Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 4
Due Monday, July 14

This week's mastery quiz has two topics. Everyone should submit M3, which is new. If you have a 4/4 on M2 (meaning you have gotten 2/2 twice), you don't need to do it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Partial Derivatives
- Major Topic 3: Optimization

Name:

M2: Partial Derivatives

(a) Set $g(x,y) = x^2y^2 - \ln(2x - y)$. Use a linear approximation to estimate $g(1.1, .9)$.

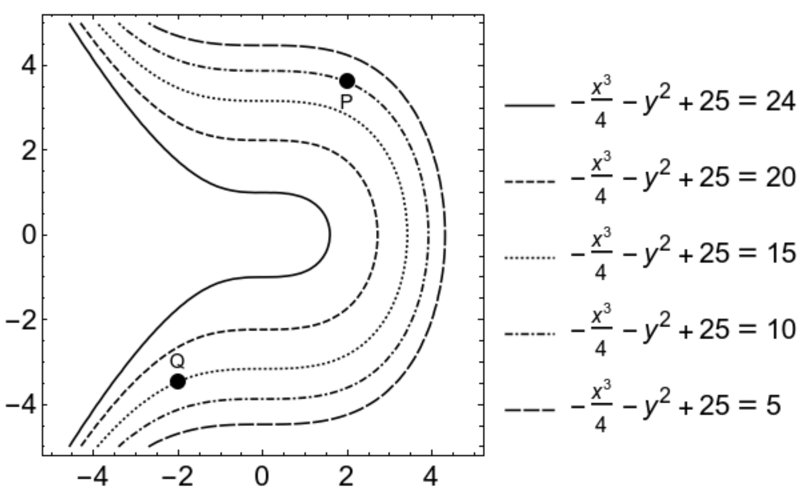
Solution: We compute

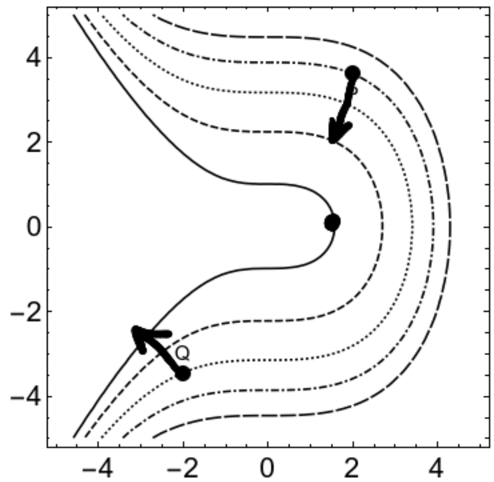
$$
\begin{aligned}
g(1,1) &= 1 - \ln(1) = 1 \\
g_x(x,y) &= 2xy^2 - \frac{2}{2x-y} \\
g_x(1,1) &= 1 - \frac{2}{1} = -1 \\
g_y(x,y) &= 2x^2y + \frac{1}{2x-y} \\
g_y(1,1) &= 1 + \frac{1}{1} = 2 \\
g(x,y) &\approx 1 - 1(x-1) + 2(y-1) \\
g(1.1,.9) &\approx 1 - 1(.1) + 2(-.1) = .7.
\end{aligned}
$$

(The exact answer is .717736 . . . so this isn't bad.)

(b) Below is a contour plot of the function $h(x,y)$.

- Sketch the gradient vector at P.
- Sketch the gradient vector at Q.
- Label a point on the diagram where $\frac{\partial h}{\partial y} = 0$.





Solution:

(c) Let $f(x, y, z) = x^2y - yz^3$. Find the directional derivative in the direction $\vec{i} + 2\vec{j} - \vec{k}$ at the point $(1, 2, 1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x, y, z) &= \langle 2xy, x^2 - z^3, -3yz^2 \rangle \\
\nabla f(1, 2, 1) &= \langle 4, 0, -6 \rangle \\
\vec{u} &= \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k} \\
f_{\vec{u}}(1, 2, 1) &= \nabla f(1, 2, 1) \cdot \vec{u} \\
&= \frac{4}{\sqrt{6}} + \frac{6}{\sqrt{6}} = \frac{10}{\sqrt{6}}.
\end{aligned}$$

M3: Optimization

(a) Find (but don't classify) the critical points of $g(x, y, z) = x^2y - xz^2$.

Solution: We have

$$\begin{aligned}
g_x(x, y, z) &= 2xy - z^2 \; g_y(x, y, z) &= x^2 \\
g_z(x, y, z) &= -2xz
\end{aligned}$$

The second equation says that $x = 0$. Then the first says that $z = 0$ and the third is totally redundant, so we have a critical point whenever $x = z = 0$.

(Important note: y is totally unconstrained here! So there are infinitely many critical points.)

(b) Find and classify the critical points of $f(x, y) = (3x + 4x^3)(y^2 + 2y)$.

Solution: We have

$$f_x(x, y) = (3 + 12x^2)(y^2 + 2y)$$
$$f_y(x, y) = (3x + 4x^3)(2y + 2).$$

The first equation tells us that we must have $y^2 + 2y = 0$ and thus $y = 0$ or $y = -2$. In either case, the second equation gives us that $3x + 4x^3 = 0$ and thus $x = 0$. (There are no other roots, since $3 + 4x^2 \neq 0$ for any real x. We don't want to think about complex numbers here.)

Thus our critical points are $(0, 0)$ and $(0, -2)$.

We have

$$\begin{aligned} f_{xx}(x, y) &= 24x(y^2 + 2y) & f_{xx}(0, 0) &= 0 & f_{xx}(0, -2) &= 0 \\ f_{xy}(x, y) &= (3 + 12x^2)(2y + 2) & f_{xy}(0, 0) &= 6 & f_{xy}(0, -2) &= -6 \\ f_{yy}(x, y) &= 6x + 8x^3 & f_{yy}(0, 0) &= 0 & f_{yy}(0, -2) &= 0. \end{aligned}$$

Then for $(0, 0)$ we have $D = 0 \cdot 0 - 6^2 = -36 < 0$, so we have a saddle point.

For $(0, -2)$ we have $D = 0 \cdot 0 - (-6)^2 = -36 < 0$, and so this is also a saddle point.