Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 11
Due Wednesday, August 6

This week's mastery quiz has three topics. Everyone should submit M6. If you have a 4/4 on M5, or a 2/2 on S5, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Line Integrals
- Major Topic 6: Surface Integrals
- Secondary Topic 5: Vector Fields

Name:

Name:

M5: Line Integrals

(a) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = x^2y^2\vec{i} + 2xy\vec{j}$ and C is the boundary of the square (clockwise) through points $(0,0), (0,3), (3,3), (3,0)$.

Solution: We could parametrize the outside of the square, but that would actually suck. Instead we integrate over the square $[0,3] \times [0,3]$, which works by Green's Theorem. We compute the curl

$$\nabla \times \vec{F}(x,y) = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^2) = 2y - 2x^2y.$$

So the integral *counterclockwise* around the square is givin by Green's Theorem:

$$\begin{aligned}\int_{-C} \vec{F} \cdot d\vec{r} &= \int_0^3 \int_0^3 2y - 2x^2y \, dy \, dx \\ &= \int_0^3 y^2 - x^2y^2 \Big|_0^3 dx \\ &= \int_0^3 9 - 9x^2 \, dx = 9x - 3x^3 \Big|_0^3 = 27 - 81 = -54.\end{aligned}$$

Thus the integral *clockwise* around the square is 54.

(b) Evaluate $\int_C \vec{G} \cdot d\vec{r}$ where $\vec{G}(x,y,z) = z\vec{i} - y\vec{j} + 2x\vec{k}$ and C is the straight line path from $(0,0,0)$ to $(1,2,3)$.

Solution: We can parametrize C with $\vec{r}(t) = (t, 2t, 3t)$ for $0 \leq t \leq 1$. Then the integral is

$$\begin{aligned}\int_C \vec{G} \cdot d\vec{r} &= \int_0^1 \left(3t\vec{i} - 2t\vec{j} + 2t\vec{k}\right) \cdot \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) dt \\ &= \int_0^1 3t - 4t + 6t \, dt = \int_0^1 5t \, dt \\ &= \frac{5}{2}t^2 \mid_0^1 = \frac{5}{2}.\end{aligned}$$

(c) Let $f(x,y,z) = 3x^2y + \frac{z}{x}$. Compute $\int_C \nabla f \, ds$ where C is parametrized by the curve $\vec{r}(t) = (t, \frac{t^2}{t+4}, t)$ for $t \in [1,4]$.

Solution: We could compute the whole line integral out, but that would be awful. Instead we use the fundamental theorem of line integrals, and we have

$$\begin{aligned}\int_C \nabla f \, ds &= f(r(4)) - f(r(1)) \\ &= f(4,2,4) - f(1,1/5,1) \\ &= 96 + 1 - (3/5 + 1) = 96 - \frac{3}{5} = 95 + \frac{2}{5} = \frac{477}{5} = 95.4\end{aligned}$$

Name: ____

M6: Surface Integrals

(a) Find the surface area of the part of the paraboloid $z = 16 - x^2 - y^2$ which lies above the xy-plane.

Solution: We want to compute $\int_S 1\, dS = \iint_R \|\vec{r}_s \times \vec{r}_t\|\, ds\, dt$. We know from class that if our surface is the graph of a function, then

$$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}.$$

We're integrating over the circle $x^2 + y^2 = 16$, so we should use polar coordinates. So we get the integral

$$\begin{aligned}\int_0^4 \int_0^{2\pi} \sqrt{1+4r^2}r\, d\theta\, dr &= \int_0^4 2\pi r\sqrt{1+4r^2}\, dr \\ &= \frac{\pi}{6}(1+4r^2)^{3/2}\Big|_0^4 = \frac{\pi}{6}(65^{3/2} - 1).\end{aligned}$$

(b) Compute the flux of the vector field $\vec{F}(x, y, z) = y\vec{i} + z\vec{k}$ upwards through portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ above the plane $z = 0$.

Solution: We can parametrize the cone with $(r\cos(t), r\sin(t), 1-r)$, where $0 \le r \le 1$ and $0 \le t \le 2\pi$, and we get Jacobian term

$$\begin{aligned}\vec{r}_r \times \vec{r}_t &= (\cos(t), \sin(t), -1) \times (-r\sin(t), r\cos(t), 0) \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(t) & \sin(t) & -1 \\ -r\sin(t) & r\cos(t) & 0 \end{vmatrix} = r\cos(t)\vec{i} + r\sin(t)\vec{j} + (r\cos^2 t + r\sin^2 t)\vec{k} \\ &= r\cos(t)\vec{i} + r\sin(t)\vec{j} + r\vec{k}.\end{aligned}$$

This is the right orientation since it is in fact pointed upwards for positive r.

Then we compute flux as

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{S} &= \int_0^1 \int_0^{2\pi} (r\sin(t), 0, 1-r)\cdot(r\cos(t), r\sin(t), r)\, dt\, dr \\ &= \int_0^1 \int_0^{2\pi} r^2\sin(t)\cos(t) + r - r^2\, dt\, dr \\ &\int_0^1 \frac{1}{2}r^2\sin^2(t) + rt - r^2 t\Big|_0^{2\pi} dr \\ &\int_0^1 2\pi r - 2\pi r^2\, dr = \pi r^2 - \frac{2\pi}{3}r^3\Big|_0^1 = \frac{\pi}{3}.\end{aligned}$$

(c) Compute $\iint_S \nabla \times \vec{F}\cdot d\vec{S}$ where $\vec{F}(x, y, z) = (z - y, x, -x)$ and S is the hemisphere $x^2 + y^2 + z^2 = 4, z \ge 0$, oriented inwards towards the center of the hemisphere.

Solution: Instead of parametrizing the spehre, we can just parametrize the boundary, which is $x^2 + y^2 = 4$ and parametrized by $(2\cos(t), 2\sin(t))$. However, we see this is the wrong orientation, so we instead use $\vec{r}(t) = (2\cos(t), -2\sin(t))$. (We could also take $(2\sin(t), 2\cos(t))$, or various other options.)

Thus we compute

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{R} \\
&= \int_0^{2\pi} (0 + 2\sin(t), 2\cos(t), -2\cos(t)) \cdot (-2\sin(t), -2\cos(t), 0)\, dt \\
&= \int_0^{2\pi} -4\sin^2(t) - 4\cos^2(t)\, dt \\
&= \int_0^{2\pi} -4\, dt = -8\pi.
\end{aligned}$$

S5: Vector Fields

(a) Which of the following is a flow line for $\vec{F}(x, y, z) = 2x\vec{i} + z\vec{j} - z^2\vec{k}$?

(i) $\vec{r}(t) = \left(e^{2t}, \ln|t|, \frac{1}{t}\right)$

(ii) $\vec{s}(t) = \left(e^t, \frac{1}{t}, t\right)$

Solution: We compute

$$\begin{aligned}
\vec{r}'(t) &= 2e^{2t}\vec{i} + \frac{1}{t}\vec{j} - \frac{1}{t^2}\vec{k} \\
\vec{F}(\vec{r}(t)) &= 2e^{2t}\vec{i} + \frac{1}{t}\vec{j} - \frac{1}{t^2}\vec{k}
\end{aligned}$$

so $\vec{r}(t)$ is a flow line.

But we also compute

$$\begin{aligned}
\vec{s}'(t) &= e^t\vec{i} - \frac{1}{t^2}\vec{j} + \vec{k} \\
\vec{F}(\vec{s}(t)) &= 2e^t\vec{i} + t\vec{j} - t^2\vec{k}
\end{aligned}$$

so $\vec{s}(t)$ is not a flow line.

(b) Find a potential function for $\vec{G}(x, y) = xy\vec{i} + (x + y)\vec{j})$, or prove none exists.

Solution: There are two approaches we could take here. One is to set up a system of differential equations:

$$\frac{\partial g}{\partial x}(x,y) = xy \qquad\qquad \frac{\partial g}{\partial y}(x,y) = x + y$$
$$g(x,y) = \frac{1}{2}x^2y + g_1(y) \qquad\qquad g(x,y) = xy + \frac{1}{2}y^2 + g_2(x)$$

and that's a contradiction, since xy isn't a function of y and x^2y isn't a function of x.

But the simpler approach is just to compute the curl

$$\begin{aligned}\nabla \times G(x,y) &= \left(\frac{\partial x + y}{\partial x} - \frac{\partial xy}{\partial y}\right)\vec{k} \\ &= (1-x)\vec{k} \neq \vec{0}.\end{aligned}$$

Since the curl isn't zero, this vector field isn't conservative, and doesn't have a potential.

(c) Find a potential field for $\vec{F}(x,y,z) = (y+z)\vec{i} + (x+z^2)\vec{j} + (x+2yz)\vec{k}$ or prove none exists.

Solution: If $\vec{F} = \nabla f$, we must have

$$\begin{aligned}f(x,y,z) &= xy + xz + g_1(y,z) \\ f(x,y,z) &= xy + yz^2 + g_2(x,z) \\ f(x,y,z) &= xz + yz^2 + g_3(x,y)\end{aligned}$$

We can satisfy all of these requirements with $f(x,y,z) = xy + xz + yz^2$.