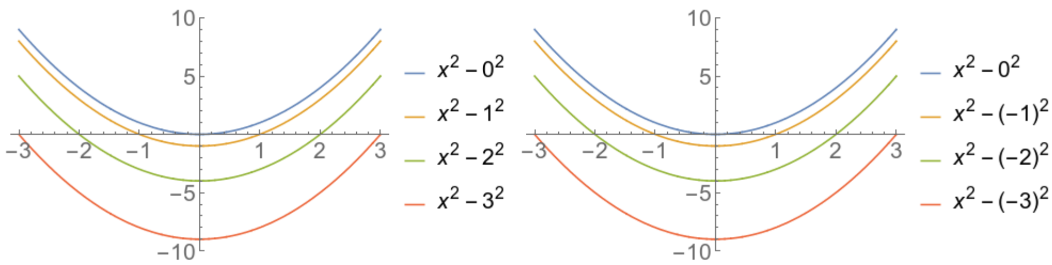


Figure 3.4: Cross sections of $x^2 - y^2$ holding y constant

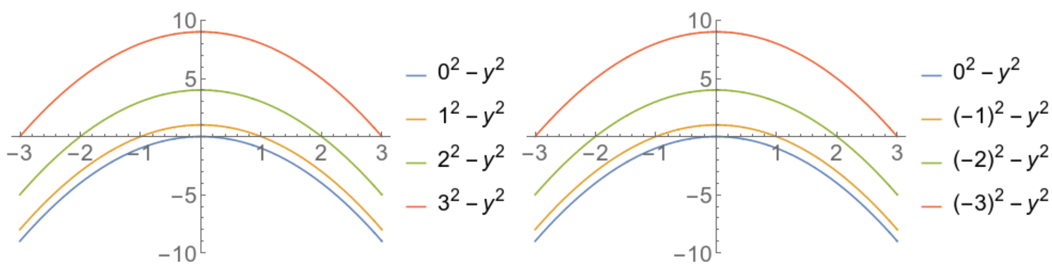


Figure 3.5: Cross sections of $x^2 - y^2$ holding x constant

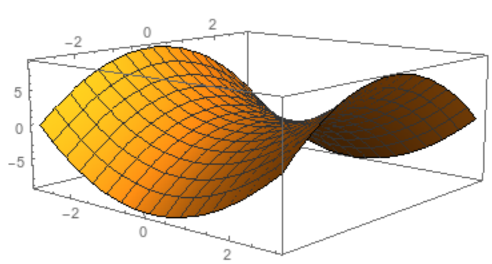


Figure 3.6: The graph of $x^2 - y^2$

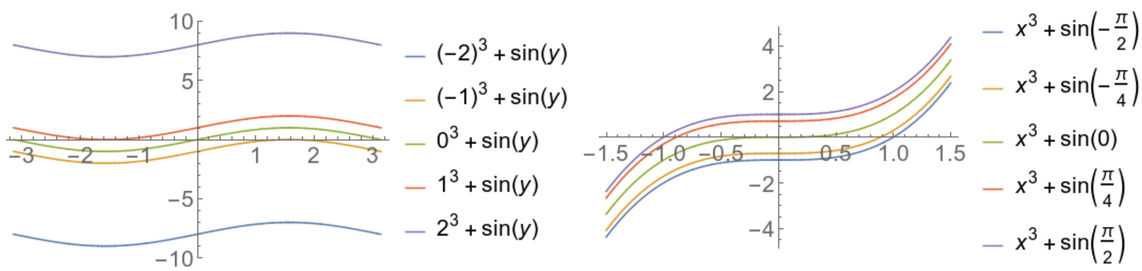


Figure 3.7: Cross sections of $x^3 + \sin(y)$, holding x constant on the left and y constant on the right

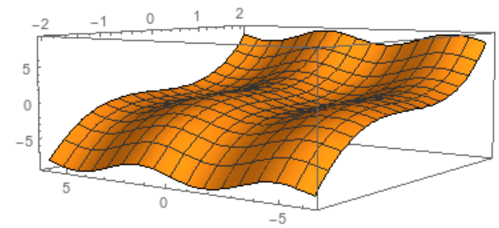


Figure 3.8: The graph of $x^3 + \sin(y)$

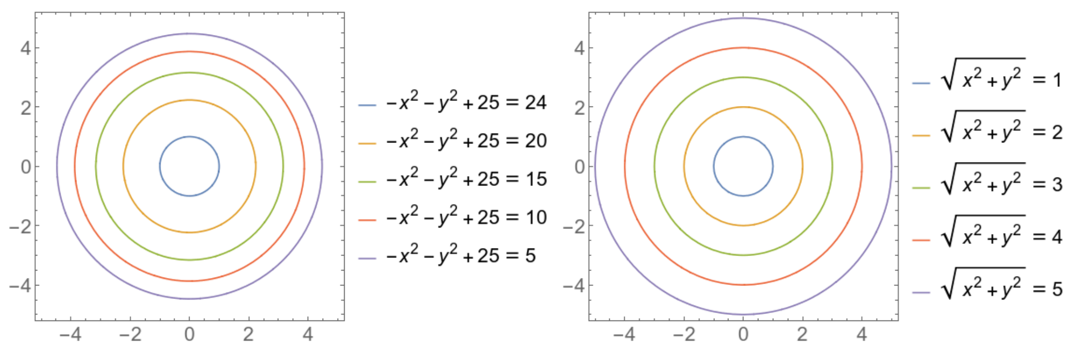


Figure 3.9: Contour diagrams for $f(x, y) = 25 - x^2 - y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$

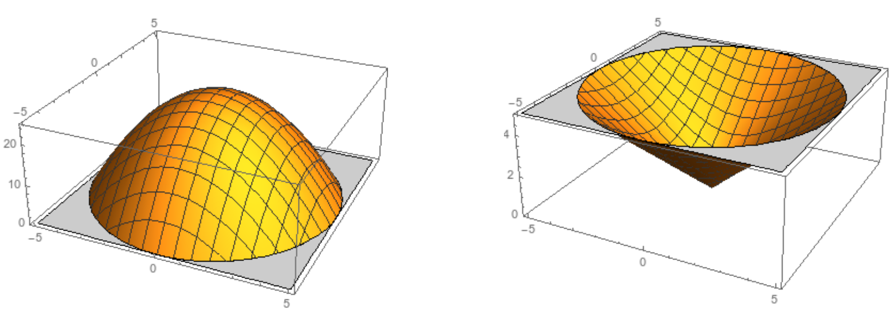


Figure 3.10: The graphs of $f(x, y) = 25 - x^2 - y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$

Example 3.8. We can also draw contour plots for some of our earlier functions. The contour plot for the saddle from example 3.4 and the sine function from example 3.5 appear in figure 3.11.

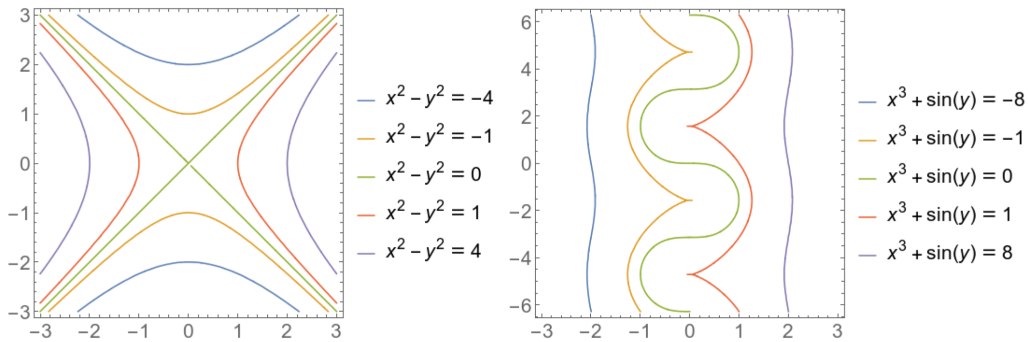


Figure 3.11: Contour plots for $x^2 - y^2$ and $x^3 + \sin(y)$

3.1.5 Graphing three-variable functions with level surfaces

We've now established a few approaches to graphically representing functions of two variables. What can we do with functions of three variables?

Simply graphing the entire function isn't really a plausible solution. As a mathematical object, the graph of a three-variable function as a subset of $\mathbb{R}^4$ is perfectly well defined; but it's almost impossible to draw or visualize these graphs, so they don't help us with our problem of visually representing three-variable functions.

In contrast, cross-sections and level sets are both useful tools. They are much tricker to implement here, because the cross-sections and level sets will themselves be two-variable functions, and thus give us two-dimensional surfaces inside threespace.

Definition 3.9. If $f(x, y, z)$ is a function of two variables, then the *level set* of f at level c is the set of all points (x, y, z) such that $f(x, y, z) = c$.

It's much harder to draw a contour diagram in this case, but we can sort of make an attempt still.

Example 3.10. Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

There are no surfaces for $c < 0$, and for $c = 0$ the level surface is a point. For larger c we get a sphere of radius $\sqrt{c}$. Thus the level sets for $c = 1, 4, 9$ are shown in figure 3.12.

Example 3.11. We can see the level surfaces of $g(x, y, z) = x^2 + y^2$ and $h(x, y, z) = x + z$, at the levels $1, 2, 3, 4$, in figure 3.13. Thbe level surfaces for g are cylinders of radius $\sqrt{c}$, and the level surfaces of h are all parallel planes.

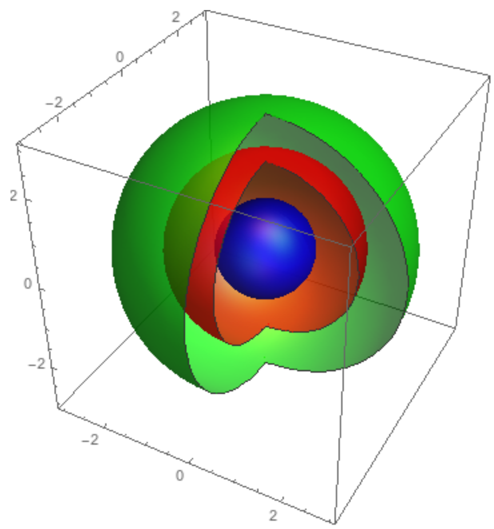


Figure 3.12: Level sets for $x^2 + y^2 + z^2$ at the levels $c = 1, 4, 9$

Example 3.12. We'd like to understand the level surfaces of $f(x, y, z) = x^2+y^2-z^2$. These will look different depending on the level of c.

It's probably easiest to think about these level surfaces by thinking about their own contour plots as z varies. If $c = 0$, then our equation is $x^2 + y^2 = z^2$. We see that for each z we get a circle of radius z in the plane perpendicular to the z-axis, and in fact at $z = 0$ we have a single point. Stacking these all together gives us two cones.

If c is positive, then we have the equation $x^2 + y^2 = z^2 + c$. Then we see that for each z we get a circle of radius $\sqrt{z^2 + c} > z$, and the radius will always be positive. If instead we take, say, the $x = 0$ cross-section, we get $y^2 - z^2 = c$, which is a hyperbola. The resulting surface is a *hyperboloid of one sheet*.

Finally, if c is negative, we get $x^2+y^2 = z^2+c$, where there is no solution when $z^2+c < 0$. Thus we'll have a stack of increasing radius circles, but it will start at $z = \pm\sqrt{c}$. This surface is a *hyperboloid of two sheets*.

Remark 3.13. We've used surfaces to represent the full graph of two-variable functions, and also to represent the level surfaces of three-variable functions. These surfaces are at least somewhat related, and in fact if we have the graph of a function $f(x, y)$, then it is also the level surface at zero of the function $f(x, y) - z$.

Thus every graph of a two-variable function is also a level surface of some three-variable function. The converse, however, is not true; many of the level surfaces we have seen cannot be the graphs of two-variable functions, since they fail the vertical line test.

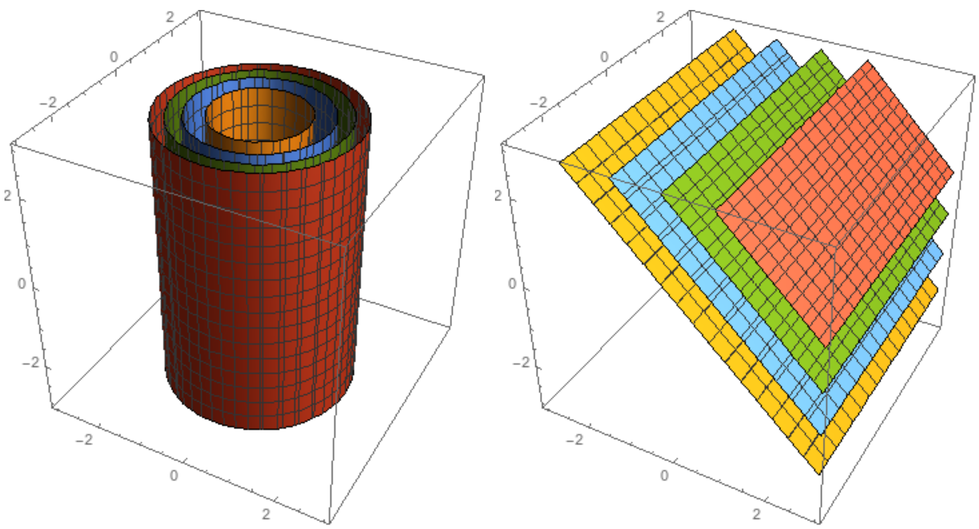


Figure 3.13: The level surfaces of g and h at the levels $1, 2, 3, 4$

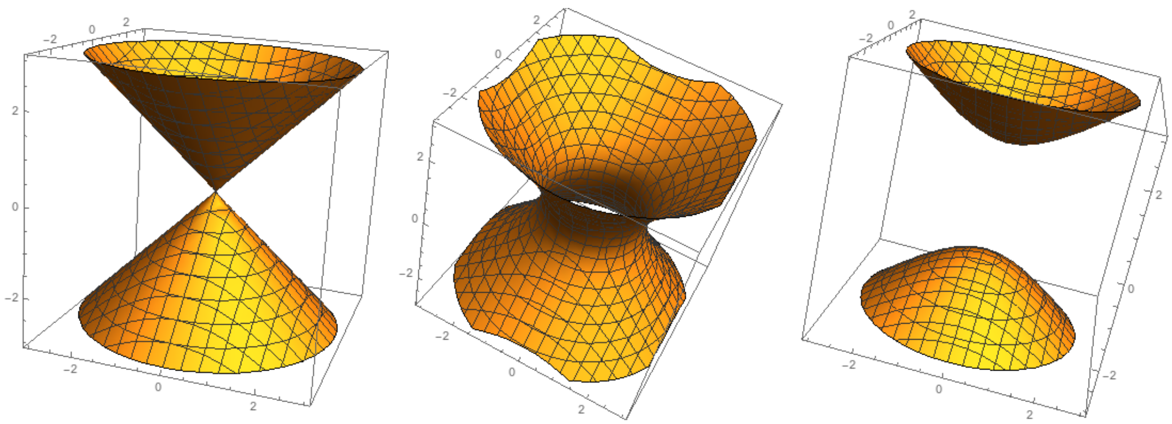


Figure 3.14: Level surfaces of $x^2 + y^2 - z^2$ at the levels $0, 2, -2$

3.2 Limits and Continuity

In calculus 1, we learned about limits, which tell us in some sense the value a function "should" have at a point—which may or may not be the value it does have, and it may not have a value at all. We can extend the same idea to multivariable functions.

Definition 3.14. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then it has a limit at the point (a, b) of L, and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if we can make $f(x, y)$ as close as we want to L, purely by requiring the distance from (x, y) to (a, b) to be small enough (but not zero).

Remark 3.15. Formally we'd write something like: for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$. We won't be drilling down into that level of rigor in this class, though. If you want to see more of this kind of thing, take Math 4239.

Example 3.16. When the function doesn't do anything weird, limits won't do anything surprising.

$\lim_{(x,y)\to(3,4)} x^2 + y^2 = 3^2 + 4^2 = 25$
$\lim_{(x,y)\to(1,-1)} x^2 - y^2 = 1^2 - (-1)^2 = 0$

Definition 3.17. A function f is *continuous* at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

A function is *continuous on a region R* if it is continuous at each point in R.

If f is not continuous at a point (a, b) then it is *discontinuous* there.

Fact 3.18. *A function defined entirely from algebraic, trigonometric, and exponential functions is continuous anywhere it is defined.*

(No function is ever continuous anywhere it is not defined).

Example 3.19. Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$. This function is continuous everywhere it is defined, which is everywhere except $(0, 0)$. So it's easy to compute, for instance, that $\lim_{(x,y)\to(1,1)} f(x,y) = \frac{1^2 \cdot 1}{1^2+1^2} = \frac{1}{2}$.

Now let's consider $\lim_{(x,y)\to(0,0)} f(x,y)$. We can't just plug $(0, 0)$ in here, so we need to do something else.

First, we can look at the graph and contour diagram of f.

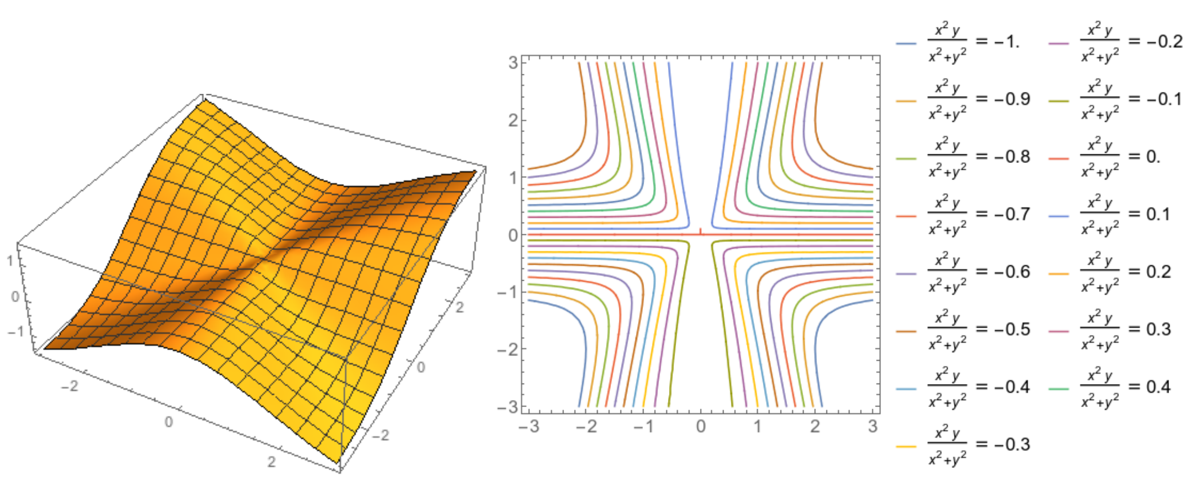


Figure 3.15: The graph and contour plot for $f(x,y) = \frac{x^2 y}{x^2+y^2}$

We see that the function seems to smoothly approach 0 as (x, y) approaches $(0, 0)$, so we suspect the limit is in fact 0.

Informally, we see that the denominator "goes to zero" "twice", while the numerator goes to zero "three times". Thus we would expect the limit to be zero.

If we want to be more rigorous, we calculate the distance between $f(x, y)$ and the guessed limit 0. Then we have

$$|f(x, y) - L| = \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| = \left| \frac{x^2}{x^2 + y^2} \right| |y| \leq |y| \leq \sqrt{x^2 + y^2}.$$

Thus the distance between $f(x, y)$ and 0 is less than the distance between (x, y) and $(0, 0)$. Clearly by making (x, y) closer to $(0, 0)$ we can make $f(x, y)$ as close as we want to 0.

Since the limit exists, we can extend this function to be continuous at the origin: the function

$$f_f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

Example 3.20. Now let's consider the similar function $g(x, y) = \frac{x^2}{x^2+y^2}$. Like in example 3.19, this is continuous everywhere it is defined, which is everywhere except at $(0, 0)$.

But at $(0, 0)$ things are tricker. The graph has a noticeable spike, and the contour plot looks terrible near $(0, 0)$, with all the contours converging onto that single point.

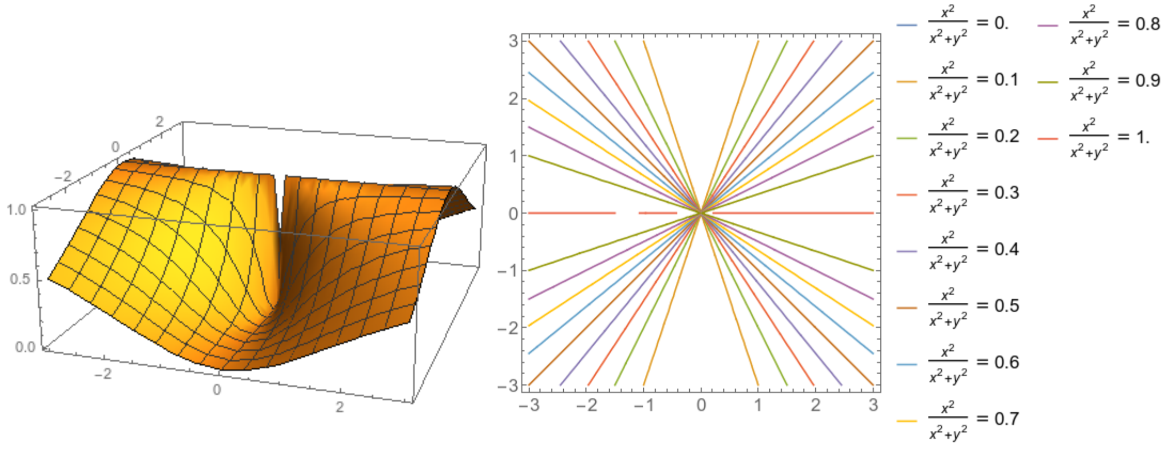


Figure 3.16: The graph and contour plot for $g(x, y) = \frac{x^2}{x^2+y^2}$

The informal algebraic argument we gave before doesn't help: both the top and the bottom go to zero "twice". So this doesn't help us find any limit.

Formally, we want to show that no limit exists, so we want to show that you can be as close to $(0, 0)$ as you want and still get very different answers for $g(x, y)$.

So first let's consider points that look like $(a, 0)$. Then $g(a, 0) = \frac{a^2}{a^2+0^2} = 1$. Since a can be as small as we want, this tells us that we can be as close to the origin as we want and have $g(x, y) = 1$.

But this doesn't mean the limit is 1! As an example, take points that look like $(0, b)$. Then $g(0, b) = \frac{0^2}{0^2+b^2} = 0$. Since b can be anything, this also tells us that we can be as close to the origin as we want, and have $g(x, y) = 0$. Thus no limit exists.

In fact, by approaching from the right direction, we can get any value between 0 and 1. And we can see this behavior both in the graph (which has an abrupt spike or dip near the origin), and in the contour plot (which shows us different directions of approach, and the values they will give).

We just saw that we can show that limits don't exist by approaching the same point from different directions. This should remind you of the one-variable case, where we might check the right- and left-sided limits and show they differ.

But the multivariable case is considerably more complex, because there are infinitely many directions. In fact it's more complicated than that: there are functions that have a consistent limit as long as you approach along any straight-line path, but that break down when you approach along the right curve. We'll see an example in recitation.

3.3 The Partial Derivative

We've already been talking about how a function changes when you change one of the input variables. This is exactly the single-variable calculus derivative and can be defined accordingly.

Definition 3.21. Let f be a function of two variables. Then we define the *partial derivatives at the point* (a, b) by

$$\begin{aligned}\frac{\partial f}{\partial x}(a, b) &= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = f_x(a, b) \\ \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} = f_y(a, b).\end{aligned}$$

If we allow (a, b) to vary, we get functions $f_x(x, y)$ and $f_y(x, y)$.

We will sometimes write $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. If we want to represent these derivatives evaluated at a point, we will write $\frac{\partial z}{\partial x}\Big|_{(a,b)}$ and $\frac{\partial z}{\partial y}\Big|_{(a,b)}$.

Remark 3.22. This isn't just analogous to the single-variable calculus derivative; it is exactly identical. If we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ and we hold the second variable fixed at $y = b$, then we get a single-variable function defined by $f_b(x) = f(x, b)$. Then $f_x(a, b) = f_b'(a)$ is just the single-variable derivative of this single-variable function.

The interesting part here is not that we can define the partial derivatives, which are basically old news. The interesting thing is that we can get the answers to genuinely multi-varaible questions out of these essentially single-variable tools.

Example 3.23. Suppose a differentiable function $f(x, y)$ has the following values:

$y \setminus x$	0	1	2	3	4	5
0	120	135	155	160	160	150
1	125	128	135	160	175	160
2	100	110	120	145	190	170
3	85	90	110	135	155	180

Then we can estimate the partial derivatives off the chart. For instance, we can estimate that $f_x(3, 1)$ is about 20: since $f(4, 1) - f(3, 1) = 15$ and $f(3, 1) - f(2, 1) = 25$. Similarly, we can estimate $f_y(3, 1) \approx -7.5$ since $f(3, 1) - f(3, 0) = 0$ and $f(3, 2) - f(3, 1) = -15$.

One way to understand partial derivatives is to think about the units of the function. For instance, in your homework (problem 12.3.26) you looked at a function $H(x, t)$ that took position and time as inputs, and had temperature as an output. Then $H_x(x, t)$ has units of degrees per meter—how quickly temperature changes when you move a foot away. And $H_t(x, t)$ has units of degrees per minute—how quickly temperature changes over time.

Partial derivatives are easy and quite boring to calculate. Since we're looking at $f(x, y)$ as a function of a single variable, while holding the other constant, we can pretend it's simply a single-variable function, and treat the other variable like a constant.

Example 3.24. Let $f(x, y) = x^2 + y^2$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$.

Let $g(x, y) = \sin(xy)$. Then $g_x(x, y) = \cos(xy) \cdot y$ and $g_y(x, y) = \cos(xy) \cdot x$.

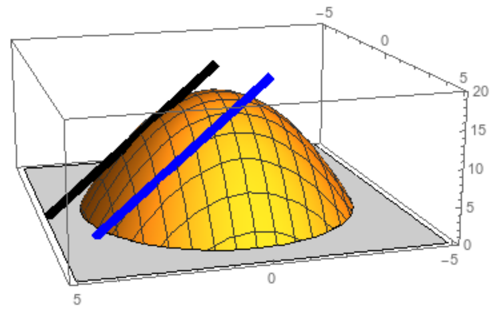
Let $h(x, y) = \frac{x^2}{y^3 - 3y}$. Then $h_x(x, y) = \frac{2x}{y^3 - 3y}$ and $h_y(x, y) = -\frac{x^2(3y^2 - 3)}{(y^3 - 3y)^2}$.

We can graphically understand partial derivatives by thinking about the cross-section.

Example 3.25. Let $f(x, y) = 16 - x^2 - y^2$. Then $f_x(x, y) = -2x$. Thus $f_x(2, 0) = -4$, and the cross-section at 0 is $f(x, 0) = 16 - x^2$ and has tangent line $z - 12 = -4(x - 2)$.

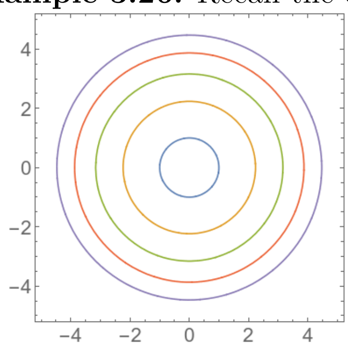
Similarly, if we look at the point $(2, 2)$, we see that the cross-section is $f(x, 2) = 12 - x^2$ and the derivative is $f_x(2, 2) = -4$, so the tangent line is $z - 8 = -4(x - 2)$.

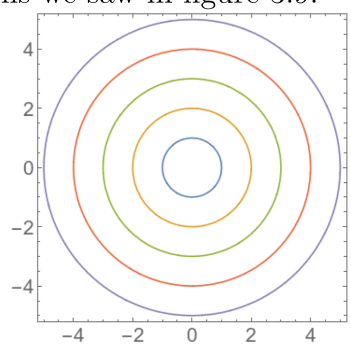
Notice that the slopes of both lines are the same, since here $f_x(x, y)$ doesn't depend on y.



In section 3.1.4 we talked about reading contour diagrams and thinking about in which directions the function was changing. We can interpret this in terms of partial derivatives.

Example 3.26. Recall the contour diagrams we saw in figure 3.9:





We can ask questions like $f_x(1,0)$ and $g_x(1,0)$. Looking at the graph, we see that $f_x(1,0) \approx -4$ since it changes from 24 to 20 between $(1,0)$ and $(2,0)$. We can see that $f_y(1,0)$ is slightly smaller, since going from $(1,0)$ to $(1,1)$ doesn't quite get us from 24 to 20.

Similarly, $g_x(-2,0)$ is about -1, since $g(-3,0) = 3, g(-2,0) = 2$, and $g(-1,0) = 1$. $g_y(-2,0)$ is positive but less than 1.

We can also define the partial derivatives in three (or more) dimensions; the only thing that changes is that the picture becomes more difficult to draw.

Example 3.27. Let $f(x,y,z) = x^2 + xyz + y/z$. Then we have

$$\begin{aligned} f_x(x,y,z) &= 2x + yz \\ f_y(x,y,z) &= zy + 1/z \\ f_z(x,y,z) &= xy - y/z^2. \end{aligned}$$

3.4 Local Linear Approximation

In many ways, the most important application of the derivative is the ability to approximate a function with a linear function. The basic idea is the same as the idea from single-variable calculus. If you zoom in enough on a 1-variable function, it will loook mostly like a line; if you zoom in on a 2-variable function, it will look like a plane.

Definition 3.28. A 2-variable *linear function* is given by a formula $f(x, y) = z_0 + m(x - x_0) + n(y - y_0)$. We might say this function has slope m in the x direction and slope n in the y direction. We could also write $f(x, y) = c + mx + ny$, but this usually isn't as helpful.

A 3-variable linear function is given by $g(x, y, z) = w_0 + m(x - x_0) + n(y - y_0) + \ell(z - z_0)$.

Remark 3.29. If you have taken linear algebra, you will notice that this is somewhat different from the definition of a linear function given there. A linear function in the linear algebra sense must also pass through the origin, and thus the equation can always be written $f(x, y) = mx + ny$.

Thus technically we have defined an "affine transformation" rather than a linear transformation. But under the same technicality, most lines in single-variable calculus are not linear functions. We'll mostly ignore that language here.

The important thing about linear functions is that changes in x and changes in y change the output completely independently. This makes everything about the functions very simple.

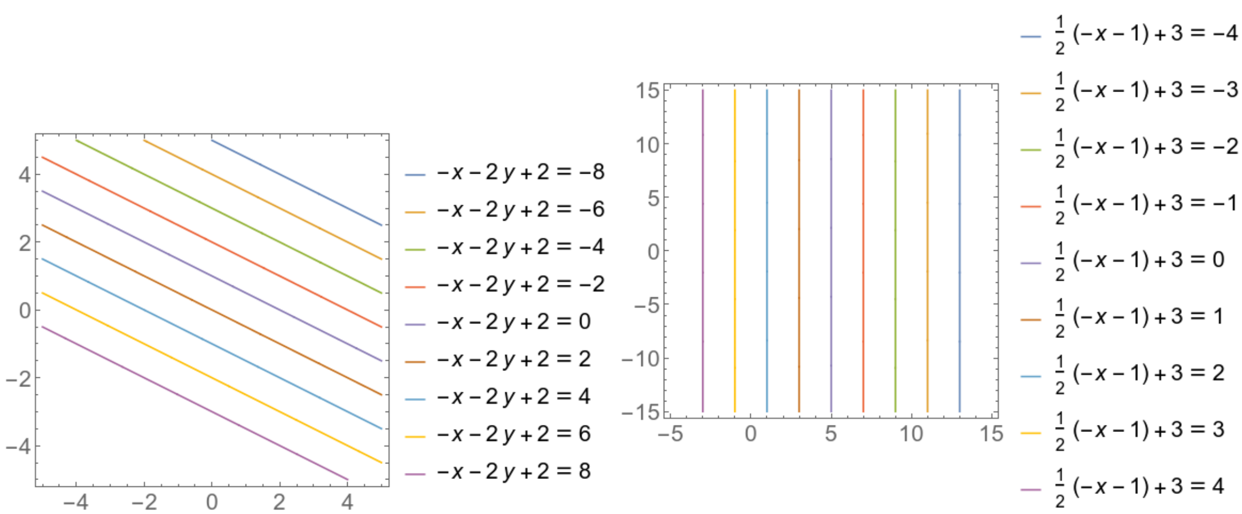


Figure 3.17: Contour diagrams for $f(x, y) = 2 - x - 2y$ and $g(x, y) = 3 - (x + 1)/2$

The graph of a linear function will look like a plane. We have

$$z = z_0 + m(x - x_0) + n(y - y_0)$$
$$0 = -(z - z_0) + m(x - x_0) + n(y - y_0)$$

so this plane goes through the point (x_0, y_0, z_0) and has normal vector $m\vec{i} + n\vec{j} - \vec{k}$.

Definition 3.30 (Informal)**.** Roughly speaking, the *tangent plane* to a surface at the point (x, y, z) is a plane that passes through the point (x, y, z), and touches the surface only at that point.

Proposition 3.31. *If $f(x, y)$ is differentiable at the point (a, b), then the equation of the tangent plane to $z = f(x, y)$ at the point (a, b) is*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

From the equation form, we see that this plane must pass through the point $(a, b, f(a, b))$. Further, the slope in the x direction is $f_x(a, b)$, which is the rate at which f is changing when you change x. Similarly, $f_y(a, b)$ is the slope in the y direction.

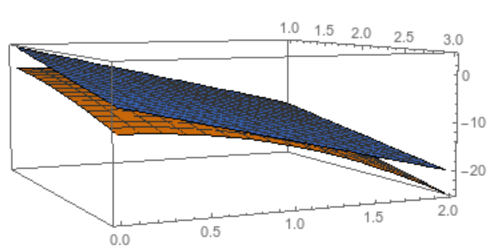
Example 3.32. Let's find the tangent plane to the function $f(x, y) = -x^2 - 4y^2$ at the point $(2, 1, -8)$.

We compute

$$f_x(x, y) = -2x \qquad f_x(2, 1) = -4$$
$$f_y(x, y) = -8y \qquad f_y(2, 1) = -8.$$

Since $f(2, 1) = -8$, the equation for the tangent plane is

$$z = -8 - 4(x - 2) - 8(y - 1)$$



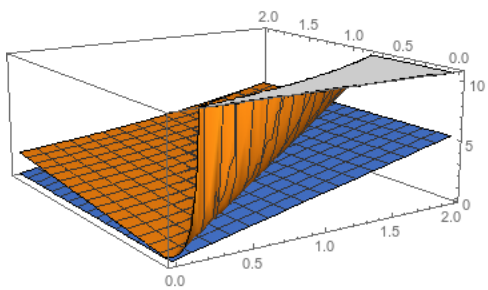
Example 3.33. Let's find the tangent plane to the function $g(x, y) = ye^{x/y}$ at the point $(1, 1)$.

We compute

$$\begin{aligned} g_x(x, y) &= ye^{x/y}\frac{1}{y} = e^{x/y} \\ g_x(1, 1) &= e \\ g_y(x, y) &= e^{x/y} + ye^{x/y}\frac{-x}{y^2} = e^{x/y} - \frac{x}{y}e^{x/y} \\ g_y(1, 1) &= e - e = 0. \end{aligned}$$

Since $g(1, 1) = e$, the equation for the tangent plane is

$$z = e + e(x - 1).$$



As with linear functions in single-variable calculus, we can use the tangent plane to approximate the values of a funtion.

Example 3.34. Let's estimate $g(1.1, 1)$.

We know that

$$\begin{aligned} g(x, y) &\approx e + e(x - 1) \\ g(1.1, 1) &\approx e + e(1.1 - 1) = e + .1e = 1.1e. \end{aligned}$$

Using Mathematica, we compute that $g(1.1, 1) \approx 3.00417$, and $1.1e \approx 2.99011$, so this is pretty good.

Definition 3.35. The *tangent plane approximation* to a function $f(x, y)$ near the point (a, b) is given by

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

It's hard to graph, but we can do the same thing in three (or more) dimensions. The *linear approximation* to a function $f(x, y, z)$ near the point (a, b, c) is given by

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

Sometimes you'll see these ideas described differently, in terms of the "differential".

Definition 3.36. The *differential* df of a function f at a point (a, b) is a linear function in the variables dx and dy, given by

$$df = f_x(a, b)dx + f_y(a, b)dy.$$

We will sometimes write $df = f_x dx + f_y dy$.

We can interpret the differential as being, for each point (a, b), a linear function that takes in a change in the x and y coordinates and outputs a change in the z coordinate. Thus

$$f(a + dx, b + dy) \approx f(a, b) + df(dx, dy) = f(a, b) + f_x(a, b)dx + f_y(a, b)dy.$$

3.5 Gradients and directional derivatives

In the previous sections we used the partial derivatives to tell us how $f(x, y)$ will change as we change the input variables x and y. But that's unnecessarily rigid; there's nothing special about *only* changing just the x input, or just the y input. We'd like to generalize this further, and see what happens when we change the input in an arbitrary direction.

Definition 3.37. Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ be a unit vector. Then we define the *directional derivative* of f in the direction $\vec{u}$ at the point (a, b, c) to be

$$f_{\vec{u}}(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b)}{h}$$

to be the rate of change of f in the direction $\vec{u}$.

If $\vec{v}$ is a non-zero non-unit vector, then we say the directional derivative in the direction of $\vec{v}$ is the directional derivative in the direction of $\frac{\vec{v}}{\|\vec{v}\|}$.

Conceptually, here we're seeing what happens if we change the input in the direction $\vec{u}$ with a change of size h, and then letting the size of the change go to zero.

Remark 3.38. If $\vec{u} = \vec{i}$, then $f_{\vec{u}} = f_x$. Similarly $f_{\vec{j}} = f_y$ and $f_{\vec{k}} = f_z$.

Example 3.39. Let's look at some of our contour plot from section 3.3 again.

We can compute these directional derivatives directly from the definition.

Example 3.40. Let $f(x) = x^2 - y^2$ (the function whose contour plot is in example 3.39). Let's compute the directional derivative in the $\vec{i} + \vec{j}$ direction at the point $(1, -3)$. Our unit vector in that direction is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, and we compute

$$\begin{aligned} f_{\vec{u}}(1,-3) &= \lim_{h \to 0} \frac{f(1 + h/\sqrt{2}, -3 + h/\sqrt{2}) - f(1,-3)}{h} \\ &= \lim_{h \to 0} \frac{(1 + h/\sqrt{2})^2 - (-3 + h/\sqrt{2})^2 - (1^2 - (-3)^2)}{h} \\ &= \lim_{h \to 0} \frac{1 + \sqrt{2}h + h^2/2 - (9 - 3\sqrt{2}h + h^2/2) - (-8)}{h} \\ &= \lim_{h \to 0} \frac{4\sqrt{2}h}{h} = \lim_{h \to 0} 4\sqrt{2} = 4\sqrt{2}. \end{aligned}$$

Computing the directional derivative directly from the limit definition is completely possible, but it's tedious. Just as we found easy ways to compute the single-variable derivative, we would like easy ways to compute the directional derivative of a multivariable function.

Fortunately, the partial derivatives give us enough information to do this. By local linearity, we see that

$$\begin{aligned} f(a + hu_1, b + hu_2) &\approx f(a,b) + hu_1 f_x(a,b) + hu_2 f_y(a,b) \\ \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h} &\approx \frac{hu_1 f_x(a,b) + hu_2 f_y(a,b)}{h} = u_1 f_x(a,b) + u_2 f_y(a,b). \end{aligned}$$

Since this approximation should get increasingly good as h gets small, we conclude that

$$f_{\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h} = u_1 f_x(a,b) + u_2 f_y(a,b).$$

Example 3.41. Let's work out our previous example this way. If $f(x,y) = x^2 - y^2$, we see that $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$. Thus $f_x(1,-3) = 2$ and $f_y(1,-3) = 6$. Then we have

$$f_{\vec{u}}(1,-3) = \frac{1}{\sqrt{2}} \cdot 2 + \frac{1}{\sqrt{2}} \cdot 6 = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

as we got before.

In this computation, you may notice that we have something that looks like a dot product of $\vec{u}$ with a vector containing the partial derivatives. This leads us to define an object that we will use in almost all of our derivative calculations in the future.

Definition 3.42. If $f(x,y)$ is differentiable at (a,b), then the *gradient vector* of f at (a,b) is

$$\operatorname{grad} f(a,b) = \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}.$$

Similarly, if $f(x,y,z)$ is differentiable at (a,b,c), then the gradient vector is

$$\operatorname{grad} f(a,b,c) = \nabla f(a,b,c) = f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}.$$

Remark 3.43. We sometimes say that

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

This is just another way of writing the same definition, but is really notationally convenient.

Proposition 3.44. *If f is differentiable at (a,b,c) and $\vec{u}$ is a unit vector, then*

$$f_{\vec{u}}(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}.$$

Example 3.45. Let $f(x,y) = xy - \sin(x)$. Then the gradient is

$$\nabla f(x,y) = (y - \cos(x))\vec{i} + x\vec{j}$$

and the gradient at the point $(\pi, 1)$ is

$$\nabla f(\pi, 1) = 2\vec{i} + \pi\vec{j}.$$

The directional derivative in the direction $3/5\vec{i} + 4/5\vec{j}$ is

$$(2\vec{i} + \pi\vec{j}) \cdot (3/5\vec{i} + 4/5\vec{j}) = \frac{6 + 4\pi}{5}.$$

The gradient tells us basically everything we want to know about the derivative of the function f; in many ways it "is" the derivative. (From a linear algebra perspective, ∇f is the matrix corresponding to the local linearization of f).

Proposition 3.46. *If f is differentiable at (a,b,c) and $\nabla f(a,b,c) \neq \vec{0}$, then:*

- *$\nabla f(a,b,c)$ is in the direction of maximum increase for f.*
- *$\|\nabla f(a,b,c)\|$ is the maximum rate of increase of f in any direction.*
- *$\nabla f(a,b,c)$ is perpendicular to the level sets of f.*

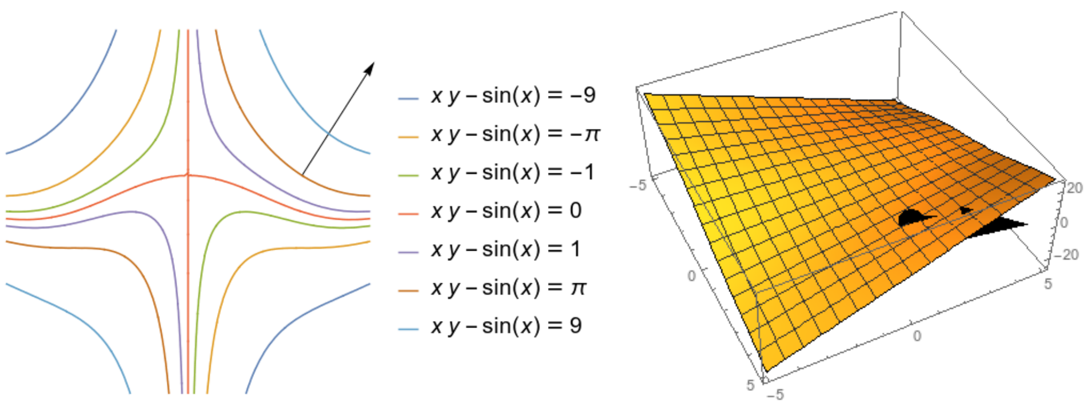
Proof. The rate of increase in the direction of a unit vector $\vec{u}$ is

$$\nabla f(a,b,c)\dot{\vec{u}} = \|\nabla f(a,b,c)\| \cdot \|\vec{u}\| \cos\theta = \|\nabla f(a,b,c)\| \cos\theta.$$

This is maximized when $\theta = 0$, which is when $\nabla f(a,b,c)$ and $\vec{u}$ point in the same direction; the maximum value is $\|\nabla f(a,b,c)\|$.

$\nabla f(a,b,c)$ is the normal vector to the tangent plane (or line) at (a,b,c), and thus is perpendicular to the tangent plane. Thus it is perpendicular to the level set. □

Example 3.47. We can look at the contour diagram and the graph for the function $f(x,y) = xy - \sin(x)$ from example 3.45.

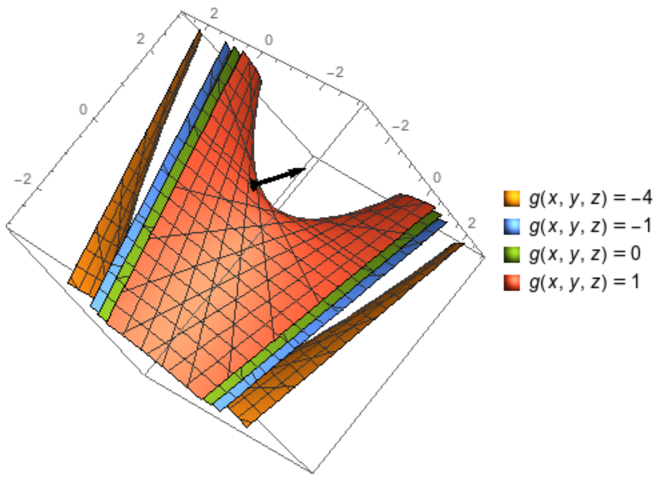


We see in the contour diagram that the gradient vector is perpendicular to the contour, and is in the direction of greatest increase. We can see the latter again in the three-dimensional graph—but this is much harder to read and see what's happening.

Example 3.48. Let's do a three-variable example next. Let $g(x,y,z) = xy + z$. Then

$$\nabla g(x,y,z) = y\vec{i} + x\vec{j} + 1\vec{k}.$$

At the point $(-1,0,1)$, we have $\nabla g(x,y,z) = -\vec{j} + \vec{k}$. Thus the direction of greatest increase is $-\vec{j} + \vec{k}$ and the magnitude of the increase in that direction is $\sqrt{2}$.



What if we want the directional derivative in the direction of, say $\vec{v} = 2\vec{i} + \vec{k}$? Then we have

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{k}$$
$$f_{\vec{u}}(-1,0,1) = (-\vec{j} + \vec{k}) \cdot \vec{u} = 0 \cdot \frac{2}{\sqrt{5}} - 1 \cdot 0 + \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

3.6 The Chain Rule

We'd like an analogue of the single-variable chain rule for multivariable functions. In the single-variable case, we ask how much f changes when you change x, and then how much g changes when you change $f(x)$, and multiply those together: $\frac{d}{dx}g(f(x)) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x)$.

The intuition in the multivariable case is basically the same; we track what effect changing each input has, and multiply them through. The expressions are more complicated pretty purely because there are more levers we can push on to change things.

To build some intuition, we'll start with the case where our composite isn't *really* a multivariable function: f depends on two variables, but each of those variables depends only on some variable t. This corresponds to, say, the force acting on a particle over time, when the force depends on position in space and the position in space depends on time.

Proposition 3.49 (Parametrized Chain Rule)**.** *If f, g, h are differentiable, and $x = g(t)$ and $y = h(t)$, then*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Conceptually, what's happening here is that we look at every way that f can change, and then see how t affects each of those factors; then we add all the separate changes together.

(This is making some implicit assumption that things are almost linear—but every time we use the derivative, we're making that assumption).

Sketch. We know that $\Delta f \approx \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y$. But further we know that $\Delta x \approx \frac{dx}{dt} \cdot \Delta t$ and $\Delta y \approx \frac{dy}{dt} \cdot \Delta t$. Putting this together gives us

$$\Delta f \approx \frac{\partial f}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial f}{\partial y}\frac{dy}{dt}\Delta t$$
$$\frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

and taking the limit as t goes to zero gives us what we want. □

Example 3.50. Suppose $z = y\cos(x)$, where $x = t^2$ and $y = t^3$. Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (-y\sin(x)) \cdot 2t + \cos(y) \cdot 3t^2 \\ &= -t^3\sin(t^2) \cdot 2t + \cos(t^3) \cdot 3t^2.\end{aligned}$$

We can generalize this sort of chain rule behavior to chaining together functions of many variables. In general, we have

$$\frac{\partial z}{\partial t} = \sum_{x_i} \frac{\partial z}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}.$$

That is, for each variable that z depends on, we multiply together the way z depends on the variable and the way the variable depends on t, and then add these all together to get the total change.

Example 3.51. Let $f(x, y) = x^2y$ where $x = 4u + v$ and $y = u^2 - v^2$. Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2xy \cdot 4 + x^2 \cdot 2u \\ &= 2(4u + v)(u^2 - v^2)4 + (4u + v)^2 2u \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2xy \cdot 1 + x^2(-2v) \\ &= 2(4u + v)(u^2 - v^2) + (4u + v)^2(-2v).\end{aligned}$$

Example 3.52. Suppose we have a function f that can be expressed as a function of x and y, or of u and v, where $x = u + v$ and $y = u - v$. (This is called a change of basis). We can express the partial derivatives in terms of each other.

We have

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 1 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot (-1) \\
&= \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.
\end{aligned}$$

If we want to go the opposite way, and express $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, then we have two options. One is to observe that $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, and then use the chain rule again:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial f}{\partial u} + \frac{1}{2}\frac{\partial f}{\partial v} \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial u} - \frac{1}{2}\frac{\partial f}{\partial v}.
\end{aligned}$$

Alternatively, we could have taken the expressions we already had and rearranged them. We knew that

$$\begin{aligned}
\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} &= 2\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} &= 2\frac{\partial f}{\partial y}
\end{aligned}$$

and dividing by 2 gives us the same answer we got before.

3.7 Second Partials

So far we've spoken explicitly only about the first-order derivatives of f. But each derivative gives us a new function, which we can also take the derivatives of. In single variable calculus this gives us "the" second derivative. In multivariable calculus, just as there is more than one first derivative, there is more than one second derivative.

Definition 3.53. We define the *second-order partial derivatives* of $f(x, y)$ to be

$$\begin{aligned}
\frac{\partial^2 z}{\partial^2 x} &= f_{xx} = (f_x)_x & \frac{\partial^2 z}{\partial x \partial y} &= f_{yx} = (f_y)_x \\
\frac{\partial^2 z}{\partial y \partial x} &= f_{xy} = (f_x)_y & \frac{\partial^2 z}{\partial^2 y} &= f_{yy} = (f_y)_y
\end{aligned}$$

Example 3.54. Let $f(x, y) = xy^2 + 3x^2e^y$. Then

$$f_x(x, y) = y^2 + 6xe^y \qquad\qquad f_y(x, y) = 2xy + 3x^2e^y$$

so we compute

$$f_{xx}(x, y) = 6e^y \qquad\qquad f_{yx}(x, y) = 2y + 6xe^y$$
$$f_{xy}(x, y) = 2y + 6xe^y \qquad\qquad f_{yy}(x, y) = 2x + 3x^2e^y.$$

You may have noticed a repetition here. Though there are four distinct mixed partials to compute, we only got three separate answers. This isn't an accident.

Theorem 3.55. *If f_{xy} and f_{yx} are continuous at the point (a, b), and (a, b) is an interior point of their domain, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

These second-order partials measure how quickly the derivatives—the first partials—change when we change our input. This is similar to your homework problem 14.1.24, which asked how the partial derivatives changed as you moved from point A to point B.

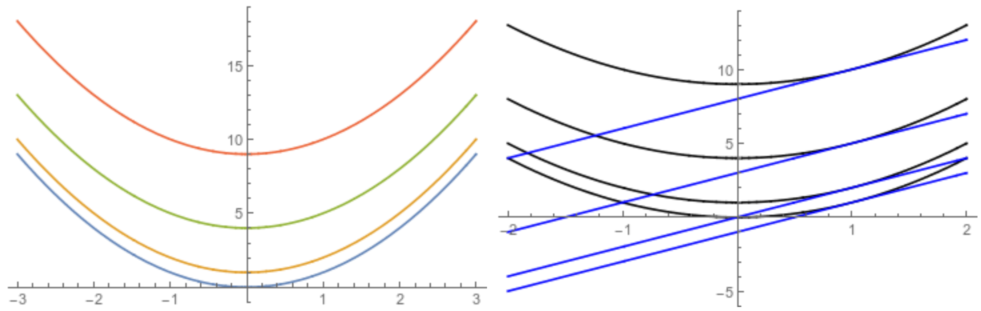
For example, if f_{xx} is positive, that means that the function gets steeper in the x direction as you increase x. If f_{xy} is positive, that means the function gets steeper in the x diretion as you increase y.

Example 3.56. Consider the function $f(x, y) = x^2 + y^2$. We see that

$$f_{xx}(x, y) = 2 \qquad f_{xy}(x, y) = 0 \qquad f_{yy}(x, y) = 2.$$

What does this tell us? Well, at any point, moving one unit in the x direction increases the x slope by about two; and similarly, moving one unit in the y direction increases the y slope by about two.

But moving in the y direction doesn't affect the x slope at all, and vice versa. Geometrically, this is because all the cross sections are identical parabolas at different heights: their levels will be different, but their slopes will all be the same at the same x value. So changing y doesn't change the x slope at all.



We can use these second partial derivatives to improve our approximations. In section 3.4 we talked about linear approximation, which the linear function that best approximates our function near a given point. With second partials, we can construct the *second-degree Taylor polynomial* or *quadratic approximation*.

Definition 3.57. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined near (a, b). The *Taylor polynomial of degree 1* for f near (a, b) is

$$T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The *Taylor polynomial of degree 2* is

$$\begin{aligned} T_2(x, y) = f(a, b) &+ f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2. \end{aligned}$$

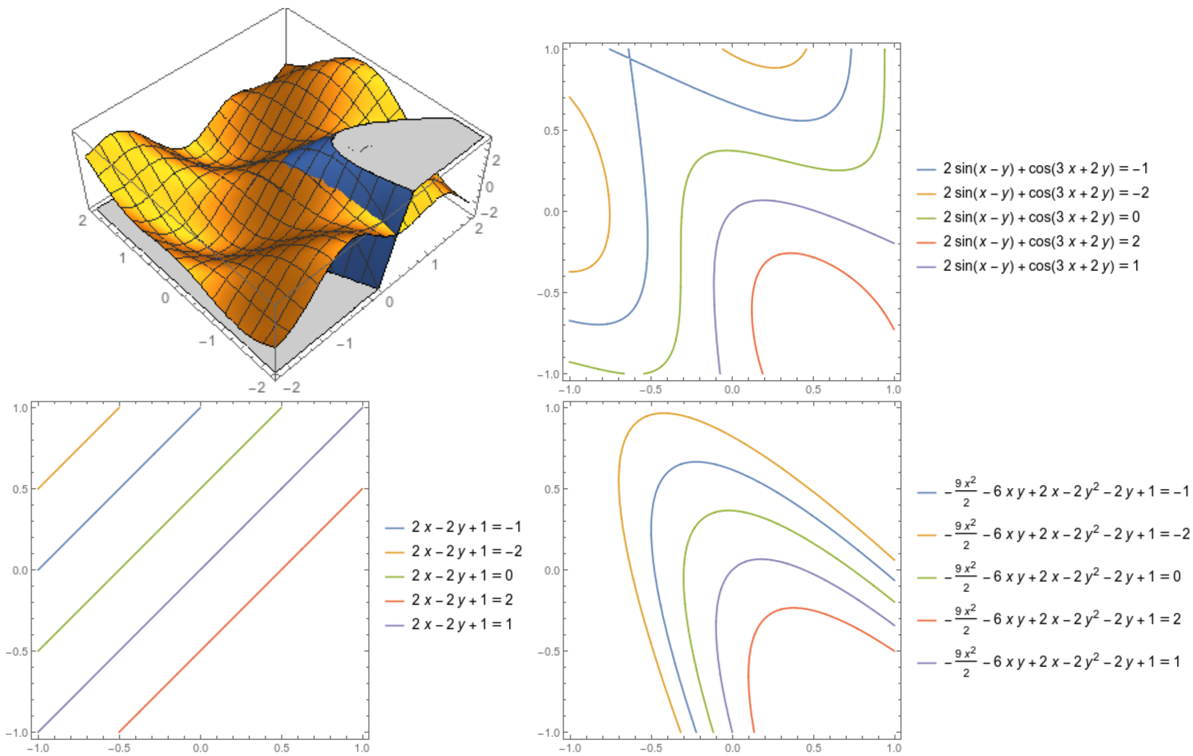
These approximations are used quite often in physics and in any other sort of numeric approximation work. It's possible to go to third-order and higher, and this works exactly like you'd expect; but third-order approximations are rarely actually useful. If the quadratic approximation isn't good enough, you generally want to just use a better tool instead.

Example 3.58. Let's find a quadratic approximation to $\cos(3x + 2y) + 2\sin(x - y)$ near $(0, 0)$.

$$\begin{aligned}
f(x, y) &= \cos(3x + 2y) + 2\sin(x - y) & f(0, 0) &= 1 \\
f_x(x, y) &= -3\sin(3x + 2y) + 2\cos(x - y) & f_x(0, 0) &= 2 \\
f_y(x, y) &= -2\sin(3x + 2y) - 2\cos(x - y) & f_y(0, 0) &= -2 \\
f_{xx}(x, y) &= -9\cos(3x + 2y) + 2\sin(x - y) & f_{xx}(0, 0) &= -9 \\
f_{xy}(x, y) &= -6\cos(3x + 2y) + 2\sin(x - y) & f_{xy}(0, 0) &= -6 \\
f_{yy}(x, y) &= -4\cos(3x + 2y) - 2\sin(x - y) & f_{yy}(0, 0) &= -4
\end{aligned}$$

so the quadratic approximation is

$$T_2(x, y) = 1 + 2x - 2y - 9x^2/2 - 6xy - 2y^2.$$



Suppose we want to find $\cos(.3 - .2) + 2\sin(.1 + .1)$. Then we have

$$f(.1, -.1) \approx T_2(.1, -.1) = 1 + .2 + .2 - .09/2 + .06 - .02 = 1.395.$$

Plugging in, the true answer is ≈ 1.39234, so this is pretty good.