Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 10
Due Monday, August 4

This week's mastery quiz has **two** topics. Everyone should submit both.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Line Integrals
- Secondary Topic 5: Vector Fields

Name:

Name:

M5: Line Integrals

(a) Compute the arc length of the curve $\vec{r}(t) = (t^2, 3, \frac{1}{3}t^3)$ between the points $(0,3,0)$ and $(1,3,1/3)$.

Solution: We know the arc length of a curve is $\int_C 1\,ds = \int_a^b \|\vec{r}'(t)\|\,dt$. We see that our two points are $\vec{r}(0)$ and $\vec{r}(1)$, so we have

$$\begin{aligned} L &= \int_0^1 \|(2t, 0, t^2)\|\,dt \\ &= \int_0^1 \sqrt{4t^2 + t^4}\,dt = \int_0^1 t\sqrt{4+t^2}\,dt \\ &= \frac{1}{3}(4+t^2)^{3/2}\Big|_0^1 = \frac{1}{3}(5^{3/2} - 4^{3/2}) = \frac{1}{3}(\sqrt{125} - 8). \end{aligned}$$

(b) Let $f(x,y,z) = xy + zx^2$. Compute $\int_C \nabla f\,ds$ where C is parametrized by the curve $\vec{r}(t) = (2t-1, t^4 + t, \sin(\pi t))$ for $t \in [0,2]$.

Solution: We could compute the whole line integral out, but that would be awful. Instead we use the fundamental theorem of line integrals, and we have

$$\begin{aligned} \int_C \nabla f\,ds &= f(r(2)) - f(r(0)) \\ &= f(3, 18, 0) - f(-1, 0, 0) \\ &= 54 + 0 - (0+0) = 54. \end{aligned}$$

(c) Let C be the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

Compute the line integral of the vector field $\vec{F}(x,y) = (xy, -x^2)$.

Solution: We parametrize the curve with $\vec{r}(t) = (t, t^2)$. Then we have

$$\begin{aligned} \int \vec{F} \cdot d\vec{r} &= \int_0^1 (t^3, -t^2) \cdot (1, 2t)\,dt = \int_0^1 t^3 - 2t^3\,dt \\ &= \int_0^1 -t^3\,dt = \frac{-t^4}{4}\Big|_0^1 = -1/4. \end{aligned}$$

S5: Vector Fields

(a) Find a flow line for the vector field $\vec{F}(x,y) = 3\vec{i} + -6x\vec{j}$ that goes through the point $(1,2)$ at time $t = 0$.

Solution: We have the system of equations

$$x'(t)3$$
$$y'(t) = -tx.$$

The first tells us that $x(t) = 3t + C$; since $x(0) = 1$ we know that $C = 1$.

Then we know that $y'(t) = -6x = -18t - 6$. Then we know that $y(t) = -9t^2 - 6t + C_2$, and since $y(0) = 2$ we know $C_2 = 2$. Thus the flow line is

$$\vec{r}(t) = (3t + 1, -9t^2 - 6t + 2).$$

(b) If $H(x, y, z) = xyz\vec{i} + \sin(xy)\vec{j} - \cos(yz)\vec{k}$, compute $\nabla \times H(x, y, z)$.

Solution:

$$\begin{aligned}\nabla \times H(x, y, z) &= \left(\frac{\partial(-\cos(yz))}{\partial y} - \frac{\partial \sin(xy)}{\partial z}\right)\vec{i} \\ &\quad + \left(\frac{\partial xyz}{\partial z} - \frac{\partial(-\cos(yz))}{\partial x}\right)\vec{j} \\ &\quad + \left(\frac{\partial \sin(xy)}{\partial x} - \frac{\partial xyz}{\partial y}\right)\vec{k} \\ &= z\sin(yz)\vec{i} + xy\vec{j} + (y\cos(xy) - xz)\,\vec{k}.\end{aligned}$$

(c) Find a potential function for $\vec{F}(x, y) = (x + y)\vec{i} + (x - y)\vec{j}$ or prove none exists.

Solution: If $\vec{F} = \nabla f$, we must have

$$f(x, y) = \frac{1}{2}x^2 + xy + g(y)$$
$$f(x, y) = xy - \frac{1}{2}y^2 + h(x)$$

and we can satisfy this with

$$f(x, y) = x^2/2 + xy - y^2/2.$$

We could also compute the curl:

$$\begin{aligned}\nabla \times F(x, y) &= \left(\frac{\partial x - y}{\partial x} - \frac{\partial x + y}{\partial y}\right)\vec{k} \\ &= (1 - 1)\vec{k} = \vec{0}.\end{aligned}$$

Since the vector field is defined everywhere, this is enough to tell us that it's conservative, and must have a potential field. But that doesn't give us a potential function, so we still need to solve the differential equations.

(d) Find a potential function for $\vec{F}(x, y, z) = y\vec{i} + x\vec{j} + xz\vec{k}$ or prove none exists.

Solution: There is no potential function, and we can prove this in two different ways. First, if $\vec{F} = \nabla f$, we must have

$$f(x,y,z) = xy + g_1(y,z)$$
$$f(x,y,z) = xy + g_2(x,z)$$
$$f(x,y,z) = xz^2/2 + g_3(x,y)$$

So we need a function that is equal to xy plus a function purely of y and z, but also equal to $xz^2/2$ plus a function purely of x and y, and you can't have both of those things at once.

Alternatively, we can just compute the curl of $\vec{F}$:

$$\begin{aligned}\nabla \times \vec{F}(x,y,z) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & xz \end{vmatrix} \\ &= 0\vec{i} - z\vec{j} + (1-1)\vec{k} = -z\vec{j} \neq \vec{0}.\end{aligned}$$

Since the curl is not $\vec{0}$ we know this must not be a conservative field.