6 Line Integrals

From this point on, our course will essentially have two topics:

1. Different types of integrals you might want to compute for one reason or another; and
2. Ways to avoid actually having to compute those integrals.

In this section we will discuss various types of one-dimensional integrals—integrals over curves. In the next section we'll discuss two-dimensional integrals, and in the final section of the course we'll discuss some important theorems that relate one-, two-, and three-dimensional integrals.

6.1 Integrating over a curve

In single-variable calculus, we studied single-variable integrals. Recall the basic idea here is that we have some function defined on real numbers, and we want to "add up" all the values our function has over some interval in the real numbers.

We want to generalize this concept to multiple dimensions. In section 5 we talked about adding up all the values of a 2-variable function in a 2-dimensional region. But sometimes we'll only take a *path* through that region, and we want to add up all the values on that path.

Example 6.1.

- If the energy it takes to move is a function of location in space, then the total energy used will be an integral of that function over the path traveled.
- If the density of a wire is a function of the point in space, then the total mass will be the integral of density over the length of the wire.
- The total length of a curve will be the integral of the function 1 over the length of the curve.

How do we compute a line integral? Let's start by working out the length of a curve. We can do this by approximating the curve with a large number of short (tangent) lines, and then adding up all their lengths. We see that as the lines get shorter and more numerous, the approximation gets better.

Suppose we have a curve parametrized by $\vec{r}(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$.

$$L \approx \sum_{i=1}^{n} \|\vec{r}(t_{i+1}) - \vec{r}(t_i)\| = \sum_{i=1}^{n} \left\| \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \right\| \Delta t.$$

As the number of line segments tends to infinity, the quotient in the middle tends to $\|\vec{r}\,'(t_n)\|$, and we so see that this sum is

$$L \approx \sum_{i=1}^{n} \|\vec{r}\,'(t_i)\|\Delta t \approx \int_a^b \|\vec{r}\,'(t)\|\, dt.$$

Proposition 6.2. *If C is a curve with parametric equation $\vec{r}(t)$ for $a \leq t \leq b$, then the length of the curve is given by*

$$\int_a^b \|\vec{r}\,'(t)\|\, dt.$$

Example 6.3. Consider the ellipse $(2\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$. What is the circumference?

We compute

$$\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} = \int_0^{2\pi} \sqrt{4\sin^2(t) + \cos^2(t)} \approx 9.69.$$

So what was "really" going on there? If we want to find the size of a shape—the length of an interval, the area of a region, the volume of a region—we can integrate the function 1 over that shape. So we just want to integrate the function 1 over the curve $C = \vec{r}(t)$.

So how do integrals like this work in general? For a single-variable integral, we chopped the interval up into a bunch of subintervals. To integrate over a curve, we want to chop the curve up into a bunch of short lines. Then we evaluate the function at a point on each short subline, multiply by the length of that subline, and add all these things up.

But we just worked out that the length of a subline was

$$\|\vec{r}(t_{n+1}) - \vec{r}(t_n)\| = \left\| \frac{\vec{r}(t_{n+1}) - \vec{r}(t_n)}{\Delta t} \right\| \Delta t \approx \|\vec{r}\,'(t_n)\|\Delta t.$$

Thus our Riemann sum is approximately given by

$$I = \sum_{i=1}^{n} f(\vec{r}(t_n))\|\vec{r}\,'(t_n)\|\Delta t \approx \int_a^b f(\vec{r}(t))\, dt.$$

Thus we define:

Definition 6.4. The *scalar line integral* of a multivariable function $f(\vec{r})$ over a curve $C = \vec{r}(t)$ for $a \leq t \leq b$ is

$$\int_C f(\vec{r})\, ds = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i)\|\Delta r_i\| = \int_a^b f(\vec{r}(t))\|\vec{r}\,'(t)\|\, dt.$$

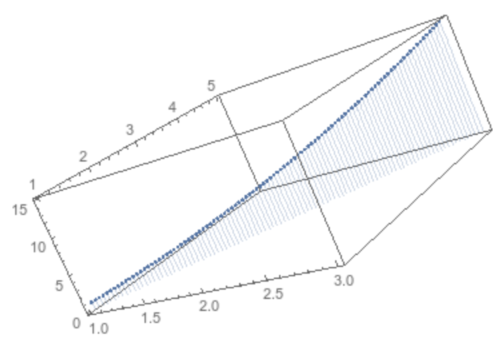
Remark 6.5. You might worry that this isn't well defined. A wire in space could be parametrized by a large number of different functions $\vec{r}(t)$. However, we will get the same answer for any parametrization! Changing the parametrization will change the output of $f(\vec{r}(t))$, but it will also change $\|\vec{r}'(t)\|$, and the two changes will exactly cancel out.

In fact, the $\|\vec{r}'(t)\|$ term is playing the same role here that the Jacobian did in our reparametrized two-variable integrals. The Jacobian measured how much our parametrization stretched the area of a surface parametrization; this term is measuring how much our parametrization stretches the length of a curve, and making sure that every bit of length is counted equally.

Example 6.6. If C is the line from $(1,1)$ to $(3,5)$, find $\int_C xy\,ds$.

We parametrize C by $\vec{r}(t) = (1+2t, 1+4t)$ for $0 \leq t \leq 1$. Then we have

$$\begin{aligned}
f(\vec{r}(t)) &= (1+2t)(1+4t) \\
\|\vec{r}'(t)\| &= \|(2,4)\| = 2\sqrt{5} \int_C xy\,ds \\
&= \int_0^1 (1+2t)(1+4t)2\sqrt{5}\,dt = 2\sqrt{5}\int_0^1 1+6t+8t^2\,dt \\
&= 2\sqrt{5}(t+3t^2+8t^3/3)|_0^1 = 2\sqrt{5}(1+3+8/3) = \frac{40\sqrt{5}}{3}.
\end{aligned}$$



Example 6.7. Suppose we have a helical spring lying along a path parametrized by $\vec{r}(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$, and suppose the density of the wire is given by $1+z$. What is the total mass of the wire?

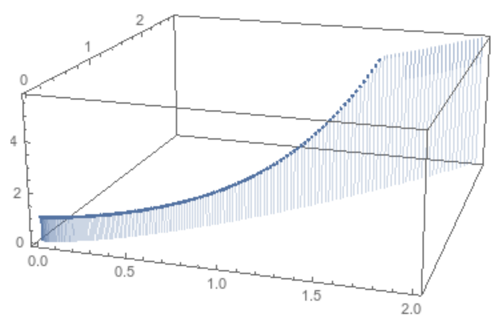
We have

$$\begin{aligned}
f(\vec{r}(t)) &= 1+t \\
\|\vec{r}\,'(t)\| &= \|(-\sin(t), \cos(t), 1)\| = |\sin^2(t) + \cos^2(t) + 1| = \sqrt{2} \\
\int_C f(\vec{r})\,ds &= \int_0^{2\pi} (1+t)\sqrt{2}\,dt \\
&= \sqrt{2}(t + t^2/2)|_0^{2\pi} = \sqrt{2}(2\pi + 2\pi^2) = 2\pi\sqrt{2}(1+\pi).
\end{aligned}$$

Example 6.8. Suppose we have a wire lying along the path given by $\vec{r}(t) = (t^2/2, t^3/3)$ for $0 \leq t \leq 2$, with density given by $1 + xy^2$. What is the total mass of the wire?

We compute

$$\begin{aligned}
f(\vec{r}(t)) &= 1 + t^8/18 \\
\|\vec{r}\,'(t)\| &= \|(t, t^2)\| = t\sqrt{1+t^2} \\
\int_C f(\vec{r})\,ds &= \int_0^2 (1 + t^8/18)t\sqrt{1+t^2}\,dt \approx 15.2.
\end{aligned}$$



6.2 Vector Fields

In section 2 we talked about vector functions, functions $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ that take in a single number and output a vector, or a position in space. We thought of these as representing motion over time. Then in section 3.1 we talked about multivariable functions $f : \mathbb{R}^3 \to \mathbb{R}$ that assign one number to each point in space. We also briefly discussed changes of coordinates in section 5.5, which are encoded by functions $T : \mathbb{R}^2 \to \mathbb{R}^2$. Now we want to combine all of those ideas.

Sometimes we have a flow, or a force field, or a current. What these all have in common is that for every point they have a direction and a magnitude, which represents the flow of the current, or the force of the force field. Thus we want to write a function that takes in a location, which is a point in $\mathbb{R}^3$, and outputs a vector in $\mathbb{R}^3$.

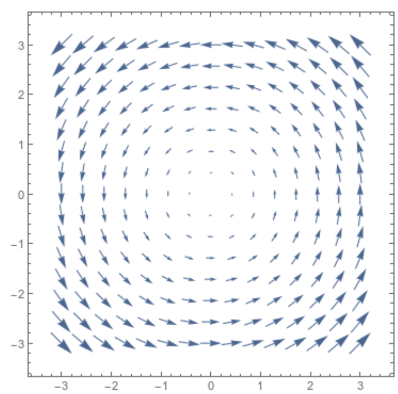
Example 6.9.
- The current direction and distance you have to go to reach your destination.
- The force exerted by the Earth's gravitic force.
- The direction and speed of the currents in a river.

Definition 6.10. A *vector field* in $\mathbb{R}^n$ is a function $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ that takes in a point in $\mathbb{R}^n$ and outputs a vector in $\mathbb{R}^n$.

Example 6.11. Consider the 2-dimensional vector field given by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$. We can get a sense for what this looks like by plotting a few points.

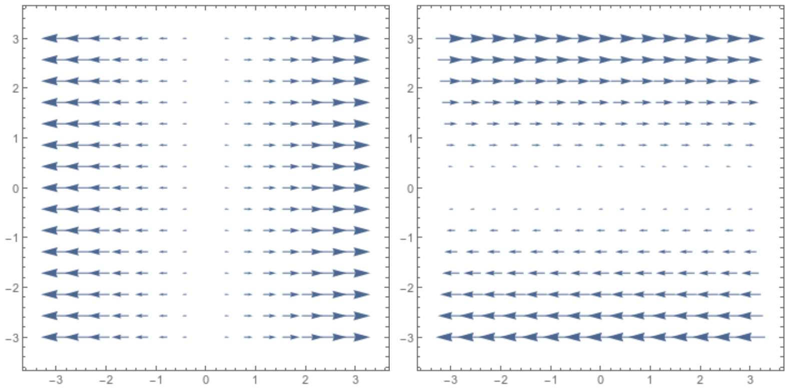
We see that $\vec{F}(x,0) = x\vec{j}$, so along the x-axis the arrows point straight up, and get longer the further away from the origin we are. Similarly, we have $\vec{F}(0,y) = -y\vec{i}$, so the arrows point to the left and get longer the further we are from the origin. And we see that, e.g. $\vec{F}(1,1) = -\vec{i} + \vec{j}$ points up and left.

This seems to be a roughly counterclockwise circular motion. And indeed, we get the plot:

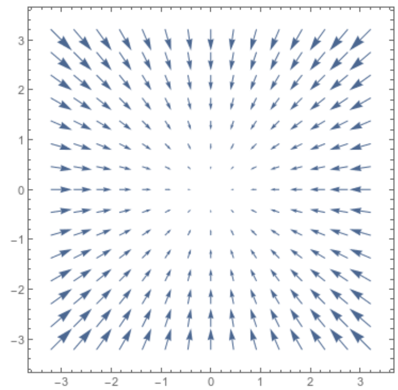


Example 6.12. Consider $\vec{F}(x,y) = x\vec{i}$. This vector field is just arrows that always point horizontally, with their size determined by their x-coodrinate.

Now consider $\vec{G}(x,y) = y\vec{i}$. This is still horizontal arrows, but now their length is determined by their y-coordinate.

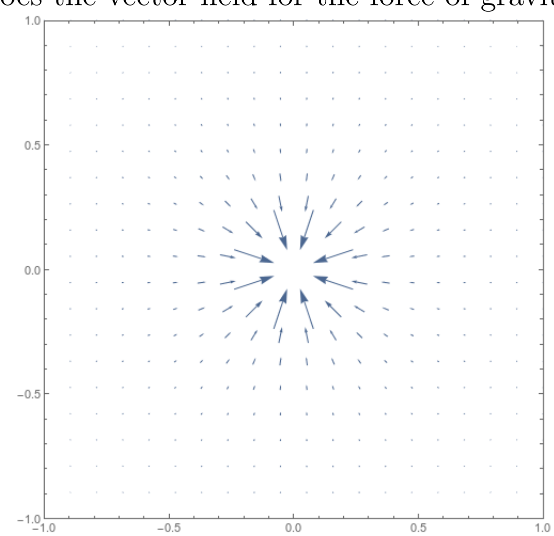


Example 6.13. What's an equation for this vector field?



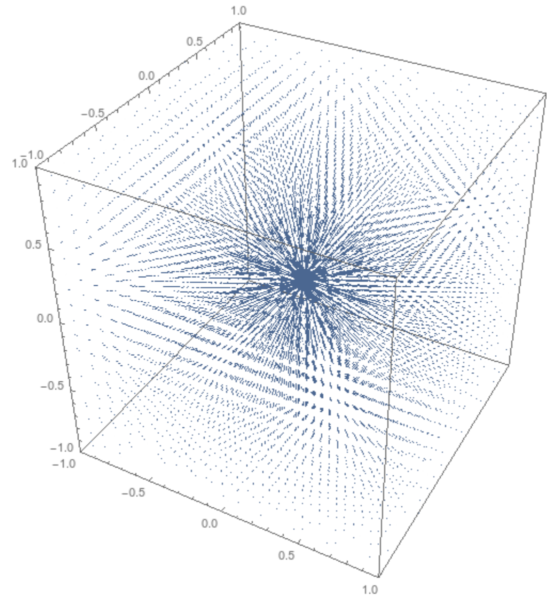
We see that the arrows are always pointing back towards the origin. This vector field has the function $\vec{F}(x, y) = -x\vec{i} - y\vec{j}$.

Example 6.14. What does the vector field for the force of gravity look like?



We can also think about vector fields in three dimensions, although the pictures are much harder to draw.

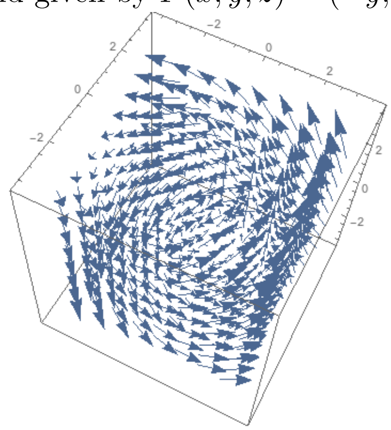
Example 6.15. What does this gravity field look like in three dimensions? We get



The formula here is

$$\vec{F}(\vec{r}) = \frac{GMm}{\|\vec{r}\|^2}\left(\frac{-\vec{r}}{\|\vec{r}\|}\right) = \frac{-GMm\vec{r}}{\|\vec{r}\|^3}.$$

Example 6.16. The vector field given by $\vec{F}(x, y, z) = (-y, x, z)$ is

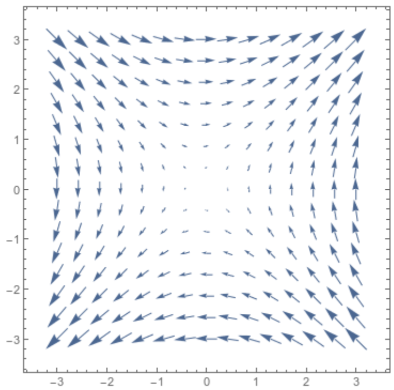


It's hard to make out from the picture, but this an upwards-twisting cylinder.

A common and important source of vector fields is the gradient function. Notice that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a multivariable function, then $\nabla f = (f_x, f_y, f_z)$ is a function from $\mathbb{R}^3 \to \mathbb{R}^3$, and thus is a vector field.

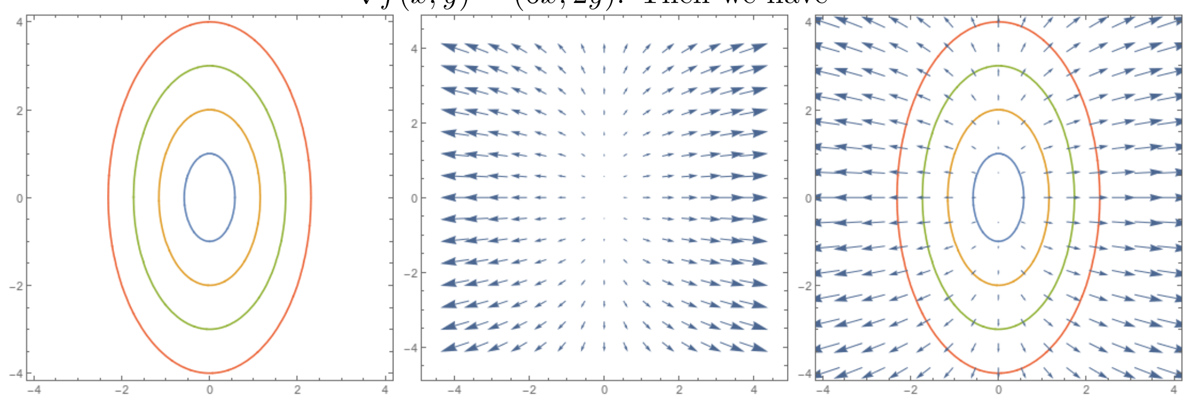
This in fact makes sense, since the gradient tells you, for every point, what the direction and magnitude of greatest increase is. Thus gradients often give us vector fields.

Example 6.17. Let $f(x, y) = xy$. Then $\nabla f(x, y) = (y, x)$ has the plot



We can infer the vector plot of a function from its contour diagram, since we know the gradient vectors are always perpendicular to the contours.

Example 6.18. Consider the function $f(x, y) = 3x^2 + y^2$. We calculate that $\nabla f(x, y) = (6x, 2y)$. Then we have



Another way to visualize vector fields is through *flow lines*. If we think of a vector field as describing the direction you will move from a given point, then flow lines tell us the path we will follow if we start at a given point.

Definition 6.19. Let $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ be a curve. We say that $\vec{r}$ is a *flow line* of $\vec{F}$ if $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$. that is, the velocity of the path is equal to the vector field.

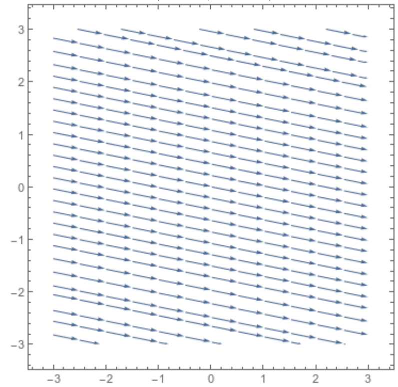
In a flow line, the vector field describes the motion of a particle along the curve.

The *flow* of a vector field is the collection of all the flow lines.

Remark 6.20. We can view this as a solution to the differential equation $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$. The choice of flow line is determined by your initial conditions.

Example 6.21. Let $\vec{v}(t) = 5\vec{i} - \vec{j}$. What does a flow line look at? What is an equation for the flow line that goes through $(3, 3)$?

A flow line would just be a line in the direction $5\vec{i} - \vec{j}$, and thus a line that looks like $\vec{r}_0 + t(5, -1)$. Then the flow line through $(3, 3)$ is $(3 + 5t, 3 - t)$.



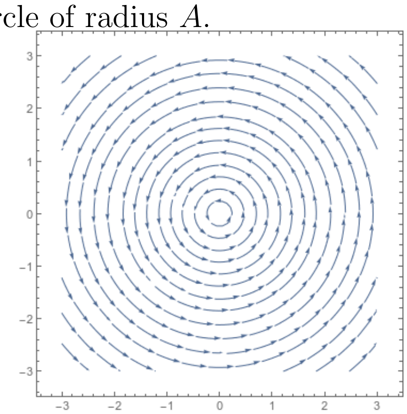
Example 6.22. Let $\vec{F}(x, y) = y\vec{i} + \vec{j}$. Find the path of an object in the flow that is at the point $(2, 2)$ at time $t = 0$.

We know that $\vec{r}'(t) = (y, 1)$, so we have $x'(t) = y(t)$ and $y'(t) = 1$. The second equation tells us that $y(t) = t + y_0$. Then the first equation tells us that since $x'(t) = t + y_0$ we have $x(t) = t^2/2 + y_0 t + x_0$.

Plugging in our initial conditions tells us that $x_0 = 2$ and $y_0 = 2$. Thus we have $\vec{r}(t) = (t^2/2 + 2t + 2, t + 2)$.

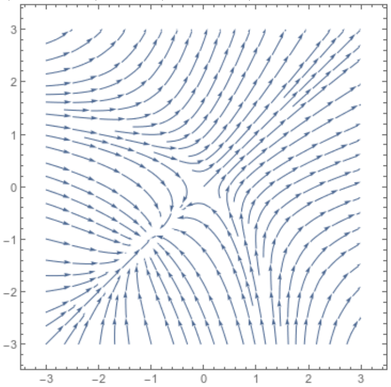
Example 6.23. Let $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$. What does the flow of this vector field look like?

We have $x'(t) = -y(t)$ and $y'(t) = x(t)$. Thus in particular we have $x''(t) = -y(t)$, and this tells us that $x(t) = A\cos(t) + B\sin(t)$. Taking the case $B = 0$, we then have $x(t) = A\cos(t)$ and then we can see that $y(t) = A\sin(t)$. Then we have $\vec{r}(t) = (A\cos(t), A\sin(t))$ and thus each flow line is a circle of radius A.



We can (often) find flow lines exactly by solving a system of differential equations. (If you want to learn more about this you should take Math 3342: Ordinary Differential Equations). But often solving them exactly is annoying or impractical, and we want to approximate the flow lines instead. We can find these by Euler's Method.

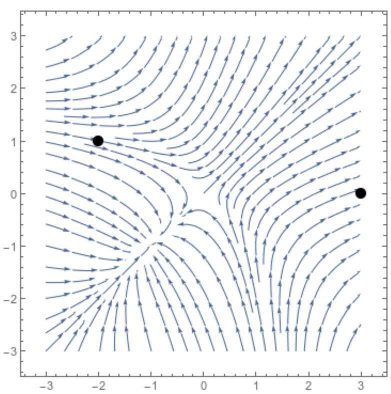
Example 6.24. Let $\vec{F}(x,y) = (x^2+y)\vec{i} + (x+y^2)\vec{j}$. Suppose $\vec{r}(0) = (-2,1)$. What is $\vec{r}(1)$?



We can estimate this with a linear approximation. If we want a very coarse estimate, we can simply calculate

$$\vec{r}\,'(0) = \vec{F}(-2,1) = 5\vec{i} - \vec{j}$$
$$\vec{r}(1) \approx \vec{r}(0) + \vec{r}\,'(0)(1-0) = (-2,1) + (5,-1) = (3,0).$$

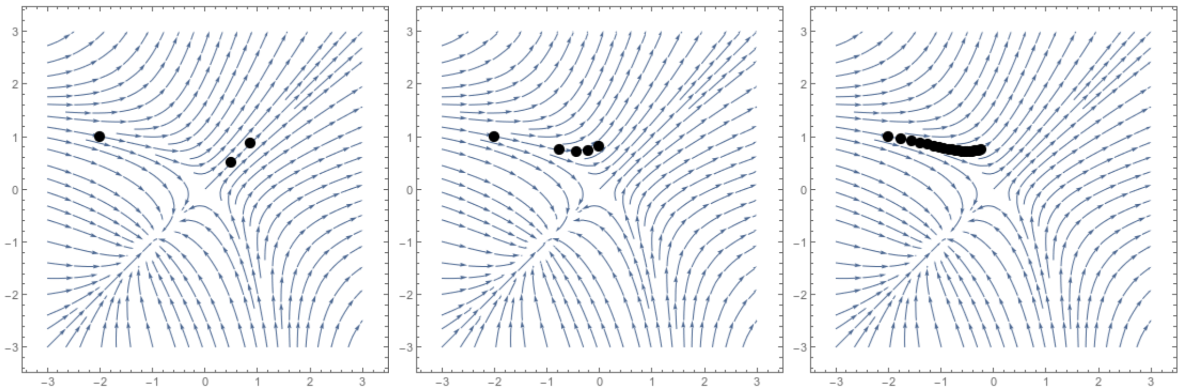
But from the picture below we see that this isn't really a very good estimate. Basically, our Δt is much too big, so our linear approximation isn't very good. (We can see this in the graph because the flow lines curve).



If we knew the second derivatives, we could fix this. But we can also fix this just by making smaller step sizes. We can compute

$$\vec{r}\,'(0) = \vec{F}(-2,1) = 5\vec{i} - \vec{j}$$
$$\vec{r}(1/2) \approx \vec{r}(0) + \vec{r}\,'(0)(1/2-0) = (-2,1) + \frac{1}{2}(5,-1) = (1/2,1/2)$$
$$\vec{r}(1) \approx \vec{r}(1/2) + \vec{r}\,'(1/2)(1-1/2) = (1/2,-1/2) + \frac{1}{2}(3/4,3/4) = (7/8,7/8)$$

and we see this is much close to the "true" answer. We can get the answer as close as we want to the true answer by taking more and more smaller and smaller steps. Below we have the pictures for doing this calculation with two, four, and 20 steps:



6.3 Integrating vector fields over a curve

In the past section, we saw how to take some scalar quantity that varies with position, and add up its value over some extended curve through space. However, it's probably more common to worry about a *field* that extends through space.

Example 6.25.
- The work done by a particle moving through a force field along some path
- The extent to which a fluid current or circulation points in the same direction as a path

What these phenomena has in common is that we want to measure the extent to which a vector field points in the same direction as our curve.

If the vector field is constant and the path is a straight line, both of these computations are easy. Work is equal to the distance traveled times the force in the direction of the distance: thus work is equal to the *dot product* of force and displacement. We might write that $W = \vec{F} \cdot d\vec{r}$.

Similarly, if we want to know the extent to which a current is flowing in the direction of some line $\vec{r}$, we compute $\vec{F} \cdot \vec{r}$. Both of these notions involve the dot product. But this computation only works when the vector field is constant, and the path is a straight line.

So let's do our usual integral approximation trick. We parametrize our curve $C = \vec{r}(t)$ for $a \leq t \leq b$, and break it up into a bunch of line segments. Then for each line segment, the direction is constant and the vector field is approximately constant, so we have work equal to

$$W_i = \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)).$$

Thus the work done over the entire path is approximately

$$\begin{aligned} W &\approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)) \\ &= \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \Delta t \\ &\approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i) \Delta t \\ &\approx \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt. \end{aligned}$$

Definition 6.26. We define the *line integral of a vector field* $\vec{F}$ over a curve $C = \vec{r}(t)$ to be

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

Remark 6.27. As before, the choice of parametrization doesn't affect the result of this integral. Proving this rigorously is incredibly annoying so we won't bother.

Example 6.28 (Recitation)**.** Let $\vec{F}(x, y) = \vec{i} + \vec{j}$ and let C be a curve going in a straight line from $(1, 1)$ to $(4, 1)$. What is $\int_C \vec{F} \cdot d\vec{r}$?

We don't actually need to set up an integral here, since the field is constant and the curve is a straight line. We have

$$\int_C \vec{F} \cdot d\vec{r} = (\vec{i} + \vec{j}) \cdot (3, 0) = 3.$$

If we do set up an integral, we have $\vec{r}(t) = (1 + t, 1)$ for $0 \leq t \leq 3$. Then our integral is

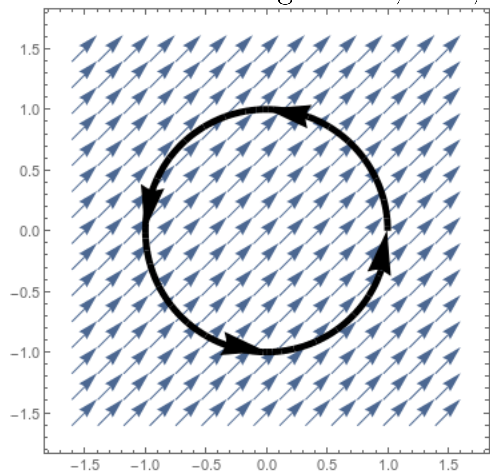
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 (\vec{i} + \vec{j}) \cdot (\vec{i})\, dt = \int_0^3 1\, dt = 3.$$

Example 6.29. If C is the circular path centered at the origin that begins and ends at $(1, 0)$, oriented counterclockwise, and $\vec{F}(x, y) = \vec{i} + \vec{j}$, what is $\int_C \vec{F} \cdot d\vec{r}$?

We can parametrize the unit circle by $\vec{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (\vec{i} + \vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\, dt \\ &= \int_0^{2\pi} \cos t - \sin t\, dt \\ &= \sin(t) + \cos(t)|_0^{2\pi} = 0 + 1 - (0 + 1) = 0. \end{aligned}$$

We could have predicted this result from the picture. We see that the current is flowing *against* the circle exactly as often as it is flowing *with* it; thus, the net is zero.



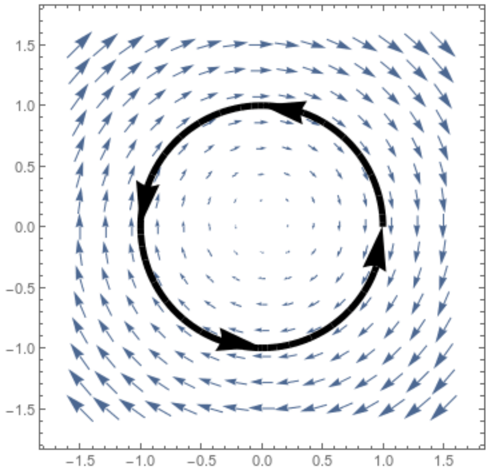
If we integrate over only the first half of the circle, we'd expect to get something negative. And in fact we do:

$$\begin{aligned}\int_C \vec{F} \cdot\, d\vec{r} &= \int_0^{\pi} (\vec{i} + \vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\, dt \\ &= \int_0^{\pi} \cos t - \sin t\, dt \\ &= \sin(t) + \cos(t)|_0^{\pi} = 0 - 1 - (0 + 1) = -2.\end{aligned}$$

Example 6.30. Let $\vec{F}(x, y) = (y, -x)$ and let C be the unit circle parametrized to go counterclockwise, as in example 6.29. Then

$$\begin{aligned}\int_C \vec{F} \cdot\, d\vec{R} &= \int_0^{2\pi} (\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} -\sin^2 t - \cos^2 t\, dt = \int_0^{2\pi} -1\, dt = -2\pi.\end{aligned}$$

Again from the picture we can tell that this integral should be negative, since the circle is running against the direction of the flow.

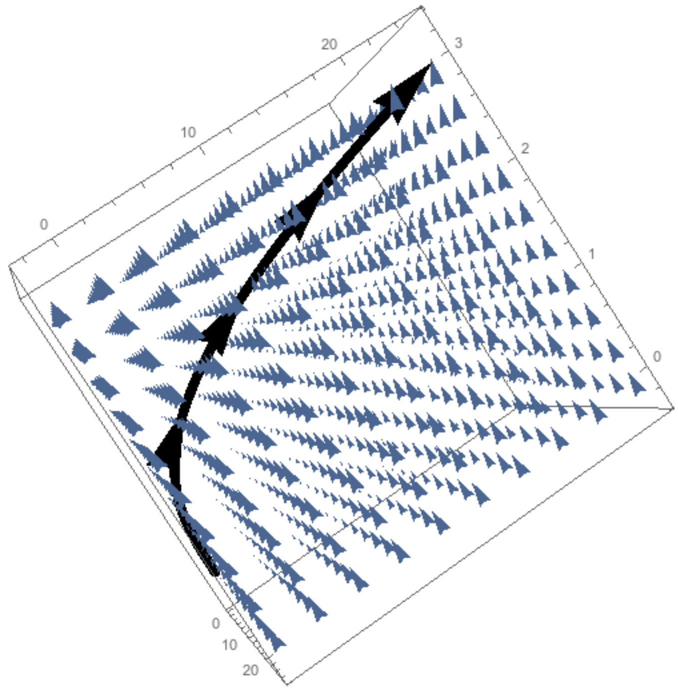


What if we parametrize the circle going the other way? Geometrically, we see that our dot product will reverse, so we'd expect to get the opposite of what we got originally. In fact that's exactly what happens: we parametrize our curve as $(\cos(t), -\sin(t))$ for $0 \leq t \leq 2\pi$ and we get

$$\int_C \vec{F} \cdot d\vec{R} = \int_0^{2\pi} (-\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t, -\cos t)\, dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

Example 6.31. Suppose a particle moves along the path $\vec{r}(t) = (t, 3t^2, t^3 - 2)$ through a force field $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + xz\vec{k}$. What is the total work done over the interval $0 \leq t \leq 3$?

$$\int_C \vec{F}\, d\vec{r} = \int_0^3 (3t^2, t^2, t^4 - 2t) \cdot (1, 6t, 3t^2)\, dt$$
$$= \int_0^3 3t^2 + 6t^3 + 3t^6 - 6t^3\, dt$$
$$= t^3 + 3t^7/7|_0^3 = 27 + 3^8/7 = \frac{6750}{7}.$$



We sometimes write our line integrals in what's known as "differential" notation. If we have $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$, and our curve is parametrized by $\vec{r}(t) = (x(t), y(t), z(t))$, then we will sometimes write

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

In this context, dx is just $\frac{dx}{dt}dt = x'(t)\,dt$.

Example 6.32. If C is a line segment from $(0,0)$ to $(3,4)$, evaluate

$$\int_C y\,dx + xy\,dy.$$

We can parametrize C by $\vec{r}(t) = (3t, 4t)$ for $0 \leq t \leq 1$. Then we have

$$\int_c y\,dx + xy\,dy = \int_0^1 4t \cdot 3\,dt + 12t^2 \cdot 4\,dt = \int_0^1 12t + 48t^2\,dt = 6t^2 + 16t^3|_0^1 = 22.$$

Notice we haven't changed anything geometrically here, and we haven't changed any of the computations you're doing; we're just writing things out differently.

Finally, here's a bit of a trolly example.

Example 6.33. Let $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ be a field of force exerted by some force field, and suppose we have a wire lying along the path $\vec{r}(t) = (2t, 3t)$ for $0 \leq t \leq 2$. What is the total force on the wire?

Since we're totaling the forces along the length of the wire, we expect to compute an integral. However, we're not asking for work but for total force, so we don't need to take a dot product. Instead we would have to compute a "scalar" integral of the vector field, and get

$$\begin{aligned}\vec{F} &= \int_0^2 (2t\vec{i} + 3t\vec{j})(t\sqrt{13})\,dt = \sqrt{13}\int_0^2 2t^2\vec{i} + 3t^2\vec{j}\,dt \\ &= \sqrt{13}\left(2t^3/3\vec{i} + t^3\vec{j}\right)\Big|_0^2 \\ &= \sqrt{13}\left(16/3\vec{i} + 8\vec{j}\right) = \frac{16\sqrt{13}}{3}\vec{i} + 8\sqrt{13}\vec{j}.\end{aligned}$$

Notice that our output here is a vector, as it should be since we're asking for the force exerted on an object.

6.4 Conservative Vector Fields

Example 6.34. Let's compute the integral of $\vec{F}(x,y) = y\vec{i} + x\vec{j}$ along the straight line from $(0,0)$ to $(1,1)$. We can parametrize this by $\vec{r}(t) = (t,t)$ for $0 \leq t \leq 1$, and we have

$$\int_0^1 (t,t)\cdot(1,1)\,dt = \int_0^1 t + t\,dt = \int_0^1 2t\,dt = t^2|_0^1 = 1.$$

But what if we went between the same two points in a different way? We could follow a path going to the right, and then up. Then we would have $\vec{r}_1(t) = (t,0)$ and $\vec{r}_2(t) = (1,t)$, each from 0 to 1. The total line integral would be

$$\int_0^1 (0,t)\cdot(1,0)\,dt + \int_0^1 (t,1)\cdot(0,1)\,dt = \int_0^1 0\,dt + \int_0^1 1\,dt = 1.$$

Or we could follow a circular path, going clockwise around the circle centered at $(1,0)$. We'd parametrize this with $\vec{r}(t) = (\cos(t)+1, -\sin(t))$ for $\pi \leq t \leq 3\pi/2$, and then we'd have

$$\begin{aligned}\int_\pi^{3\pi/2} (-\sin(t), \cos(t)+1)\cdot(-\sin(t), -\cos(t))\,dt &= \int_\pi^{3\pi/2} \sin^2(t) - \cos^2(t) - \cos(t)\,dt \\ &= -\sin(t) - \frac{1}{2}\sin(2t)|_\pi^{3\pi/2} \\ &= -\sin(3\pi/2) - \frac{1}{2}\sin(3\pi) + \sin(\pi) + \frac{1}{2}\sin(2\pi) \\ &= 1 - 0 + 0 + 0 = 1.\end{aligned}$$

This is the same thing we got before! This isn't an accident.

Definition 6.35. We say that a vector field $\vec{F}$ is *conservative* or *path-independent* if whenever C_1 and C_2 are two curves with the same starting point and the same end point, then $\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C2} \vec{F} \cdot d\vec{r}$.

We say a curve is *closed* if it has the same starting point and ending point. A vector field is conservative if whenever C is a closed curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Remark 6.36. We call these fields conservative because in physics, they represent a field that follows some sort of conservation of energy law.

Visually, we can tell whether a vector field is conservative by seeing whether we can draw a closed curve with positive line integral. For instance, we see that the field from example 6.29 is conservative, since in any closed curve the work will exactly balance out and we'll wind up with a zero integral. In contrast, the vector field from example 6.30 is clearly not conservative, since we integrated a closed circle over it and got an answer of 2π.

If we know we have a conservative vector field, we can dramatically simplify some integrals.

Example 6.37. Compute the integral done by the force field $\vec{F}(x, y) = y\vec{i} + x\vec{j}$, which we know to be conservative, on a particle following the path given by $\vec{r}(t) = (t+\sin^8(\pi t)\cos^5(t), \cos^4(2\pi t)+2t)$ for $0 \leq t \leq 1$.

Obviously we don't want to actually do anything with that parametric path. However, we see that $\vec{r}(0) = (0, 1)$ and $\vec{r}(1) = (1, 3)$. Since $\vec{F}$ is path independent, we can just compute the integral over the path $\vec{r}_1(t) = t, 1 + 2t$. And we have

$$\int_0^1 (1 + 2t, t) \cdot (1, 2)\, dt = \int_0^1 1 + 4t\, dt = t + 2t^2|_0^1 = 3.$$

But we can actually make our job even easier after a little bit of work and a clever observation.

We first observe that if $\vec{F} = \nabla f$ for some multivariable function f, then $\vec{F}$ is conservative. In particular, we prove the following:

Proposition 6.38 (Fundamental Theorem of Calculus for Line Integrals)**.** *If C is a piecewise smooth oriented path from the point P to the point Q, and f is some function that is continuously differentiable on the path C, then*

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P).$$

In particular, this implies that any gradient field is conservative, since the line integral depends only on the starting and ending points of the curve.

Remark 6.39. 1. This theorem often allows us to avoid having to compute a line integral at all.

2. This theorem is the analogue to the single-variable Fundamental Theorem of Calculus: in some sense, f is the antiderivative of ∇f.

3. If $\nabla f = \vec{F}$, we sometimes say that f is a *potential function* for $\vec{F}$. For instance, if $\vec{F} = \nabla f$ is the gravitic force field, then f measures your gravitational potential energy.

Proof. We can view $f(Q) - f(P)$ as the change in f as the input moves from Q to P. We know from linear approximation that $f(Q) \approx f(P) + \nabla f(P) \cdot (Q - P)$. But this approximation won't be very good if P and Q are far apart.

We can improve the approximation by dividing the path along the curve up into a bunch of short line segments. Then the total change in the value of f along the whole path is exactly $f(Q) - f(P)$, but it is approximately the sum of the approximate change along each of these line segments. Thus we have

$$\begin{aligned} f(Q) - f(P) &\approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t)) \\ &= \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \Delta t \\ &\approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i) \Delta t \approx \int_C \nabla f \cdot d\vec{r}. \end{aligned}$$

□

Example 6.40. If we look back at example 6.34 that began this subsection, we may observe that $\vec{F}(x, y) = y\vec{i} + x\vec{j} = \nabla(xy)$. Thus for each problem, we were computing $f(1,1) - f(0,0) = 1 \cdot 1 - 0 \cdot 0 = 1$.

In example 6.37 we computed $f(1,3) - f(0,1) = 1 \cdot 3 - 0 \cdot 1 = 3 - 0 = 3$.

Example 6.41. If $f(x, y) = x^2y - y^2$, and C is a spiral that begins at $(1, 2)$ and ends at $(3, 1)$, we compute

$$\int_C \nabla f \cdot d\vec{r} = f(3,1) - f(1,2) = 8 - (-2) = 10.$$

We just showed that every gradient field is conservative. It turns out that the converse is also true: every conservative vector field is the gradient of some function. In fact, we can be very (uselessly) specific about this:

Proposition 6.42. *Let $\vec{F}$ be a conservative vector field. Then we can define a function f by picking any point P, and define*

$$f(Q) = \int_C \vec{F} \cdot d\vec{r}$$

where C is any curve that begins at P and ends at Q. Then $\vec{F} = \nabla f$.

Remark 6.43. Recall that this is related to the fundamental theorem of calculus for single variable integrals. There, we said that if f is a continuous function, it has an antiderivative $F(x) = \int_a^x f(t)\,dt$.

We can get a number of different potential functions here, depending on our choice of starting point P. This choice corresponds to the choice of a in the single-variable case, which corresponds to the choice of constant in the $+C$ of the antiderivative.

Thus a vector field $\vec{F}$ is conservative if and only if it is the gradient of some potential function f.

Example 6.44. Find a potential function for $\vec{F}(x, y) = y\cos x\vec{i} + (\sin x + y)\vec{j}$.

Suppose $\vec{F}$ has a potential function. Then we have some function $f(x, y)$ such that $f_x(x, y) = y\cos x$ and $f_y(x, y) = \sin x + y$.

The first equation tells us that $f(x) = y\sin x + g(y)$ for some g. The second equation tells us that $f(x, y) = y\sin(x) + y^2/2 + h(x)$ for some h. Combining these two facts gives us $f(x) = y\sin(x) + y^2/2 + C$ for some constant C.

Example 6.45. Is $\vec{F}(x, y) = 2y\vec{i} + x\vec{j}$ conservative?

Suppose $\nabla f(x, y) = \vec{F}(x, y)$. Then we have $f_x(x, y) = 2y$ and $f_y(x, y) = x$. The first equation tells us that $f(x, y) = 2xy + g(y)$, and the second equation tells us that $f(x, y) = xy + h(x)$.

These two equations are incompatible, and thus $\vec{F}$ isn't the gradient of any potential function, and thus is not conservative.

This gives us a hint for how to figure out if a vector field is conservative. If $\vec{F}$ is conservative, then we have $\vec{F}(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$. In particular, if we take the y derivative of the first term and the x derivative of the second term, we will get the mixed partial in either case.

That is, if we have some f such that $f_x = 2y$, then $f_{xy} = 2$. But if $f_y = x$, then $f_{xy} = 1$, which is a contradiction.

Thus we see that if $\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j}$, then $\vec{F}$ is conservative if and only if $\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}$. Generalizing this statement leads into our next topic of discussion.

6.5 The Curl of a Vector Field

Let's step back to consider the geometry of a vector field for a bit. We've been considering vector fields that are conservative, which means the integral over any closed loop is zero. What does it look like for a vector field to be non-conservative?

A non-conservative vector field will have some closed loops where the line integral is non-zero. Thus, the vector field will have some component that looks like it is rotating. We can compute "how much" the vector field is rotating around some point by taking the line integral of a circle centered at that point; as the circle gets smaller, we ignore everything except that point and get a number that represents the rotation there.

Definition 6.46. The *circulation density* of a vector field at a point (x, y, z) around the unit vector $\vec{n}$ is

$$\operatorname{circ}_{\vec{n}} \vec{F}(x, y, z) = \lim_{\text{area}\to 0} \frac{\int_C \vec{F} \cdot d\vec{r}}{\text{area of } C}$$

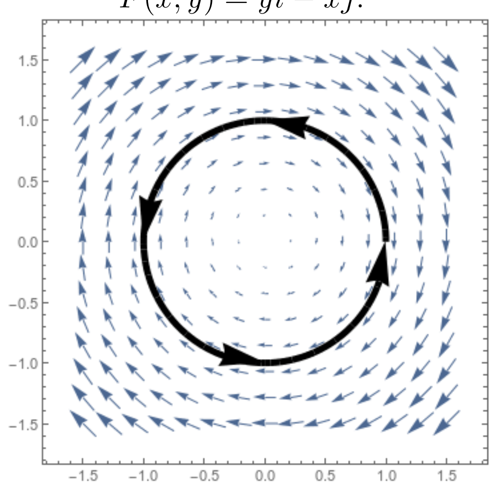
Where C is a circle perpendicular to $\vec{n}$ oriented by the right-hand rule.

Remark 6.47. Recall that we describe rotation with a vector perpendicular to the plane of rotation, according to the right-hand rule. So rotation that appears clockwise to us is represented by a vector pointing away from us; rotation that appears counterclocwise is represented by a vector pointing towards us.

Thus if $\vec{n}$ is pointing towards us, $\operatorname{circ}_{\vec{n}}$ measures the circulation in the direction we would identify as counterclockwise.

Example 6.48. Let's look at the circulation density at the origin of the vector field

$$\vec{F}(x, y) = y\vec{i} - x\vec{j}.$$



If a circle centered at the origin has radius a then we can parametrize it by $\vec{r}(t) =$

$(a\cos t, a\sin t)$. Then our line integral is

$$\int_0^{2\pi} (a\sin(t), -a\cos(t)) \cdot (-a\sin(t), a\cos(t))\, dt = \int_0^{2\pi} -a^2\, dt = -2\pi a^2.$$

The area of the circle is of course πa^2, so the circulation density is

$$\lim_{a\to 0} \frac{-2\pi a^2}{\pi a^2} = -2.$$

What does this tell us? It tells us that at the origin, the vector field is rotating counterclockwise with magnitude -2—or, in other words, it's rotating clockwise with magnitude 2. Which is exactly what we see from the picture.

Now, this process makes perfect sense geometrically, but is not fun to compute with. Fortunately, there is a much easier way to deal with this.

Definition 6.49. Let $\vec{F}(x,y,z) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$ be a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$. We define the *curl* of $\vec{F}$ to be

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

If $\vec{F}$ is a vector field from $\mathbb{R}^2 \to \mathbb{R}^2$, this reduces to

$$\nabla \times F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

and we often treat it as a scalar quantity.

How is the curl related to the circulation density?

- The direction of $\nabla \times \vec{F}$ is the direction $\vec{n}$ which maximizes $\operatorname{circ}_{\vec{n}} \vec{F}$.
- The magnitude of $\nabla \times \vec{F}$ is the circulation density in that direction.

Thus the curl tells you in which direction, if any, a vector field is rotating at any given point.

Example 6.50. Looking back at the field $\vec{F}(x,y) = y\vec{i} - x\vec{j}$ of example 6.48, we can compute the curl:

$$\nabla \times \vec{F} = \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} = -1 - 1 = -2.$$

Thus not only is the circulation -2 at the origin; it is in fact -2 everywhere.

We saw in section 6.4 that a conservative 2-dimensional vector field will have zero curl. We can actually make this statement much stronger:

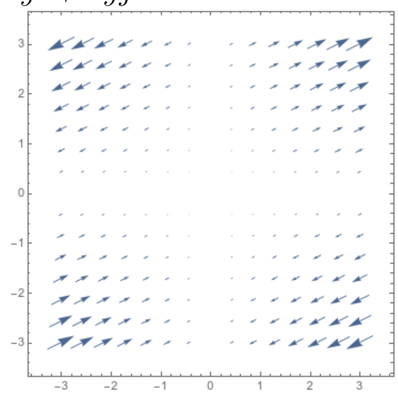
Proposition 6.51. *Let $\vec{F}$ be a vector field. If $\vec{F}$ is conservative, then $\nabla \times F = \vec{0}$.*

If $\vec{F}$ is also defined everywhere, and $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

That is, we can almost say that a vector field is conservative if and only if its curl is zero. However, if a vector field has a singularity, it may have zero curl and still not be conservative.

We will effectively prove this proposition in section 6.6.

Example 6.52. Is $\vec{F}(x, y) = 2xy\vec{i} + xy\vec{j}$ conservative?

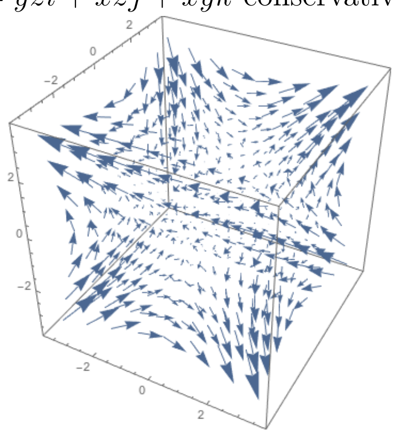


We compute

$$\nabla \times \vec{F} = \frac{\partial xy}{\partial x} - \frac{\partial 2xy}{\partial y} = y - 2x \neq 0.$$

Thus $\vec{F}$ is not conservative.

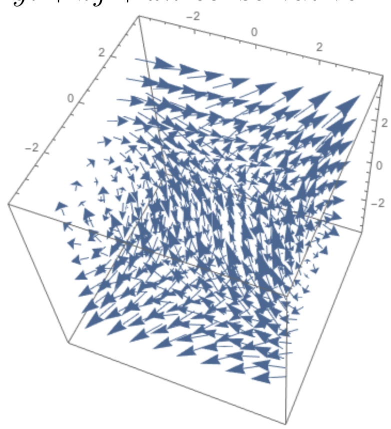
Example 6.53. Is $\vec{F}(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ conservative?



$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\vec{i} + (y - y)\vec{j} + (z - z)\vec{k} = \vec{0}.$$

Thus $\vec{F}$ is conservative. In fact, we can check that $\vec{F}(x, y, z) = \nabla xyz$.

Example 6.54. Is $\vec{F}(x, y, z) = y\vec{i} + z\vec{j} + x\vec{k}$ conservative?



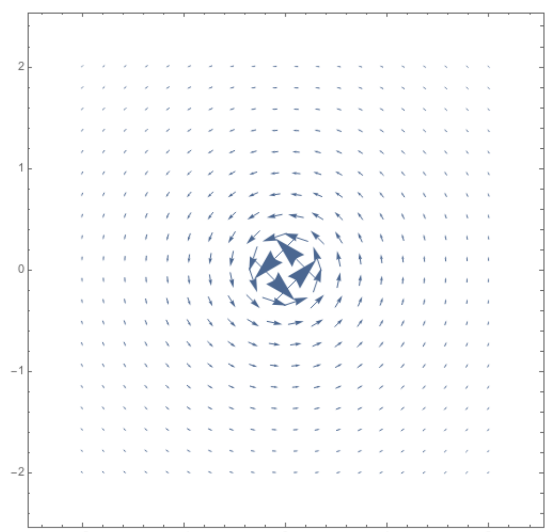
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (0-1)\vec{i} + (0-1)\vec{j} + (0-1)\vec{k} = -\vec{i} - \vec{j} - \vec{k} \neq \vec{0}.$$

Thus this vector field is not conservative.

In contrast, we see that $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ is conservative. It is the gradient of $x^2/2 + y^2/2 + z^2/2$.

However, remember that if $\vec{F}$ isn't defined everywhere, the curl test does not work.

Example 6.55. Let $\vec{F}(x, y) = \frac{-y\vec{i}+x\vec{j}}{x^2+y^2}$. This function is undefined, and in fact has an infinite singularity, at the origin.



We calculate that

$$\begin{aligned}
\nabla \times \vec{F}(x,y) &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\
&= \frac{1(x^2+y^2) - (2x)x}{(x^2+y^2)^2} - \frac{(-1)(x^2+y^2) + y(2y)}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.
\end{aligned}$$

But we can *see* from this picture that the vector field isn't conservative: a circle counterclockwise around the origin will have positive line integral.

In particular, we can also take the line integral around the unit circle. We parametrize $\vec{r}(t) = (\cos(t), \sin(t))$ and calculate

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \frac{1}{\cos^2(t) + \sin^2(t)} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt \\
&= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} dt = 2\pi.
\end{aligned}$$

(We also could have noticed, geometrically, that this vector field is always tangent to the unit circle, and thus the integral is equal to the magnitude of the vector field times the length of the curve, which is $1 \cdot 2\pi$).

Thus the curl of $\vec{F}$ is zero, but $\vec{F}$ is not conservative, since it is undefined at zero. This idea is actually very important. In complex analysis it gives us the idea of *residues*, which are used in a large number of computations; my Ph.D. research used the residue of a particular type of function around an infinite singularity at zero to give information about number theory.

In geometry and topology it gives us the idea of *(de Rham) cohomology*, which reverses the idea: if we don't know where our domain has holes, we can compute the integral of a zero-curl vector field over a closed loop and see if it is equal to zero.

6.6 Green's Theorem

We have now seen that the curl of a vector field measures, in some sense, the value of a small line integral. It seems like we should then be able to relate the curl to a line integral—if the curl at each point gives the value of a line integral at that point, then perhaps adding up many values of the curl corresponds to adding up many line integrals? In fact this is precisely the case.

Theorem 6.56 (Green)**.** *Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane, and oriented so that the region is to the left as we move around the curve, and suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing R and C. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\nabla \times \vec{F}(x,y)\right) \cdot \vec{k}\, dx\, dy.$$

Remark 6.57. We sometimes write, equivalently, that

$$\int_C F_1\, dx + F_2\, dy = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

These pieces are interchangeable: the left-hand sides are different ways of writing the same line integral, and the right-hand sides are different ways of writing the same two-dimensional integral.

Proof. The basic idea of the proof is that the curl gives the counterclockwise line integral at every point, so taking the integral of the curl will give you the sum of a lot of little line integrals, which add up into one big line integral.

First we'll prove the theorem for a rectangular region. Consider the region $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, and let $\vec{F}(x,y)$ be defined on this region. Let C be the path counterclockwise around the outside of this rectangle, with C_1 going from (a,c) to (a,d), C_2 from (a,d) to (b,d), and so on.

Then we can see easily that

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_a^b (F_1(x,c), F_2(x,c)) \cdot (1,0)\, dx = \int_a^b F_1(x,c)\, dx \\
\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_a^b -F_1(x,d)\, dx \\
\int_{C_1+C_3} \vec{F} \cdot d\vec{r} &= \int_a^b F_1(x,c) - F_1(x,d)\, dx
\end{aligned}$$

But $F_1(x,c) - F_1(x,d) = \int_d^c \frac{\partial F_1}{\partial y}\, dy$ by the Fundamental Theorem of Calculus, so after switching the order of the bounds of integration, we have

$$\int_{C_1+C_3} F \cdot d\vec{r} = \int_a^b \int_c^d -\frac{\partial F_1}{\partial y}\, dy\, dx.$$

We can use a similar argument to work out that

$$\int_{C_2+C_4} F \cdot d\vec{r} = \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx.$$

Adding these together gives the integral around the whole rectangle:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F} \cdot d\vec{r} = \int_a^b \int_c^d \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dy\, dx.$$

To prove this for an arbitrary region: we divide the region up into a large number of small rectangles. On each rectangle, the theorem is true. The integral over the whole region is approximately the sum of the integrals of the small regions (with an error term coming from the fact that the rectangles don't cover the region exactly).

Less obviously, the integral over the boundary of R is approximately the sum of the integrals of the boundaries of each rectangle: two adjacent rectangles will share a boundary segment, but oriented in opposite ways so they cancel out. After adding the boundaries of all the rectangles together, everything will cancel except the segments on the boundary (which again only approximate the boundary).

Thus we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &\approx \sum_i \int_{C_i} \vec{F} \cdot d\vec{r} \\ &= \sum_i \int_{R_i} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dy\, dx \\ &\approx \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dy\, dx. \end{aligned}$$

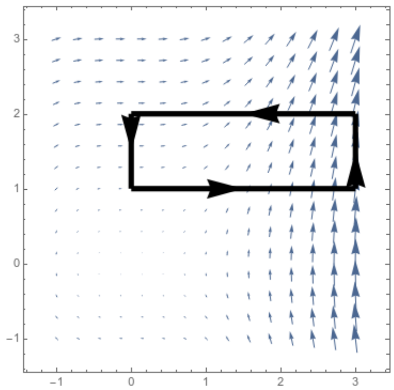
In the limit as we take infinitely many rectangles, these approximations become equalities.

□

Remark 6.58. Green's Theorem only works if $\vec{F}$ is defined on the entire region R. If $\vec{F}$ is undefined somewhere, then the integral over the region is undefined, and so Green's Theorem doesn't make sense.

Green's theorem is a really useful tool for converting otherwise difficult integrals into much easier ones.

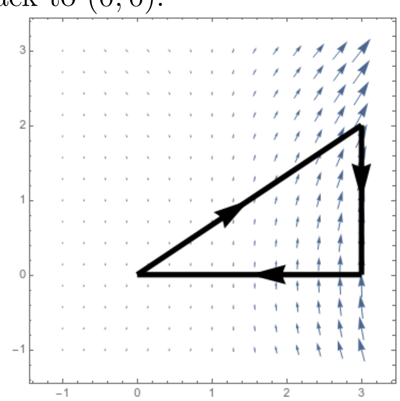
Example 6.59. Let $\vec{F}(x, y) = y\vec{i} + x^2\vec{j}$, and let C be the counterclockwise path around the rectangle R described by $0 \leq x \leq 3, 1 \leq y \leq 2$. Compute $\int_C \vec{F} \cdot d\vec{r}$.



This would be really annoying to do directly: we'd have to do four different integrals over the line segments comprising the perimeter of R. Fortunately, Green's Theorem makes our life easier.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy \\ &= \int_1^2\int_0^3 2x-1\,dx\,dy = \int_1^2 x^2 - x|_0^3\,dy = \int_1^2 6\,dy = 6.\end{aligned}$$

Example 6.60. Let's integrate $(yx^2-y)\,dx + (x^3+4)\,dy$ over a path that goes from $(0,0)$ to $(3,2)$, then to $(3,0)$, then back to $(0,0)$.



We could parametrize each segment individually, but that seems annoying. Instead, we use Green's theorem. We have

$$\begin{aligned}\frac{\partial F_2}{\partial x} &= \frac{\partial x^3+4}{\partial x} = 3x^2 \\ \frac{\partial F_1}{\partial y} &= \frac{\partial yx^2-y}{\partial y} = x^2-1 \\ \int_C \vec{F}\cdot d\vec{r} &= \int_R 3x^2-(x^2-1)\,dA = \int_0^3\int_0^{2x/3} 2x^2+1\,dy\,dx \\ &= \int_0^3 4x^3/3+2x/3\,dx = 3^4/3+3^2/3 = 30.\end{aligned}$$

Except we did one thing subtly wrong. Recall the curve needs to be oriented so the region is on the left; but the curve we want the line integral of has the region on the right. Thus we get the opposite of what we want, and the true answer is -30.

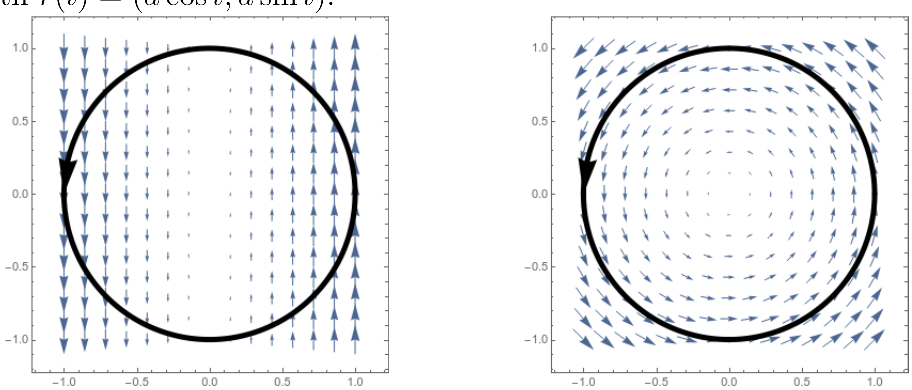
Less often, we use Green's Theorem to go in the other direction, and replace a double integral with a line integral. One fun example is using Green's Theorem to compute area.

Recall that the area of a region is $\int_R 1\, dA$. Is there a vector field such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$? In fact there are several; common choices include $x\vec{j}$, $-y\vec{i}$, and $-y/2\vec{i} + x/2\vec{j}$. Then by Green's Theorem we have

$$A = \int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\int x\, dy - y\, dx$$

where C is any curve that traverses the boundary of the region, keeping the region on the left.

Example 6.61. Let's use this to find the area of a circle of radius a. We can parametrize the circle with $\vec{r}(t) = (a\cos t, a\sin t)$.



Then if we integrate $x\vec{j}$ around the outside, we have

$$A = \int_C x\, dy = \int_0^{2\pi} (a\cos t) \cdot a\cos t\, dt = a^2 \int_0^{2\pi} \cos^2(t)\, dt.$$

We could do that integral, knowing the antiderivative of cosine, but it's annoying. So instead we use the more complicated-looking vector field $(-y/2\vec{i} + x/2\vec{j})$:

$$A = \frac{1}{2}\int_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} a\cos(t)a\cos(t) - a\sin(t)(-a\sin t)\, dt$$
$$= \frac{a^2}{2}\int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt = \frac{a^2}{2}\int_0^{2\pi} 1\, dt = a^2\pi.$$