Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 2
Due Monday, July 7

This week's mastery quiz has three topics. Everyone should submit both M1 and S2, but if you got a 2/2 last week on S1 you don't have to submit it. (Check Blackboard when I get those grades up over this weekend.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Vectors
- Secondary Topic 1: Lines and Planes
- Secondary Topic 2: Vector Functions

Name:

Name:

M1: Vectors

(a) Find the area of the **triangle** with vertices $(0,0,0), (5,3,1), (7,2,2)$.

Solution: This triangle is spanned by the vectors $5\vec{i}+3\vec{j}+\vec{k}$ and $7\vec{i}+2\vec{j}+2\vec{k}$, so we compute

$$\begin{aligned}\left(5\vec{i}+3\vec{j}+\vec{k}\right)\times\left(7\vec{i}+2\vec{j}+2\vec{k}\right) &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 5 & 3 & 1\\ 7 & 2 & 2\end{vmatrix}\\ &= 6\vec{i}+7\vec{j}+10\vec{k}-21\vec{k}-2\vec{i}-10\vec{j}\\ &= 4\vec{i}-3\vec{j}-11\vec{k}.\end{aligned}$$

Thus the area of the triangle is $\frac{1}{2}\|4\vec{i}-3\vec{j}-11\vec{k}\| = \frac{1}{2}\sqrt{16+9+121} = \frac{1}{2}\sqrt{146} = \sqrt{73/2}$.

(b) Find $\cos\theta$ where θ is the angle between $\vec{u}=\vec{i}-2\vec{j}+\vec{k}$ and $\vec{v}=5\vec{i}+2\vec{j}-6\vec{k}$.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\\ &= \frac{5-4-6}{\sqrt{6}\sqrt{65}}\\ &= \frac{-5}{\sqrt{390}}.\end{aligned}$$

(c) Find the orthogonal decomposition of $\vec{v}=6\vec{i}+2\vec{j}-3\vec{k}$ with repsect to $\vec{u}=2\vec{i}+\vec{j}+2\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned}\operatorname{Proj}_{\vec{u}}\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\\ &= \frac{8}{9}(2\vec{i}+\vec{j}+2\vec{k})\\ &= \frac{16}{9}\vec{i}+\frac{8}{9}\vec{j}+\frac{16}{9}\vec{k}\end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned}\vec{v}_\perp &= \vec{v}-\operatorname{Proj}_{\vec{u}}\vec{v}\\ &= 6\vec{i}+2\vec{j}-3\vec{k}-\left(\frac{16}{9}\vec{i}+\frac{8}{9}\vec{j}+\frac{16}{9}\vec{k}\right)\\ &= \frac{38}{9}\vec{i}+\frac{10}{9}\vec{j}-\frac{43}{9}\vec{k}.\end{aligned}$$

Name:

S1: Lines and Planes

(a) Find an equation for the plane that passes through the points $(1,4,2),(3,2,2)$, and $(7,1,5)$.

Solution:

$$\begin{aligned} z &= 2 + m(x-1) + n(y-4) \\ 2 &= 2 + m(3-1) + n(2-4) \\ 5 &= 2 + m(7-1) + n(1-4) \end{aligned}$$

which gives us the system of equations

$$\begin{aligned} 0 &= 2m - 2n \\ 3 &= 6m - 3n \end{aligned}$$

The first equation implies that $m = n$ and thus the second gives us $m = n = 1$. Then the equation of our plane is

$$z = 2 + (x-1) + (y-4).$$

Alternatively, we can take the cross product of the vectors $2\vec{i} - 2\vec{j}$ and $6\vec{i} - 3\vec{j} + 3\vec{k}$. This gives us

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 0 \\ 6 & -3 & 3 \end{vmatrix} = -6\vec{i} - 6\vec{k} + 12\vec{k} - 6\vec{j} = -6\vec{i} - 6\vec{j} + 6\vec{k}$$

and thus we get the equation

$$-6(x-1) - 6(y-4) + 6(z-2) = 0.$$

(b) Find an equation for the plane perpendicular to $\vec{n} = \vec{i} + \vec{4}j - \vec{2}k$ that passes through the point $(5,-3,0)$.

Solution:

$$(x-5) + 4(y+3) - 2(z-0) = 0.$$

(c) Find a vector perpendicular to the plane given by the equation

$$z = 5 - 4(x-2) + 7(y-3).$$

Solution: This equation is the same as

$$-4(x-2) + 7(y-3) - 1(z-5) = 0$$

and so a normal vector is $-4\vec{i} + 7\vec{j} - \vec{k}$.

Name:

S2: Vector Functions

(a) Suppose two particles follow the paths $\vec{r}_1(t) = (3t, t^2, 4-t)$ and $\vec{r}_2(t) = (t^2+2, 6-t, 3t-4)$. Do the two particles collide? Do their paths intersect?

Solution: The particles collide if and only if there's a solution to

$$\begin{aligned} 3t &= t^2 + 2 \\ t^2 &= 6 - t \\ 4 - t &= 3t - 4 \end{aligned}$$

The last equation tells us that $t = 2$. We can check this in the two other equations, and indeed $3 \cdot 2 = 2^2 + 2$ and $2^6 = 6 - 2$. So the particles do collide. (And thus their paths also intersect.)

(b) Find a (parametric) equation for the line tangent to the curve $\vec{r}(t) = (t^3, \frac{1}{t+1}, t-2)$ at the time $t = 1$.

Solution: We have $\vec{r}'(t) = (3t^2, \frac{-1}{(t+1)^2}, 1)$ and thus $\vec{r}'(1) = (3, -1/4, 1)$. We compute $\vec{r}(1) = (1, 1/2, -1)$. So a parametric equation for the line is

$$T(t) = (1 + 3t, 1/2 - t/4, -1 + t).$$

(c) If a particle moves with velocity $\vec{v}(t) = \langle 6t, \cos(\pi t/2) \rangle$ then what is the displacement between times $t = 1$ and $t = 4$?

Solution:

$$\begin{aligned} \vec{r}(t) &= \int_1^4 6t\vec{i} + \cos(\pi t/2)\vec{j}\, dt \\ &= 3t^2\vec{i} + \frac{2}{\pi}\sin(\pi t/2)\vec{j}\Big|_1^4 \\ &= (48 - 3)\vec{i} + \frac{2}{\pi}(\sin(2\pi) - \sin(\pi/2))\vec{j} = 45\vec{i} - \frac{2}{\pi}\vec{j}. \end{aligned}$$