7 Surface Integrals

7.1 Surfaces

In section 2 we studied curves in space, which are a one-dimensional object and require one "parameter" to define. A surface is a two-dimensional object, and thus requires two parameters.

Definition 7.1. A parametrization of a surface is a function $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^n$.

Sometimes write in components: $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$. Each component is a multivariable function $\mathbb{R}^2 \to \mathbb{R}$.

We've already seen some very important examples.

Example 7.2. The graph of a function $z = f(x,y)$ is given by the parametrization $\vec{r}(s,t) = (s,t,f(s,t))$.

Parametrizations and coordinate systems are the same idea—describing a point with a collection of numbers. Thus the alternate coordinate systems we've seen can be viewed as parametrizations.

Example 7.3. We can parametrize a sphere using spherical coordinates. A sphere of radius 5 is parametrized by $\vec{r}(\theta,\phi) = (5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi)$.

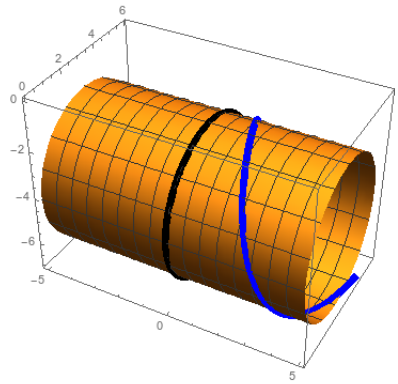
Example 7.4. Let's parametrize a cylinder of radius 1, centered at the origin. We can do this, effectively, with cylindrical coordinates. We have the parametrization $\vec{r}(\theta,z) = (\cos(\theta), \sin(\theta), z)$.

If we want to parametrize a cylinder of radius 3 centered at the line $y = 3, z = -4$, then we just need to tweak this. Notice that this cylinder is pointing in a new direction! We get $\vec{r}(\theta,s) = (s, 3\cos(\theta)+3, 3\sin(\theta)-4)$.

As a final remark, we can see that parametrizations aren't unique. Obviously we could instead do something like $\vec{r}(\theta,s) = (s, 3\sin(\theta)+3, 3\cos(\theta)-4)$, which would just have the circles oriented in the opposite direction.

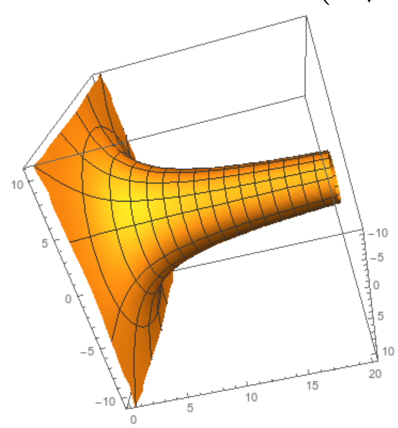
But we could also do something like $\vec{r}(\theta,s) = (s+\theta, 3\sin(\theta)+3, 3\cos(\theta)-4)$. This parametrizes exactly the same cylinder! But in the previous parametrizations, holding s constant gives a circle parallel to the yz plane. In this parametrization, holding s constant gives a helix, as in example 2.4.

In the following diagram, the black curve is the image of the first parametrization with $z = 0$. The blue curve is the second parametrization with $z = 0$.



Example 7.5. Let's parametrize a trumpet shape narrowing from a bell at the origin along the direction of the y axis. The radius, as a function of y, is given by $f(x) = \frac{10}{\sqrt{x}}$.

Then we can parametrize the surface by $\vec{r}(y,\theta) = \left(\frac{10\cos(\theta)}{\sqrt{y}}, y, \frac{10\sin(\theta)}{\sqrt{y}}\right)$.



If we fix one parameter on a surface then we get a curve. This is the same idea as level sets and contours that we discussed in section 3.1.

Example 7.6. Consider the cylinder parametrized by $\vec{r}(t,z) = (\cos(t), \sin(t), z)$. The two parameter curves through the point $(0,1,1)$ are given by $\vec{r}(t,1)$, which is a circle of radius 1 in the $z = 1$ plane; and $\vec{r}(\pi/2, z)$, which is a vertical line through the point $(0,1,0)$.

In general every parameter curve will be either a circle (if we fix the second parameter) or a vertical line (if we fix the first parameter).

7.2 Scalar surface integrals

In section 6.1 we looked at integrating a function over a curve, where we added up the value of the function at all points on that curve. We used this to find average values and to find total mass of a wire from its density.

Often we have a 2-dimensional object or "surface" that we want to do the same adding-up process for. In this section we'll see how to do this.

As usual, we want to break our region up into rectangles, evaluate the function on each rectangle, multiply by the area of the rectangle, and then add them all up. So how do we do this?

If our surface were a region in the plane, we'd already know. So describe with a region in the plane. This is exactly what a parametrization does!

Suppose our surface is parametrized by $\vec{r}(s,t)$ for $a \leq s \leq b, c \leq t \leq d$. We can certainly divide the st rectangle into a bunch of subrectangles. This corresponds on the surface to a bunch of (approximate) parallelograms. So we want to multiply the value of the function on each parallelogram by the area of each parallelogram.

For a given parallelogram, the value of the function f is going to be $f(\vec{r}(s,t))$. So we just need to find the area of the parallelogram.

Recall from section 1.4 that the area of a parallelogram is the magnitude of the cross product of the two sides, $\|\vec{u} \times \vec{v}\|$. In 5.5 we used this to work out the area of a parallelogram parametrized by $\vec{r}(s,t)$. We saw that the sides were

$$\frac{\Delta x}{\Delta s}\vec{i} + \frac{\Delta y}{\Delta s}\vec{j} + \frac{\Delta z}{\Delta s}\vec{k} \approx \frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\Delta s\vec{j} + \frac{\partial z}{\partial s}\Delta s\vec{k} = \vec{r}_s(s,t)\Delta s$$
$$\frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta x}{\Delta t}\vec{k} \approx \frac{\partial x}{\partial t}\Delta t + \frac{\partial y}{\partial t}\Delta t\vec{j} + \frac{\partial z}{\partial t}\Delta t\vec{k} = \vec{r}_t(s,t)\Delta t$$

so the area of the parallelogram is

$$\|\vec{r}_s(s,t)\Delta s \times \vec{r}_t(s,t)\Delta t\| = \|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|\Delta s\Delta t.$$

Thus we define

Definition 7.7. The *surface integral* of the function $f(x,y,z)$ on the surface S parametrized by $\vec{r}(s,t)$ over a planar region R is

$$\begin{aligned}\int_S f\,dS &= \lim \sum_{i,j} f(\vec{r}(s_i,t_i))\|\vec{r}_s(s_i,t_i) \times \vec{r}_t(s_i,t_i)\|\Delta s\Delta t \\ &= \int_R f(\vec{r}(s,t))\|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|\,ds\,dt.\end{aligned}$$

Example 7.8. Integrate x^2 over cylinder of radius 2, height 3, with base at $z = 0$.

Parametrization: $\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$.

$$\begin{aligned}
\vec{r}_\theta(\theta, z) &= (-2\sin\theta, 2\cos\theta, 0) \\
\vec{r}_z(\theta, z) &= (0, 0, 1) \\
\vec{r}_\theta \times \vec{r}_z &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\sin\theta, 0) \\
\|\vec{r}_\theta \times \vec{r}_z\| &= \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2.
\end{aligned}$$

So

$$\begin{aligned}
\int_S x^2 z \, ds &= \int_0^{2\pi} \int_0^3 4\cos^2\theta \cdot 2 \, dz \, d\theta \\
&= 8 \int_0^{2\pi} 3\cos^2\theta \, d\theta = 24 \left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta) \right) \Big|_0^{2\pi} = 24\pi.
\end{aligned}$$

Example 7.9. Find the mass of a hemisphere (the half of the sphere with $z \geq 0$ of radius 1 centimeter with density of z grams per square centimeter.

As with all of our integrals, we need to find a parametrization; then we'll compute bounds and the area correction term, and then have a straightforward integral.

We can parametrize the sphere using spherical coordinates with $\rho = 1$. Thus we take

$$\vec{r}(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$. We compute the parallelogram area by

$$\begin{aligned}
\vec{r}_\theta(\theta, \phi) &= (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \\
\vec{r}_\phi(\theta, \phi) &= (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) \\
\vec{r}_\theta \times \vec{r}_\phi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} \\
&= (-\cos\theta\sin^2\phi - 0)\vec{i} + (0 - \sin\theta\sin^2\phi)\vec{j} + (-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)\vec{k} \\
&= -\cos\theta\sin^2\phi i - \sin\theta\sin^2\phi\vec{j} - \sin\phi\cos\phi\vec{k} \\
\|\vec{r}_\theta \times \vec{r}_\phi\| &= \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi} \\
&= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi} = |\sin\phi|
\end{aligned}$$

Since $0 \leq \phi \leq \pi/2$ we know that $\sin\phi \geq 0$ and can drop the absolute values.

Thus our integral is

$$\begin{aligned}\int_S f\,dS &= \int_0^{2\pi}\int_0^{\pi/2} \cos\phi\cdot\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\,d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi.\end{aligned}$$

Example 7.10. Set up an integral for the surface area of the graph of $z = x^2 - y^2$ over the square $-1 \leq x \leq 1, -1 \leq y \leq 1$.

To find surface area, we need to integrate the function 1 over the surface. Since this surface is a graph, parametrization is easy: we can take $\vec{r}(x,y) = (x, y, x^2 - y^2)$ for $1 \leq x, y \leq 1$. We compute

$$\begin{aligned}\vec{r}_x &= (1, 0, 2x) \\ \vec{r}_y &= (0, 1, -2y) \\ \vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = -2x\vec{i} - 2y\vec{j} + \vec{k} \\ \|\vec{r}_x \times \vec{r}_y\| &= \sqrt{1 + 4x^2 + 4y^2}.\end{aligned}$$

Then our integral is

$$\int_{-1}^{1}\int_{-1}^{1} \sqrt{1+4x^2+4y^2}\,dy\,dx = 4 - \frac{1}{3}\arctan(4/3) + \frac{7\ln(5)}{3} \approx 7.45.$$

Working out some of these cross product terms is really annoying. Fortunately, we can precompute a bunch of them so we don't have to do it again.

Proposition 7.11. *If we parametrize a sphere of radius r with*

$$\vec{r}(\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi),$$

then

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = r^2\sin\phi$$

If we parametrize a cylinder of radius r with

$$\vec{r}(\theta, z) = (r\cos\theta, r\sin\theta, z),$$

then

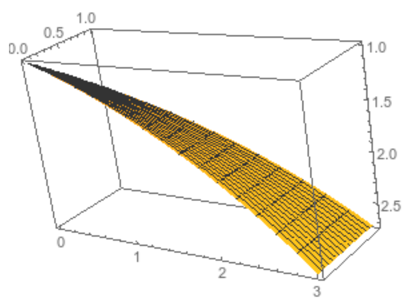
$$\|\vec{r}_\theta \times \vec{r}_z\| = r.$$

If we parametrize the graph of a function $f(x,y)$ with $\vec{r}(x,y) = (x, y, f(x,y))$, then

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

However, we sometimes still need to do surface integrals over non-standard parametrizations.

Example 7.12. Integrate x^2/y over the surface parametrized by $\vec{r}(s,t) = (e^s, st, 3s)$ for $1 \leq s, t \leq 2$.



We compute

$$\begin{aligned}
\vec{r}_s(s,t) &= (e^s, t, 3) \\
\vec{r}_t(s,t) &= (0, s, 0) \\
\vec{r}_s \times \vec{r}_t &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^s & t & 3 \\ 0 & s & 0 \end{vmatrix} = -3s\vec{i} + se^s\vec{k} \\
\|\vec{r}_s \times \vec{r}_t\| &= \sqrt{9s^2 + s^2e^{2s}} = s\sqrt{9 + e^{2s}} \\
\int_S x^2/y \, dS &= \int_1^2 \int_1^2 \frac{e^{2s}}{st} \cdot s\sqrt{9 + e^{2s}} \, ds \, dt \\
&= \int_1^2 \frac{1}{3t} \left(9 + e^{2s}\right)^{3/2} |_1^2 \, dt = \left((9 + e^4)^{3/2} - (9 + e^2)^{3/2}\right) \int_1^2 \frac{1}{3t} \, dt \\
&= \left((9 + e^4)^{3/2} - (9 + e^2)^{3/2}\right) \frac{\ln(t)}{3} |_1^2 \\
&= \left((9 + e^4)^{3/2} - (9 + e^2)^{3/2}\right) \frac{\ln(2)}{3} \approx 101.855.
\end{aligned}$$

7.3 Flux Integrals

A more common thing we want to do with surface integrals is compute flux of a vector field.

Definition 7.13. The *orientation* of a surface is a continuous choice of normal vector at every point. For a rectangle this just means choosing which side is the "front"; for more complex surfaces it often tells you which side is "up".

The *area vector* of an oriented surface is vector $\vec{A}$ with direction the orientation, and magnitude area of the surface.

The *flux* of a vector $\vec{v}$ through a flat oriented surface is $\vec{v} \cdot \vec{A}$.

Remark 7.14. Not every surface can be given a consistent orientation, but we won't really be worrying about non-orientable surfaces in this course.

Flux measures amount of flow through surface. What if surface isn't flat? Or flow isn't constant? Approximate by a bunch of flat surfaces, flow is locally constant, so can use constant flux. Then add up.

Definition 7.15. The *flux integral* of the vector field $\vec{F}$ through the oriented surface $\vec{S}$ is

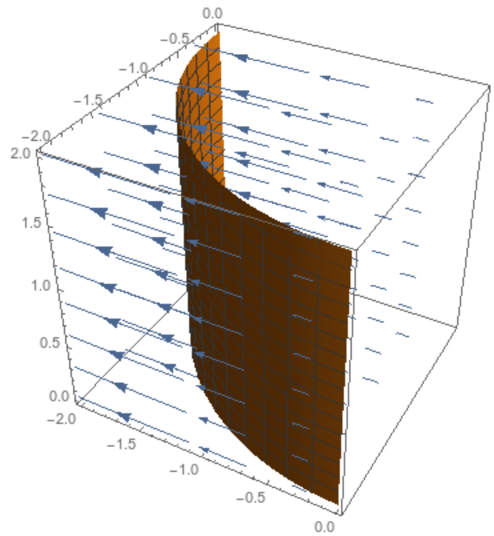
$$\int_S \vec{F} \cdot d\vec{A} = \lim \sum \vec{F} \cdot \Delta \vec{A}.$$

If S is a closed surface oriented outwards, we call this the flux out of $\vec{S}$.

How to compute? Parametrize surface with $\vec{r}(s,t)$. Divide up into small parallelograms. As in section 7.2, each has area $\|\vec{r}_s \times \vec{r}_t\| \Delta s \Delta t$. But direction of $\vec{r}_s \times \vec{r}_t$ is perpendicular to the parallelogram, so we can take $\Delta \vec{A} = (\vec{r}_s \times \vec{r}_t) \Delta s \Delta t$. Thus

$$\begin{aligned} \sum \vec{F} \cdot \Delta \vec{A} &\approx \sum \vec{F} \cdot (\vec{r}_s \times \vec{r}_t) \Delta s \Delta t \\ &\approx \int_a^b \int_c^d \vec{F} \cdot (\vec{r}_s \times \vec{r}_t) \, dt \, ds. \end{aligned}$$

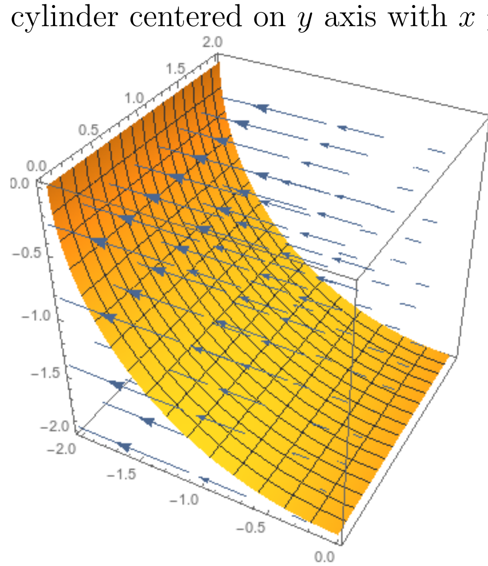
Example 7.16. Suppose we want to compute the flux of $\vec{F}(x,y,z) = x\vec{i}$ outwards through a portion of the cylinder of radius 2 centered on the z-axis with $x \leq 0$, $y \leq 0$, and $0 \leq z \leq 2$.



We can parametrize this by $\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$. From proposition 7.11 we know that $\vec{r}_\theta \times \vec{r}_z = (2\cos\theta, 2\sin\theta, 0)$ (and we check that this is oriented outwards), so we set up the integral

$$\begin{aligned}\int_0^2 \int_\pi^{3\pi/2} (2\cos\theta, 0, 0) \cdot (2\cos\theta, 2\sin\theta, 0)\, d\theta\, dz &= \int_0^2 \int_\pi^{3\pi/2} 4\cos^2\theta\, d\theta\, dz \\ &= \int_0^2 2\theta + \sin(2\theta)|_\pi^{3\pi/2}\, dz \\ &= \int_0^2 \pi\, dz = 2\pi.\end{aligned}$$

What if instead we take cylinder centered on y axis with $x \leq 0, z \leq 0, 0 \leq y \leq 2$?



We now parametrize it with $\vec{r}(\theta, y) = (2\cos\theta, y, 2\sin\theta)$ for $\pi \leq \theta \leq 3\pi/2$. We compute that

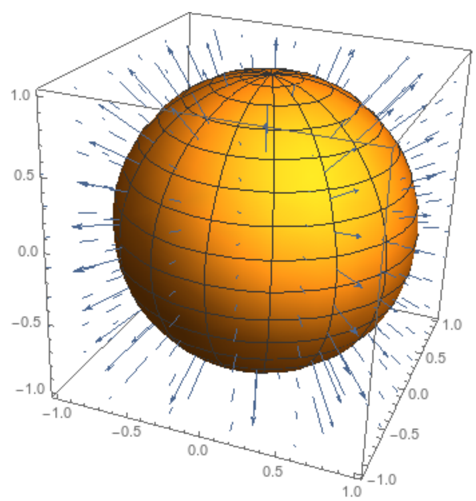
$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \end{vmatrix} = (-2\cos\theta, 0, -2\sin\theta).$$

But this vector is oriented inwards: if we take $\theta = \pi$ then our cross product vector is $(2, 0, 0)$ which points inwards from $\vec{r}(\pi, 0) = (-2, 0, 0)$. So we take the negative of this, and our cross product vector should be $\vec{r}_y \times \vec{r}_\theta = (2\cos\theta, 0, 2\sin\theta)$.

From there, we have a similar integral

$$\begin{aligned} \int_0^2 \int_\pi^{3\pi/2} (2\cos\theta, 0, 0) \cdot (2\cos\theta, 0, 2\sin\theta)\, d\theta\, dy &= \int_0^2 \int_\pi^{3\pi/2} 4\cos^2\theta\, d\theta\, dy \\ &= \int_0^2 2\theta + \sin(2\theta)\Big|_\pi^{3\pi/2} dy \\ &= \int_0^2 \pi\, dy = 2\pi. \end{aligned}$$

Example 7.17 (Gauss's Law)**.** Flux of vector field $\frac{\vec{r}}{\|\vec{r}\|^3}$ through a sphere of radius R (oriented outwards).



We don't actually need to compute an integral here. The flux is always perpendicular to the surface of the sphere, so we have

$$\vec{F} \cdot d\vec{A} = \|\vec{F}\| \cdot \|d\vec{A}\| = \frac{dA}{\|\vec{r}^2\|} = \frac{dA}{R^2}$$

since we're evaluating on the sphere of radius R. Then the flux integral is just $\frac{1}{R^2} \int_S dA$ where the integral is just the area of the sphere of radius R, and thus the flux integral is equal to $4\pi R^2 / R^2 = 4\pi$.

But suppose we want to compute the integral normally. We parametrize the unit sphere by $\vec{r}(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$, and we worked out in proposition 7.11 that

$$\vec{r}_\phi \times \vec{r}_\theta = R^2\cos\theta\sin^2\phi\vec{i} + R^2\sin\theta\sin^2\phi\vec{j} + R^2\sin\phi\cos\phi\vec{k},$$

and this direction is oriented outwards. So the flux integral is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} \frac{1}{R^3}(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)\cdot(R^2\cos\theta\sin^2\phi\vec{i} + R^2\sin\theta\sin^2\phi\vec{j} + R^2\sin\phi\cos\phi\vec{k})\,d\phi\,d\theta \\
&\quad = \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \sin\phi\cos^2\phi\,d\phi\,d\theta \\
&\quad = \int_0^{2\pi}\int_0^{\pi} \sin\phi\,d\phi\,d\theta = \int_0^{2\pi} -\cos\phi|0^\pi\,d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi.
\end{aligned}$$

Proposition 7.18.
- *If S is the graph of $z = f(x,y)$ over R, oriented upwards, then*

$$\int_S \vec{F}\cdot d\vec{A} = \int_R \vec{F}(x,y,f(x,y))\cdot(-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dx\,dy.$$

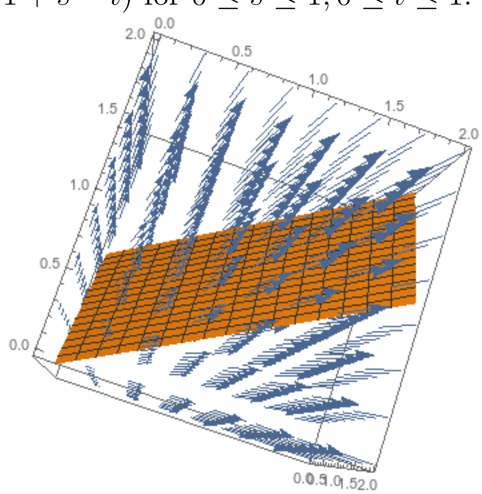
- *If S is a cylinder oriented away from the z-axis of radius R, then*

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R,\theta,z)\cdot(\cos\theta\vec{i} + \sin\theta\vec{j})R\,dz\,d\theta.$$

- *If S is a sphere of radius R oriented outwards, then*

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_S \vec{F}\cdot\frac{\vec{r}}{\|\vec{r}\|}\,dA \\
&= \int_T \vec{F}(R,\theta,\phi)\cdot(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})R^2\sin\phi\,d\phi\,d\theta.
\end{aligned}$$

Example 7.19. Find flux of $\vec{F}(x,y,z) = x\vec{i} + y\vec{j}$ through the surface oriented downwards given by $\vec{r}(s,t) = (2s, s+t, 1+s-t)$ for $0 \le s \le 1, 0 \le t \le 1$.



We compute

$$\begin{aligned}
\vec{r}_s(s,t) &= (2,1,1) \\
\vec{r}_t(s,t) &= (0,1,-1) \\
\vec{r}_s \times \vec{r}_t &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = (-1-1)\vec{i} + (0+2)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}.
\end{aligned}$$

But this vector is oriented upwards, so we take its opposite $2\vec{i} - 2\vec{j} - 2\vec{k}$. Then our integral is

$$\begin{aligned}
\int_0^1 \int_0^1 (2s, s+t, 0) \cdot (2,-2,-2)\, ds\, dt &= \int_0^1 \int_0^1 2s - 2t\, ds\, dt \\
&= \int_0^1 s^2 - 2st|_0^1\, dt = \int_0^1 1 - 2t\, dt \\
&= t - t^2|_0^1 = 0.
\end{aligned}$$

Thus there is no net flux.

7.4 Stokes's Theorem

First we have to define the boundary of a surface. I can give a technical definition: a point P is on the boundary of S if no open ball $B_\epsilon = \{Q : \|Q - P\| < \epsilon\}$ centered at P is entirely contained in S. But more generally we understand what the boundary of a surface is: it's the set of all the points on the edge.

We're going to want to worry about the orientation of a surface relative to its boundary. We want the orientations to be compatible: we determine compatibility via the right-hand rule. We pick an orientation for S by choosing a normal vector for every point. The boundary is oriented compatibly if the (clockwise or counterclockwise) circulation corresponding to this normal vector points in the same direction as the boundary.

Once our boundaries are oriented compatibly, we can make an argument very similar to Green's theorem. We can compute the circulation around the boundary with a line integral. Or, instead, we can compute the integral of the curl over the entire surface; this integrates all the circulation density, and thus gives us the total circulation.

Theorem 7.20 (Stokes)**.** *If S is a smooth oriented surface with piecewise smooth oriented boundary C (with matching orientations of S and C), and $\vec{F}$ is a smooth vector field on an* *open region containing S and C, then*

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot d\vec{A}.$$

Remark 7.21. If S is a region within the xy plane, then this is precisely Green's Theorem.

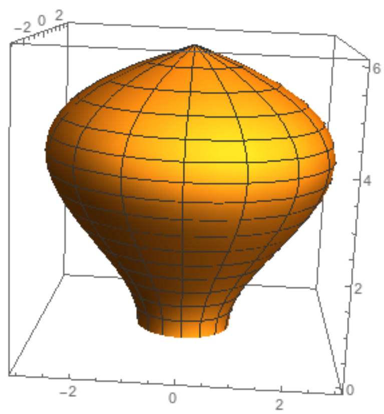
Example 7.22. Let $\vec{F}(x, y, z) = -2y\vec{i} + 2x\vec{j}$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C is a circle parallel to yz plane centered in x axis.

Compute $\nabla \times F = 4\vec{k}$. Curl is parallel to circle, so flux of curl through disk is zero. Thus circulation is zero.

Where C is parallel to the xy plane, centered on z axis, radius r.

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times F \cdot d\vec{A} = \|\nabla \times \vec{F}\| \cdot \text{area of } S = 4\pi r^2.$$

Example 7.23. Let's consider the surface of a lightbulb, whose base is given $x^2 + y^2 = 1$. Let $\vec{F}(x, y, z) = e^{z^2-2z}x\vec{i} + (\sin(xyz) + y + 1)\vec{j} + e^{z^2}\sin(z^2)\vec{k}$, and find flux of $\nabla \times \vec{F}$ outward through the lightbulb's surface.



Attempting to do this surface integral would clearly be terrible even if we had a good parametrization for the surface. Fortunately we can avoid this, since the entire boundary of the lightbulb is just $x^2 + y^2 = 1$. Thus

$$\begin{aligned}
\int_S \nabla \times \vec{F} \cdot d\vec{A} &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} \vec{F}(\cos\theta, \sin\theta, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta \\
&= \int_0^{2\pi} (\cos\theta, \sin\theta + 1, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta \\
&= \int_0^{2\pi} -\cos\theta\sin\theta + \cos\theta\sin\theta + \cos\theta\, d\theta \\
&= \int_0^{2\pi} \cos\theta\, d\theta = \sin\theta|_0^{2\pi} = 0.
\end{aligned}$$

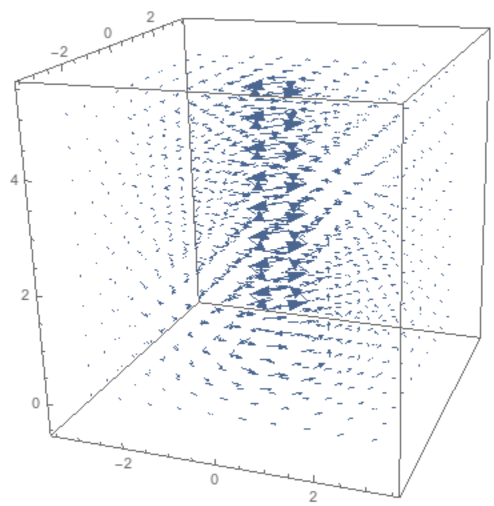
Notice that this doesn't depend on the specific shape of the lightbulb!

Stokes's theorem is particularly nice when we're studying an irrotational field—one with zero curl.

Example 7.24. Let

$$\vec{B}(x,y,z) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2}.$$

This vector field is, among other things, the magnetic field induced by a current running down a wire along the z-axis.



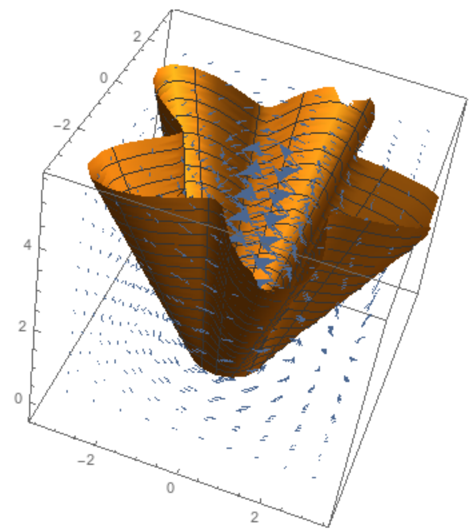
We saw this field in example 6.55, where we calculated that $\nabla \times \vec{B}(x,y,z) = \vec{0}$. Let's compute the circulation of $\vec{B}$ counterclockwise around C_2, a five-pointed star at the height $z = \pi + e$, centered at the z-axis.

We don't want to try to compute that directly; fortunately, we don't have to. We first might try using Stokes's theorem to integrate the curl (which is zero) over the interior; but we can't do that, because $\vec{B}$ isn't actually defined at $x = y = 0$. So we have to do something more complex.

Let's start by computing a relatively easy integral, over C_1, the counterclockwise circle of radius 1 in the xy plane. We parametrize this with $\vec{r}(t) = (\cos t, \sin t, 0)$, and we get the integral

$$\int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

Now let's consider the cylindrical-ish surface S whose base is C_1 and whose top is C_2.



Then we see that the boundary of S (oriented outwards) is $C_1 - C_2$ (since we need to reverse the orientation of C_2 to match the orientation of S). Thus by Stokes's theorem we have

$$\int_S \nabla \times \vec{B} \cdot d\vec{A} = \int_{C_1 - C_2} \vec{B} \cdot d\vec{r}.$$

But $\nabla \times \vec{B} = \vec{0}$, so this tells us that

$$\begin{aligned} 0 &= \int_{C_1 - C_2} \vec{B} \cdot d\vec{r} \\ &= \int_{C_1} \vec{B} \cdot d\vec{r} - \int_{C_2} \vec{B} \cdot d\vec{r} \\ \int_{C_2} \vec{B} \cdot d\vec{r} &= \int_{C_1} \vec{B} \cdot d\vec{r} = 2\pi. \end{aligned}$$

Remark 7.25. This last example shows us something even more dramatic. The actual details of the curve C_2 are completely irrelevant; only the fact that it can combine with C_1 to form the boundary of a tube that doesn't intersect the z-axis. This argument shows that any closed curve that winds around the z-axis once will have an integral of $\pm 2\pi$, with the sign depending on the relative orientation of C_1 and C_2.

This idea is really powerful, and is the basis of something called "cohomology". With luck we'll talk about this on the last day of class, in section 8.4

We can also turn this process backwards.

Definition 7.26. Let $\vec{G}$ be a vector field. If $\nabla \times \vec{F} = \vec{G}$, then we say that $\vec{F}$ is a *vector potential* for $\vec{G}$. If $\vec{G}$ has a vector potential, we say it is a *curl field*—which just means that it is the curl of some vector field.

Example 7.27. Let $\vec{F}(x,y,z) = (-xy, 1, yz - 3x^2)$. Find a vector potential for $\vec{F}$ or prove none exists.

Let $\vec{G}(x,y,z) = (G_1(x,y,z), G_2(x,y,z), G_3(x,y,z))$ be a vector potential for $\vec{F}$, meaning that $\nabla \times \vec{G} = \vec{F}$. We have a lot of degrees of freedom here—if there's one vector potential, then there are many—so we can make some simplifying assumptions up front. In particular, we'll assume $G_3(x,y,z) = 0$, and then we have

$$\begin{aligned}
\vec{F}(x,y,z) = \nabla \times \vec{G}(x,y,z) &= \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & 0 \end{bmatrix} \right| \\
&= -\frac{\partial G_2}{\partial z}\vec{i} + \frac{\partial G_1}{\partial z}\vec{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\vec{k} \\
-\frac{\partial G_2}{\partial z} &= -xy \\
\frac{\partial G_1}{\partial z} &= 1 \\
\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= yz - 3x^2.
\end{aligned}$$

The first equation tells us that $G_2(x,y,z) = xyz + g_2(x,y)$, and the second equation tells us that $G_1(x,y,z) = z + g_1(x,y)$.

The third equation gives us that $yz + \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} = yz - 3x^2$. We can try taking $G(x,y,z) = (z + 3x^2 y, xyz, 0)$, and we see this satisfies all our constraints. Alternatively, we could take $G(x,y,z) = (z, xyz - x^3, 0)$, and that would also work.

Proposition 7.28. *If $\vec{G}$ is a curl field, then any two oriented surfaces with the same oriented boundary have the same flux integral.*

Proof. Suppose $\vec{G}$ is a curl field, and S_1 and S_2 are two oriented surfaces with the same oriented boundary C. Then by Stokes's theorem, we have

$$\int_{S_1} \vec{G} \cdot d\vec{A} = \int_C \vec{F}\, d\vec{r} = \int_{S_2} \vec{G} \cdot d\vec{A}.$$

□

This result should remind you of the path-independence result for line integrals. Two paths with the same boundary have the same integral if the vector field is a gradient field; two surfaces with the same boundary have the same integral if the vector field is a curl field.

We used the curl to test whether a vector field is a gradient field. Now we need a similar tool to test whether a vector field is a curl field.