Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 4
Due Monday, July 14

This week's mastery quiz has three topics. Everyone should submit M3, which you're seeing for the second time, and M4, which is new. If you have a 4/4 on M2 (meaning you have gotten 2/2 twice), you don't need to do it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Partial Derivatives
- Major Topic 3: Optimization
- Major Topic 4: Integration

Name:

Name:

M2: Partial Derivatives

(a) Give an equation for the plane tangent to $f(x, y) = 3 + x^2y + \ln(x + xy)$ at the point $(1, 0)$.

Solution: We compute

$$\begin{aligned} f_x(x,y) &= 2xy + \frac{1+y}{x+xy} \\ f_x(1,0) &= 0 + \frac{1}{1} = 1 \\ f_y(x,y) &= x^2 + \frac{x}{x+xy} \\ f_y(1,0) &= 1 + \frac{1}{1} = 2 \\ f(1,0) &= 3 + 0 + \ln(1+0) = 3 \\ z &= 3 + 1(x-1) + 2(y-0). \end{aligned}$$

(b) Find all three second partial derivatives of $g(x, y) = \sqrt{x^2 + y}$.

Solution:

$$\begin{aligned} g_x(x,y) &= \frac{x}{\sqrt{x^2+y}} \\ g_y(x,y) &= \frac{1}{2\sqrt{x^2+y}} \\ g_{xx}(x,y) &= \frac{\sqrt{x^2+y} - \frac{x^2}{\sqrt{x^2+y}}}{(x^2+y)} \\ g_{xy}(x,y) &= \frac{-2x}{4(x^2+y)^{3/2}} \\ g_{yy}(x,y) &= \frac{-1}{4(x^2+y)^{3/2}}. \end{aligned}$$

(c) Let $f(x, y, z) = \frac{e^{x\sqrt{y}}}{z}$. Find the directional derivative in the direction $\vec{i} + 3\vec{j} - 2\vec{k}$ at the point $(0, 1, 1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x,y,z) &= \langle \sqrt{y}e^{x\sqrt{y}}/z, \frac{x}{2\sqrt{y}z}e^{x\sqrt{y}}, -e^{x\sqrt{y}}/z^2\rangle \\
\nabla f(0,1,1) &= \langle 1,0,-1\rangle \\
\vec{u} &= \frac{1}{\sqrt{14}}\vec{i} + \frac{3}{\sqrt{14}}j - \frac{2}{\sqrt{2}}\vec{k} \\
f_{\vec{u}}(0,1,1) &= \nabla f(0,1,1)\cdot \vec{u} \\
&= \langle 1,0,-1\rangle \cdot \langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\rangle \\
&= \frac{1}{\sqrt{14}} + \frac{2}{\sqrt{14}} = \frac{3}{\sqrt{14}}
\end{aligned}$$

M3: Optimization

(a) Find and classify the critical points of $f(x,y) = x^2 + xy^2 + y^2$.

Solution: We have

$$\begin{aligned}
f_x(x,y) &= 2x + y^2 \\
f_y(x,y) &= 2xy + 2y
\end{aligned}$$

The second equation tells us that either $y = 0$ or $x = -1$. If $y = 0$ then the first equation gives $x = 0$. If $x = -1$ then the first equation gives $y^2 = 2$ so $y = \pm\sqrt{2}$. Thus our critical points are $(0,0), (-1,\sqrt{2}), (-1,-\sqrt{2})$.

We have

$$\begin{array}{llll}
f_{xx}(x,y) = 2 & f_{xx}(0,0) = 2 & f_{xx}(-1,\sqrt{2}) = 2 & f_{xx}(-1,-\sqrt{2}) = 2 \\
f_{xy}(x,y) = 2y & f_{xy}(0,0) = 0 & f_{xy}(-1,\sqrt{2}) = 2\sqrt{2} & f_{xy}(-1,-\sqrt{2}) = -2\sqrt{2} \\
f_{yy}(x,y) = 2x+2 & f_{yy}(0,0) = 2 & f_{yy}(-1,\sqrt{2}) = 0 & f_{yy}(-1,-\sqrt{2}) = 0
\end{array}$$

Then for $(0,0)$ we have $D = 2\cdot 2 - 0^2 = 4 > 0$, and since $f_{xx}(0,0) = 2 > 0$ this is a local minimum.

For $(-1,\sqrt{2})$ we have $D = 2\cdot 0 - (2\sqrt{2})^2 = -8 < 0$, so this is a saddle point.

For $(-1,-\sqrt{2})$ we have $D = 2\cdot 0 - (-2\sqrt{2})^2 = -8 < 0$, so this is a saddle point.

(b) Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint that $x^2 + 4y^2 \leq 1$.

Solution: We need to think about interior critical points, and also critical points on the boundary.

We have $\nabla f = \langle y, x\rangle$ which is zero only when $x, y = 0$. This gives us one critical point at $(0,0)$, and $f(0,0) = 0$.

Now we consider the boundary. We have

$$y = \lambda 2x$$
$$x = \lambda 8y.$$

This gives us $x = \lambda^2 \cdot 16x$ and so $\lambda = \pm 1/4$. (Or $x = 0$, but then we have $y = 0$ and that contradicts the constraint equation.)

Then we have $x = \pm 2y$ and thus our equation is $4y^2 + 4y^2 = 1$, so $y = \pm\sqrt{1/8}$. Then $x = \pm\sqrt{1/2}$, and all four combinations are possible. We calculate

$$f(1/\sqrt{2}, 1/\sqrt{8} = \frac{1}{\sqrt{16}} = 1/4$$
$$f(1/\sqrt{2}, -1/\sqrt{8}) = -1/4$$
$$f(-1/\sqrt{2}, 1/\sqrt{8} = -1/4$$
$$f(-1/\sqrt{2}, -1/\sqrt{8}) = 1/4.$$

Thus the absolute maximum of f on this region is $1/4$, and the absolute minimum is $-1/4$.

M4: Integration

(a) Find the volume of the solid bounded above by $f(x, y) = e^{-x^2}$, below by the plane $z = 0$, and over the triangle formed by $x = 1, y = 0, y = x$.

Solution: We have $\int_0^1 \int_0^x e^{-x^2}\,dy\,dx$. Then we compute

$$\int_0^1 \int_0^x e^{-x^2}\,dy\,dx = \int_0^1 ye^{-x^2}\Big|_0^x dx$$
$$= \int_0^1 xe^{-x^2}\,dx$$
$$= \frac{-1}{2}e^{-x^2}\Big|_0^1 = -\frac{1}{2}(e^{-1} - 1) = \frac{1}{2} - \frac{1}{2e}.$$

We could set it up in the other order: we get $\int_0^1 \int_y^1 e^{-x^2}\,dx\,dy$. But there's no closed-form antiderivative for e^{-x^2}, so we get kinda stuck.

(b) Sketch the region of integration and compute $\int_{-1}^0 \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2\,dx\,dy$. (Do not use a calculator!)

Solution: This is much easier if we change the order of integration. We see that x varies from -1 to 1, and then the equation of the curve is $x^2 = y + 1$ and thus $y = x^2 - 1$. So we have

$$\begin{aligned}
\int_{-1}^{1}\int_{x^2-1}^{0} y^2\,dy\,dx &= \int_{-1}^{1} \frac{y^3}{3}\Big|_{x^2-1}^{0}\,dx \\
&= \int_{-1}^{1} -\frac{(x^2-1)^3}{3}\,dx = \frac{1}{3}\int_{-1}^{1} 1 - 3x^2 + 3x^4 - x^6\,dx \\
&= \frac{1}{3}\left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right)\Big|_{-1}^{1} \\
&= \frac{1}{3}\left(1 - 1 + 3/5 - 1/7 - (-1 + 1 - 3/5 + 1/7)\right) = 6/15 - 2/21 = \frac{32}{105}.
\end{aligned}$$

(c) Integrate the function $f(x, y, z) = xyz$ over the region bounded by $x = 0, x = 4, y = 0, z = 0$, and $y = 4 - z^2$.

Solution: x goes from 0 to 4. Then z goes from 0 to 2 and y goes from 0 to $4 - z^2$, and we have

$$\begin{aligned}
\int_0^2\int_0^{4-z^2}\int_0^4 xyz\,dx\,dy\,dz &= \int_0^2\int_0^{4-z^2} \frac{x^2yz}{2}\Big|_0^4\,dy\,dz \\
&= \int_0^2\int_0^{4-z^2} 8yz\,dy\,dz \\
&= \int_0^2 4y^2z\Big|_0^{4-z^2}\,dz \\
&= \int_0^2 4z(4-z^2)^2\,dz = 4\int_0^2 z^5 - 8z^3 + 16z\,dz \\
&= 4\left(\frac{z^6}{6} - 2z^4 + 8z^2\right)\Big|_0^2 \\
&= 4\left(\frac{32}{3} - 32 + 32\right) = 128/3.
\end{aligned}$$