Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 3
Due Wednesday, July 9

This week's mastery quiz has *four* topics. Everyone should submit both M2 and S3. If you have a 2/2 on S2, you don't need to do it again. If you have a 4/4 on M1 (meaning you have gotten 2/2 twice), you don't need to do it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Vectors
- Major Topic 2: Partial Derivatives
- Secondary Topic 2: Vector Functions
- Secondary Topic 3: Multivariable Functions

Name:

Name:

M1: Vectors

(a) Find a *unit* vector perpendicular perpendicular to both $\vec{v} = 3\vec{i} - 2\vec{j} + 5\vec{k}$ and $\vec{w} = 7\vec{j} + \vec{k}$.

Solution:

$$\begin{aligned}
\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 5 \\ 0 & 7 & 1 \end{vmatrix} \\
&= -2\vec{i} + 0\vec{j} + 21\vec{k} + 0\vec{k} - 35\vec{i} - 3\vec{j} \\
&= -37\vec{i} - 3\vec{j} + 21\vec{k}
\end{aligned}$$

This is perpendicular to both $\vec{v}$ and $\vec{w}$, but isn't a unit vector; we need to normalize. So our final vector is

$$\begin{aligned}
\vec{u} &= \frac{-37\vec{i} - 3\vec{j} + 21\vec{k}}{\| - 37\vec{i} - 3\vec{j} + 21\vec{k}\|} \\
&= \frac{-37\vec{i} - 3\vec{j} + 21\vec{k}}{\sqrt{37^2 + 3^2 + 21^2}} \\
&= \frac{-37\vec{i} - 3\vec{j} + 21\vec{k}}{\sqrt{1819}} \\
&= \frac{-37}{\sqrt{1819}}\vec{i} - \frac{3}{\sqrt{1819}}\vec{j} + \frac{21}{\sqrt{1819}}\vec{k} \\
&\approx -.8675\vec{i} - .0703\vec{j} + .4924\vec{k} \\
&\approx \frac{-37}{42.6497}\vec{i} - \frac{3}{42.6497}\vec{j} + \frac{21}{42.6497}\vec{k}
\end{aligned}$$

(b) Find the cosine of the angle between the vectors $\vec{v} = 6\vec{i} - 2\vec{j} + 2\vec{k}$ and $\vec{u} = 4\vec{i} - 1\vec{j} + 2\vec{k}$.

Solution: We know that

$$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \cdot \|\vec{u}\|} = \frac{24 + 2 + 4}{\sqrt{44}\sqrt{21}} = \frac{30}{\sqrt{924}} = \frac{5\sqrt{3}}{\sqrt{77}}.$$

(c) Find the orthogonal decomposition of $\vec{v} = 4\vec{i} + \vec{j} - \vec{k}$ with respect to $\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned}
\text{Proj}_{\vec{u}}\, \vec{v} &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{8 - 1 - 3}{4 + 1 + 9}\langle 2, -1, 3\rangle \\
&= \frac{4}{14}\langle 2, -1, 3\rangle = \frac{4}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}.
\end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

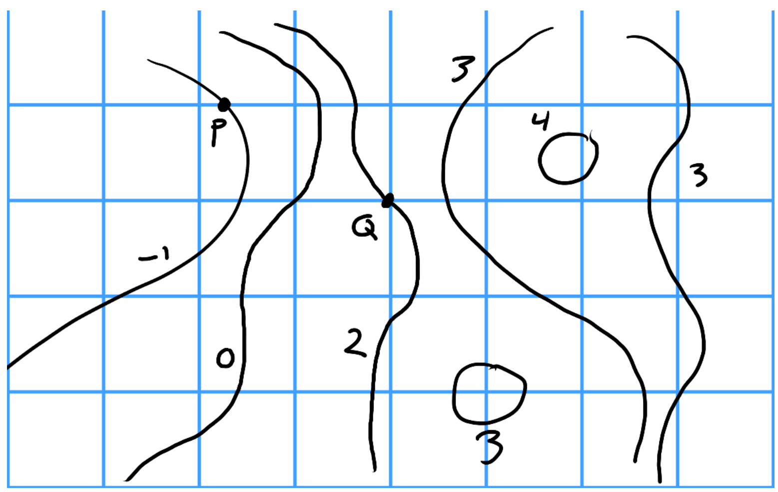
$$\begin{aligned}\vec{v}_\perp &= \vec{v} - \mathrm{Proj}_{\vec{u}}\,\vec{v} \\ &= 4\vec{i} + \vec{j} - \vec{k} - \left(\frac{4}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}\right) \\ &= \frac{24}{7}\vec{i} + \frac{9}{7}\vec{j} - \frac{13}{7}\vec{k}.\end{aligned}$$

M2: Partial Derivatives

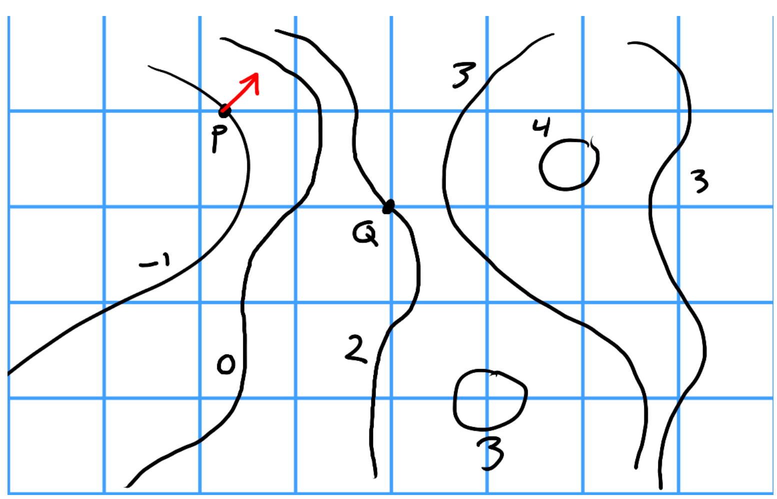
(a) Below is a contour plot of the function $f(x, y)$.

  (i) Sketch the gradient vector at P.

  (i) Estimate $\frac{\partial h}{\partial x}$ at the point Q. Explain your reasoning in a sentence or so.



Solution:



It looks like $\frac{\partial f}{\partial x}$ is about 2. If you move one unit to the left, your output drops by about 2. If you move about half a unit to the right, your value increases by 1.

(b) Let $h(x, y, z) = x^3yz^2 + \frac{xy}{y^2+z}$. Compute $h_x(x, y, z)$, $h_y(x, y, z)$, and $h_z(x, y, z)$.

Solution:

$$\begin{aligned} h_x(x, y, z) &= 3x^2yz^2 + \frac{y}{y^2+z} \\ h_y(x, y, z) &= x^3z^2 + \frac{x(y^2+z) - 2y(xy)}{(y^2+z)^2} \\ h_z(x, y, z) &= 2x^3yz - \frac{xy}{(y^2+z)^2}. \end{aligned}$$

(c) Give an equation for the plane tangent to $f(x, y) = x\sin(xy)$ at the point $(1, \pi)$.

Solution: We compute

$$\begin{aligned} f_x(x, y) &= \sin(xy) + xy\cos(xy) \\ f_x(1, \pi) &= -\pi \\ f_y(x, y) &= x^2\cos(xy) \\ f_y(1, \pi) &= -1 \\ f(1, \pi) &= 0 \\ z &= 0 - \pi(x - 1) - (y - \pi). \end{aligned}$$

S2: Vector Functions

(a) (a) Find a parametric equation for a particle moving in a straight line, starting at $(5, 2, -2)$ and moving towards $(4, -3, 2)$.

(b) Suppose another particle follows the path $\vec{r}_2(t) = (t - 3, 5 - t, t)$. Does this particle's path intersect the path of the particle from part (a)?

Solution:

(a) There are many correct answers, but one is $\vec{r}_1(t) = (5 - t, 2 - 5t, -2 + 4t)$.

(b) The paths intersect if and only if

$$\begin{aligned} 5 - t_1 &= t_2 - 3 \\ 2 - 5t_1 &= 5 - t_2 \\ -2 + 4t_1 &= t_2 \end{aligned}$$

The last equation tells us that $t_2 = 4t_1 - 2$; substituting that into the first equation gives $5 - t_1 = 4t_1 - 5$ and thus $t_1 = 2$, which implies $t_2 = 6$. Plugging these numbers into the second equation gives $-8 = -1$, which is false, so the paths do not intersect.

(b) Find an equation for the line tangent to the curve $\vec{r}(t) = (3t, \ln(t^2 + 1), 5t^2 + 2)$ at the time $t = 3$.

Solution: We have $\vec{r}'(t) = (3, \frac{2t}{t^2+1}, 10t)$ and thus $\vec{r}'(3) = (3, 6/10, 30)$. So a parametric equation for the line is

$$T(t) = (9 + 3t, \ln(10) + 6t/10, 47 + 30t).$$

(c) Suppose a particle is at position $(3, 1, -1)$ at time $t = -1$ and has velocity vector $\vec{v}(t) = \cos(\pi t/6)\vec{i} + \sin(\pi t/6)\vec{j} + 2t\vec{k}$. Where is the particle at time $t = 3$?

Solution: The displacement is

$$\begin{aligned}
\int_{-1}^{3} \vec{v}(t)\, dt &= \int_{-1}^{3} \cos(\pi t/6)\vec{i} + \sin(\pi t/6)\vec{j} + 2t\vec{k}\, dt \\
&= \frac{6}{\pi}\sin(\pi t/6)\vec{i} - \frac{6}{\pi}\cos(\pi t/6)\vec{j} + t^2\vec{k}\Big|_{-1}^{3} \\
&= \left(\frac{6}{\pi}\vec{i} - 0\vec{j} + 9\vec{k}\right) - \left(\frac{-3}{\pi}\vec{i} - \frac{3\sqrt{3}}{\pi}\vec{j} + 1\vec{k}\right) \\
&= \frac{9}{\pi}\vec{i} + \frac{3\sqrt{3}}{\pi}\vec{j} + 8\vec{k}.
\end{aligned}$$

Since this is the displacement vector, we just need to add it to the starting position, to get

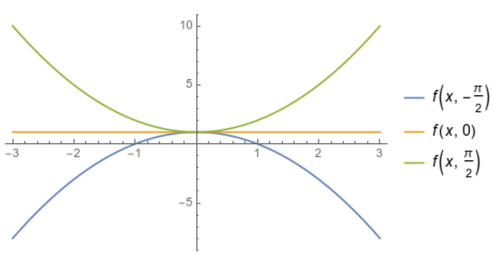
$$\begin{aligned}
\vec{r}(3) &= (3, 1, -1) + \left(\frac{9}{\pi}, \frac{3\sqrt{3}}{\pi}, 8\right) \\
&= \left(3 + \frac{9}{\pi}, 1 + \frac{3\sqrt{3}}{\pi}, 7\right).
\end{aligned}$$

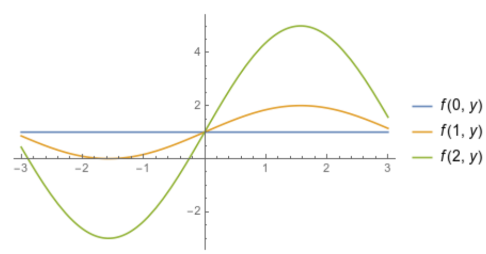
S3: Multivariable Functions

(a) Let $f(x, y) = x^2 \sin(y) + 1$. Sketch cross-sections of f for $x = 0, 1, 2$ and $y = -\pi/2, 0, \pi/2$.

Solution:

$$f(0, y) = 1 \qquad f(1, y) = \sin(y) + 1 \qquad f(2, y) = 4\sin(y) + 2$$
$$f(x, -\pi 2) = -x^2 + 1 \qquad f(x, 0) = 1 \qquad f(x, 2) = x^2 1$$



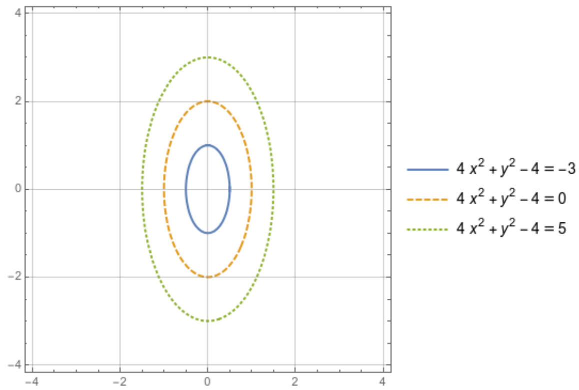


(b) Let $g(x, y) = 4x^2 + y^2 - 4$. Sketch a clearly labeled contour diagram that includes contours for $c = -3, c = 0, c = 5$. In a few words can you describe the shape of the graph of g?

Solution: Our contours work out to be

$$4x^2 + y^2 = 1 \qquad 4x^2 + y^2 = 4 \qquad 4x^2 + y^2 = 9$$

which are all ellipses, so we get the contour diagram



This gives us a sort of elongated bowl shape.

(c) Let $h(x, y, z) = \frac{x^2-y^2}{xz-yz}$. Compute $\lim_{(x,y,z)\to(1,1,1)} h(x, y, z) =$

Solution:

$$\lim_{(x,y,z)\to(1,1,1)} h(x, y, z) = \lim_{(x,y,z)\to(1,1,1)} \frac{x^2 - y^2}{xz - yz} = \lim_{(x,y,z)\to(1,1,1)} \frac{x + y}{z} = 2.$$