Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 4
Due Monday, July 14

This week's mastery quiz has three topics. Everyone should submit M2. If you have a 2/2 on S3, you don't need to do it again. If you have a 4/4 on M1 (meaning you have gotten 2/2 twice), you don't need to do it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Vectors
- Major Topic 2: Partial Derivatives
- Secondary Topic 3: Multivariable Functions

Name:

Name:

M1: Vectors

(a) Find the area of the parallelogram with vertices $(0,0,0),(3,1,2),(2,4,3),(5,5,5)$.

Solution: This parallelogram is spanned by the vectors $3\vec{i}+\vec{j}+2\vec{k}$ and $2\vec{i}+4\vec{j}+3\vec{k}$, so we compute

$$\begin{aligned}3\vec{i}+\vec{j}+2\vec{k}\times 2\vec{i}+4\vec{j}+3\vec{k} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & 2\\ 2 & 4 & 3\end{vmatrix}\\ &= 3\vec{i}+4\vec{j}+12\vec{k}-2\vec{k}-8\vec{i}-9\vec{j}\\ &= -5\vec{i}-5\vec{j}+10\vec{k}.\end{aligned}$$

Thus the area of the parallelogram is $\|-5\vec{i}-5\vec{j}+4\vec{k}\| = \sqrt{25+25+100} = \sqrt{150}$.

(b) Find $\cos\theta$ where θ is the angle between $\vec{u} = -2\vec{i}+4\vec{j}+\vec{k}$ and $\vec{v} = 3\vec{i}+\vec{j}+4\vec{k}$.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\\ &= \frac{-6+4+4}{\sqrt{21}\sqrt{26}}\\ &= \frac{2}{\sqrt{546}}.\end{aligned}$$

(c) Let $\vec{v} = 2\vec{i}+3\vec{j}+2\vec{k}$ and $\vec{u} = \vec{i}-\vec{j}+\vec{k}$. Compute the orthogonal decomposition of $\vec{v}$ with respect to $\vec{u}$. That is, write $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\perp}$.

Solution:

$$\begin{aligned}\vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{2-3+2}{1+1+1}\vec{u}\\ &= \frac{1}{3}\vec{u} = \frac{1}{3}\vec{i}-\frac{1}{3}\vec{j}+\frac{1}{3}\vec{k}\\ \vec{v}_{\perp} = \vec{v}-\vec{v}_{\text{parallel}} &= \frac{5}{3}\vec{i}+\frac{10}{3}\vec{j}+\frac{5}{3}\vec{k}.\end{aligned}$$

M2: Partial Derivatives

(a) Let $f(x,y,z) = \ln(xy+z)$. Find the directional derivative in the direction $-\vec{i}-\vec{j}+\vec{k}$ at the point $(0,3,1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x,y,z) &= \langle \frac{y}{xy+z}, \frac{x}{xy+z}, \frac{1}{xy+z} \rangle \\
\nabla f(0,3,1) &= \langle 3,0,1 \rangle \\
\vec{u} &= \frac{-1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} \\
f_{\vec{u}}(0,3,1) &= \nabla f(0,3,1) \cdot \vec{u} \\
&= -\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{-2}{\sqrt{3}}.
\end{aligned}$$

(b) Find all three second partial derivatives of $g(x,y) = x^2y + xy^3$.

Solution:

$$\begin{aligned}
g_x(x,y) &= 2xy + y^3 \\
g_y(x,y) &= x^2 + 3xy^2 \\
g_{xx}(x,y) &= 2y \\
g_{xy}(x,y) &= 2x + 3y^2 \\
g_{yy}(x,y) &= 6xy.
\end{aligned}$$

(c) Set $g(x,y) = ye^{2x+y}$. Use a linear approximation to estimate $g(-.9, 2.1)$.

Solution: We compute

$$\begin{aligned}
g(-1,2) &= 2 \\
g_x(x,y) &= 2ye^{2x+y} \\
g_x(-1,2) &= 4 \\
g_y(x,y) &= e^{2x+y} + ye^{2x+y} \\
g_y(-1,2) &= 3 \\
g(x,y) &\approx 2 + 4(x+1) + 3(y-2) \\
g(-.9, 2.1) &\approx 2 + .4 + .3 = 2.7
\end{aligned}$$

(The exact answer is $2.8347\ldots$ so this isn't great but isn't too bad.)

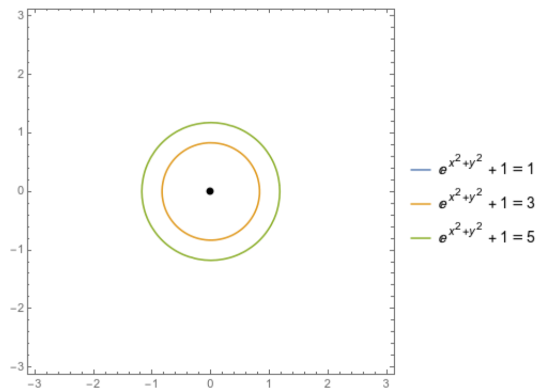
S3: Multivariable Functions

(a) Let $g(x,y) = 1 + e^{x^2+y^2}$. Sketch a clearly labeled contour diagram that includes contours for $c = 1, c = 3, c = 5$. In a few words can you describe the shape of the graph of g?

Solution: Our contours work out to be

$$
\begin{aligned}
1 + e^{x^2+y^2} &= 1 \qquad & e^{x^2+y^2} &= 0 \\
1 + e^{x^2+y^2} &= 3 \qquad & e^{x^2+y^2} &= 2 \\
1 + e^{x^2+y^2} &= 5 \qquad & e^{x^2+y^2} &= 4.
\end{aligned}
$$

The first equation requires $x^2 + y^2 = 0$ and thus the only solution is the origin. The other two equations give circles, of radius $\sqrt{\ln(2)}$ and $\sqrt{\ln(4)}$ respectively. This gives us a bowl that opens upwards quickly and dramatically.



(b) Let $h(x, y) = \frac{x^2y^2}{2x^4+y^4}$. Show that $\lim_{(x,y)\to(0,0)}$ does not exist.

Solution: If we approach along the line $y = 0$, we have

$$\lim_{x\to 0} \frac{0}{2x^4} = 0,$$

but if we approach along the line $y = x$ we have

$$\lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{3}.$$

Since these two answers don't agree, there's no overall limit.

(c) Let $f(x, y) = 3x^2 + xy - y$. Sketch cross-sections of f for $x = -1, 0, 1$ and $y = -2, 0, 2$.

Solution:

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\begin{aligned}
&f(-1, y) = 3 - y - y = 3 - 2y \qquad & f(0, y) = -y \qquad & f(1, y) = 3 + y - y = 3 \\
&f(x, -2) = 3x^2 - 2x + 2 \qquad & f(x, 0) = 3x^2 \qquad & f(x, 2) = 3x^2 + 2x - 2
\end{aligned}
$$