Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 7
Due Wednesday, July 23

This week's mastery quiz has two topics. Everyone should submit M4, which you're seeing for the second time. If you have a 4/4 on M3 (meaning you have gotten 2/2 twice), you don't need to do it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Major Topic 4: Integration

Name:

Name:

M3: Optimization

(a) Find and classify the critical points of $f(x, y) = x^2 + xy^2 + y^2$.

Solution: We have

$$f_x(x, y) = 2x + y^2$$
$$f_y(x, y) = 2xy + 2y$$

The second equation tells us that either $y = 0$ or $x = -1$. If $y = 0$ then the first equation gives $x = 0$. If $x = -1$ then the first equation gives $y^2 = 2$ so $y = \pm\sqrt{2}$. Thus our critical points are $(0, 0), (-1, \sqrt{2}), (-1, -\sqrt{2})$.

We have

$$\begin{array}{llll} f_{xx}(x,y) = 2 & f_{xx}(0,0) = 2 & f_{xx}(-1,\sqrt{2}) = 2 & f_{xx}(-1,-\sqrt{2}) = 2 \\ f_{xy}(x,y) = 2y & f_{xy}(0,0) = 0 & f_{xy}(-1,\sqrt{2}) = 2\sqrt{2} & f_{xy}(-1,-\sqrt{2}) = -2\sqrt{2} \\ f_{yy}(x,y) = 2x+2 & f_{yy}(0,0) = 2 & f_{yy}(-1,\sqrt{2}) = 0 & f_{yy}(-1,-\sqrt{2}) = 0 \end{array}$$

Then for $(0, 0)$ we have $D = 2 \cdot 2 - 0^2 = 4 > 0$, and since $f_{xx}(0, 0) = 2 > 0$ this is a local minimum.

For $(-1, \sqrt{2})$ we have $D = 2 \cdot 0 - (2\sqrt{2})^2 = -8 < 0$, so this is a saddle point.

For $(-1, -\sqrt{2})$ we have $D = 2 \cdot 0 - (-2\sqrt{2})^2 = -8 < 0$, so this is a saddle point.

(b) Use the method of Lagrange multipliers to find the point on the circle $x^2 + y^2 = 40$ closest to the point $(1, 3)$.

Solution: Our constraint function is $g(x, y) = x^2 + y^2 = 40$. Our objective function is the distance $\sqrt{(x-1)^2 + (y-3)^2}$, but it's okay to just minimize $f(x, y) = (x-1)^2 + (y-3)^2$. Then we have

$$2(x-1) = \lambda 2x$$
$$2(y-3) = \lambda 2y.$$

Rearranging gives

$$\begin{aligned} (1-\lambda)x &= 1 \\ (1-\lambda)y &= 3 \\ 1/x &= 3/y \\ y &= 3x. \end{aligned}$$

Substituting that into our constraint equation gives

$$\begin{aligned} x^2 + 9x^2 &= 40 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

and thus our critical points are $(2,6)$ and $(-2,-6)$ which we check are indeed on the circle.

Then we compute $f(2,6) = 1^2 + 3^2 = 10$, so this point is at distance $\sqrt{10}$ from $(1,3)$. And we have $f(-2,-6) = 3^2 + 9^2 = 90$ so this point is at distance $\sqrt{90}$ from $(1,3)$. Thus the point closest to $(1,3)$ is $(2,6)$—which maybe becomes obvious in retrospect if you look at a picture.

(c) Find (but don't classify) the critical points of $g(x,y,z) = x^3+y^3-3x^2-y^2-z^2+2z-1$.

Solution: We have

$$\begin{aligned} g_x(x,y,z) &= 3x^2 - 6x \\ g_y(x,y,z) &= 3y^2 - 2y \\ g_z(x,y,z) &= -2z + 2 \end{aligned}$$

The first equation gives $x = 0$ or $x = 2$; the second equation gives $y = 0$ or $y = 2/3$; and the third equation gives $z = 1$. So the critical points are

$$(0,0,1), (2,0,1), (0,2/3,1), (2,2/3,1).$$

M4: Integration

(a) We want to find the volume of the region enclosed by the portion of the cylinder $x^2 + y^2 = 9$ with $y \leq 0, z \geq 0$, and the sphere $x^2 + y^2 + z^2 = 25$. Set up three different iterated integrals to compute this, in cartesian, cylindrical, and spherical coordinates. Choose one of the integrals you set up and evaluate it.

Solution: The two surfaces intersect when $9 + z^2 = 25$, so $z = \pm 4$ and we have a circle of radius 3. Thus we get

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{0} \int_{0}^{\sqrt{25-x^2-y^2}} 1 \, dz \, dy \, dx.$$

I do not want to do this integral.

Cylindrical looks much better. We worked out above that the radius of the circle of intersection is 3; and theta goes from π to 2π to stay in the half-space where y is negative. Not forgetting the Jacobian we have

$$\int_{0}^{3} \int_{\pi}^{2\pi} \int_{0}^{\sqrt{25-r^2}} 1 \cdot r \, dz \, d\theta \, dr.$$

This one is very doable! We compute

$$\begin{aligned} I &= \int_0^3 \int_\pi^{2\pi} r\sqrt{25 - r^2}\, d\theta\, dr \\ &= \pi \int_0^3 r\sqrt{25 - r^2}\, dr \\ &= -\frac{\pi}{3}(25 - r^2)^{3/2}\Big|_0^3 \\ &= \frac{-\pi}{3}\left(16^{3/2} - 25^{3/2}\right) \\ &= \frac{-\pi}{3}(64 - 125) = \frac{61\pi}{3}. \end{aligned}$$

In spherical things don't work out quite so well. The equation of the top is easy, but the side wall of the cylindar is hard to handle.

We again have θ going from π to 2π, and in each direction ρ varies from 0 to 4. But we have to split our other bounds into two parts. When ϕ is between 0 and $\arctan(3/4)$, then ρ goes from 0 to 5. But then when ϕ is between $\arctan(3/4)$ and $\pi/2$, we need to do some trigonometry: we get $\rho = 3/\sin(\phi)$. Then our integral is

$$\int_\pi^{2\pi} \int_0^{\arctan(3/4)} \int_0^5 1 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta + \int_\pi^{2\pi} \int_{\arctan(3/4)}^{\pi/2} \int_0^{3/\sin(\phi)} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

I don't want to do this one either.

(b) Sketch the region of integration and compute $\iint_R y\sqrt{x^2 + y^2}\, dA$ where R is the region given by $x^2 + y^2 \leq 4$ and $0 \leq y \leq x$.

Solution: We change to polar coordinates because this is a wedge of a circle. We have $0 \leq r \leq 2$ (be careful here, $x^2 + y^2$ is the radius squared!) and $0 \leq theta \leq \pi/4$. We know that $\sqrt{x^2 + y^2} = r$ and $y = r\sin\theta$, so we get

$$\begin{aligned} \iint_R y\sqrt{x^2 + y^2}\, dA &= \int_0^2 \int_0^{\pi/4} r^2 \sin(\theta) \cdot r\, d\theta\, dr \\ &= \int_0^2 -r^3 \cos(\theta)\Big|_0^{\pi/4} dr \\ &= \int_0^2 -r^3(\sqrt{2}/2 - 1)\, dr \\ &= (1 - \sqrt{2}/2)r^4/4\Big|_0^3 = 4 - 4\sqrt{2}/2. \end{aligned}$$

(c) Let R be the parallelogram with vertices $(0, 0), (1, 2), (3, 3), (4, 5)$. Find a transformation that translates to the square with vertices $(0, 0), (0, 1), (1, 0), (1, 1)$. Use this transformation to compute $\iint_R xy\, dA$.

Solution: We want $s = 1, t = 0$ to give us the point $(1, 2)$, and $s = 0, t = 1$ to give us the point $(3, 3)$. So we take $x = s + 3t$ and $y = 2s + 3t$. Then if $s = 1, t = 1$ we get $(4, 5)$, and this gives us our mapping of the square to the parallelogram.

We compute the Jacobian

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right| = |1 \cdot 3 - 3 \cdot 2| = 3.$$

Then

$$\begin{aligned}
\iint_R xy\, dA &= \int_0^1 \int_0^1 (s+3t)(2s+3t) \cdot 3\, ds\, dt \\
&= 3 \int_0^1 \int_0^1 2s^2 + 9st + 9t^2\, ds\, dt \\
&= 3 \int_0^1 2s^3/3 + 9s^2t/2 + 9t^2 s\Big|_0^1\, dt \\
&= 3 \int_0^1 2/3 + 9t/2 + 9t^2\, dt \\
&= 3\left(2t/3 + 9t^2/4 + 3t^3\right)\Big|_0^1 \\
&= 3(2/3 + 9/4 + 3) = 11 + 27/4 = 17 + 3/4 = 71/4.
\end{aligned}$$